

APPROXIMATION OF FIXED POINTS
OF STRONGLY PSEUDOCONTRACTIVE MAPS
WITHOUT LIPSCHITZ ASSUMPTION

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In the present paper, the following result is shown: Let X be a real Banach space with a uniformly convex dual X^* , and let K be a nonempty closed convex and bounded subset of X . Assume that $T : K \rightarrow K$ is a continuous strong pseudocontraction. Let $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be two real sequences satisfying

- (i) $0 < \alpha_n, \beta_n < 1$ for all $n \geq 1$;
- (ii) $\sum_{n=1}^\infty \alpha_n = \infty$; and
- (iii) $\alpha_n \rightarrow 0, \beta_n \rightarrow 0$ as $n \rightarrow \infty$.

Then the Ishikawa iterative sequence $\{x_n\}_{n=1}^\infty$ generated by

$$(I) \begin{cases} x_1 \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{cases}$$

converges strongly to the unique fixed point of T .

1. INTRODUCTION AND PRELIMINARIES

Let X be a normed linear space, $K \subseteq X$. A mapping $T : K \rightarrow K$ is said to be strongly pseudocontractive if there exists $t > 1$ such that the inequality

$$(1) \quad \|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

holds for every $x, y \in K$ and $r > 0$. A mapping U with domain $D(U)$ and range $R(U)$ in X is called accretive if the following inequality

$$(2) \quad \|x - y\| \leq \|x - y + s(Ux - Uy)\|$$

holds for every $x, y \in D(U)$ and for all $s \geq 0$. Browder [1, Prop. 1] proved that T is pseudocontractive if and only if $(I - T)$ is accretive, where I denotes the identity operator.

Let X be a real Banach space with a dual X^* . The normalized duality mapping $J : X \rightarrow 2^{X^*}$ is defined by

$$J(x) = \{f \in X^*; \langle x, f \rangle = \|x\|^2, \|x\| = \|f\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if X^* is strictly convex, then J is single valued and such that $J(-x) = -J(x)$, $J(tx) =$

Received by the editors December 5, 1995.

1991 *Mathematics Subject Classification*. Primary 47H17.

Key words and phrases. The Ishikawa iteration, strong pseudocontraction, strictly convex Banach space.

$tJ(x)$ for all $t \geq 0$, $x \in X$. If X^* is uniformly convex, then J is uniformly continuous on the bounded sets of X (see, e.g., Barbu [2])

By means of the normalized duality mapping J , we can give the following equivalent definition for accretive mapping.

A mapping $A : K \rightarrow K$ is said to be accretive if for each x, y in K there exists $j \in J(x - y)$ such that

$$(3) \quad \langle Ax - Ay, j \rangle \geq 0.$$

A mapping $A : K \rightarrow K$ is called strongly accretive if for each $x, y \in K$ there exists $j \in J(x - y)$ such that

$$(4) \quad \langle Ax - Ay, j \rangle \geq k\|x - y\|^2$$

for some constant $k > 0$. Without loss of generality we shall assume that $k \in (0, 1)$.

In [3], Bogin proved that U is strongly pseudocontractive if and only if $I - U$ is strongly accretive.

Very recently, Chidume [4] proved the following result: If X is a real Banach space with a uniformly convex dual X^* , and K is a nonempty closed convex and bounded subset of X , and $T : K \rightarrow K$ is a Lipschitz strongly pseudocontractive mapping, the Ishikawa iterative sequence $\{x_n\}_{n=1}^{\infty}$ generated by (I) converges strongly to the unique fixed point of T . However, when T is continuous strongly pseudocontractive, it is still an open question whether or not the Ishikawa iterative sequence $\{x_n\}_{n=1}^{\infty}$ defined by (I) converges strongly to a fixed point of T .

It is our purpose in this paper to solve the above question by proving the following much more general result: If X is a real Banach space with a uniformly convex dual X^* , K is a nonempty bounded closed convex subset of X , and $T : K \rightarrow K$ is a continuous strongly pseudocontractive mapping, then the Ishikawa iteration sequence $\{x_n\}_{n=1}^{\infty}$ defined by (I) converges strongly to the unique fixed point of T .

We need the following lemmas.

Lemma 1.1. *Let X^* be strictly convex; then*

$$(5) \quad \|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle$$

for all $x, y \in X$.

Proof. For arbitrary fixed $x, y \in X$, let $\varphi(s) = \|x + sy\|^2$, $s \in [0, +\infty)$; then $\varphi : [0, +\infty) \rightarrow [0, \infty)$ is a convex function.

To see this, for every $s_1, s_2 \in [0, +\infty)$, $t \in [0, 1]$, we have

$$\begin{aligned} \varphi(ts_1 + (1-t)s_2) &= \|t(x + s_1y) + (1-t)(x + s_2y)\|^2 \\ &\leq t^2\|x + s_1y\|^2 + (1-t)^2\|x + s_2y\|^2 + 2t(1-t)\|x + s_1y\|\|x + s_2y\| \\ &\leq (t^2 + t(1-t))\|x + s_1y\|^2 + ((1-t)^2 + t(1-t))\|x + s_2y\|^2 \\ &= t\varphi(s_1) + (1-t)\varphi(s_2). \end{aligned}$$

Since $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is a convex function, as $s > t > 0$, we have

$$\begin{aligned} \varphi(t) &= \varphi\left(\frac{t}{s} \cdot s + \left(1 - \frac{t}{s}\right) \cdot 0\right) \\ &\leq \frac{t}{s}\varphi(s) + \left(1 - \frac{t}{s}\right)\varphi(0) \\ &= \frac{t}{s}(\varphi(s) - \varphi(0)) + \varphi(0), \end{aligned}$$

and then

$$t^{-1}(\varphi(t) - \varphi(0)) \leq s^{-1}(\varphi(s) - \varphi(0)).$$

Hence, function $t \mapsto t^{-1}(\varphi(t) - \varphi(0))$ is increasing. As a result, $\lim_{t \rightarrow 0^+} t^{-1}(\varphi(t) - \varphi(0))$ exists and

$$\|x + y\|^2 - \|x\|^2 \geq \lim_{t \rightarrow 0^+} t^{-1}(\varphi(t) - \varphi(0)) = \langle y, \text{grad}\|x\|^2 \rangle = 2\langle y, J(x) \rangle,$$

so that, for all $x, y \in X$ we have

$$(6) \quad \|x + y\|^2 \geq \|x\|^2 + 2\langle y, J(x) \rangle.$$

Now by replacing x by $x - y$ in (6) we get

$$(7) \quad \|x\|^2 \geq \|x - y\|^2 + 2\langle y, J(x - y) \rangle.$$

Replacing x by $-x$ in (7), we obtain

$$(8) \quad \|x\|^2 \geq \|x + y\|^2 - 2\langle y, J(x + y) \rangle.$$

Thus, we have

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle$$

for all $x, y \in X$. □

Lemma 1.2. *Let $\{\rho_n\}_{n=1}^\infty$ be a nonnegative real sequence satisfying*

$$(9) \quad \rho_{n+1} \leq (1 - \lambda_n)\rho_n + \sigma_n,$$

where $\lambda_n \in [0, 1]$, $\sum_{n=1}^\infty \lambda_n = \infty$, and $\sigma_n = o(\lambda_n)$. Then $\rho_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Let $\alpha = \inf\{\rho_n : n \geq 1\}$; then $\alpha = 0$. If not, assume that $\alpha > 0$; then for all $n \geq 1$, $\rho_n \geq \alpha > 0$.

Since $\sigma_n = o(\lambda_n)$ we know that

$$\sigma_n < \frac{1}{2}\rho_n\lambda_n$$

for all large n .

From (9), we get

$$\rho_{n+1} < (1 - \lambda_n)\rho_n + \frac{1}{2}\rho_n\lambda_n = (1 - \frac{1}{2}\lambda_n)\rho_n.$$

By a simple iteration we get that

$$0 \leq \rho_{n+1} \leq \exp\{-\frac{1}{2}\sum_{j=1}^n \lambda_j\}\rho_1 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

i.e., a contradiction. Thus there exists some subsequence of $\{\rho_n\}$, say $\{\rho_{n_j}\}$, such that $\rho_{n_j} \rightarrow 0$ as $j \rightarrow \infty$. Hence for arbitrary given $\epsilon > 0$, there exists large integer $j_0 \geq 1$ such that

$$\sigma_n < \lambda_n\epsilon, \rho_{n_{j_0}} < \epsilon$$

for all large $n \geq n_{j_0}$.

By virtue of (9) we have

$$\rho_{n_{j_0}+1} \leq (1 - \lambda_{n_{j_0}})\epsilon + \lambda_{n_{j_0}}\epsilon = \epsilon;$$

by induction, we have

$$\rho_{n_{j_0}+k} \leq \epsilon, \text{ for all } k \geq 1.$$

This shows $\rho_n \rightarrow 0$ as $n \rightarrow \infty$. □

2. MAIN RESULTS

Now we prove the main results.

Theorem 2.1. *Let X be a real Banach space with a uniformly convex dual X^* , K be a nonempty closed convex bounded subset of X , and $T : K \rightarrow K$ be a continuous strongly pseudocontractive mapping. Then the Ishikawa iterative sequence $\{x_n\}_{n=1}^{\infty}$ defined by (I) converges strongly to the unique fixed point of T .*

Proof. The existence of a fixed point follows from Deimling [5]. Let q be a fixed point of T . Since $T : K \rightarrow K$ is strongly pseudocontractive, $I - T$ is strongly accretive, and for every $x, y \in K$

$$(10) \quad \langle (I - T)x - (I - T)y, J(x - y) \rangle \geq k\|x - y\|^2,$$

where $k = (t - 1)t^{-1}$ (see, e.g., [3]).

From (10) we obtain

$$(11) \quad \langle Tx - Ty, J(x - y) \rangle \leq (1 - k)\|x - y\|^2$$

for all $x, y \in K$.

Using (5), (11) and (I), we get

$$(12) \quad \begin{aligned} \|x_{n+1} - q\|^2 &= \|(1 - \alpha_n)(x_n - q) + \alpha_n(Ty_n - Tq)\|^2 \\ &\leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle Ty_n - Tq, J(x_{n+1} - q) \rangle \\ &= (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle Ty_n - Tq, J(x_{n+1} - q) - J(y_n - q) \rangle \\ &\quad + 2\alpha_n\langle Ty_n - Tq, J(y_n - q) \rangle \\ &\leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n a_n + 2\alpha_n(1 - k)\|y_n - q\|^2, \end{aligned}$$

where

$$a_n = \langle Ty_n - Tq, J(x_{n+1} - q) - J(y_n - q) \rangle.$$

Now we shall show $a_n \rightarrow 0$ as $n \rightarrow \infty$. Observe that $\{Ty_n - Tq\}$ is bounded; then to show $a_n \rightarrow 0$ as $n \rightarrow \infty$, it suffices to show

$$J(x_{n+1} - q) - J(y_n - q) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Indeed, since X^* is uniformly convex, J is uniformly continuous on any bounded subset of X . Noting that $(x_{n+1} - q) - (y_n - q) = x_{n+1} - y_n = (\beta_n - \alpha_n)x_n + \alpha_n Ty_n - \beta_n q \rightarrow 0$ as $n \rightarrow \infty$, we see that

$$J(x_{n+1} - q) - J(y_n - q) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Again using (5) and (I), we have

$$(13) \quad \begin{aligned} \|y_n - q\|^2 &= \|(1 - \beta_n)(x_n - q) + \beta_n(Tx_n - Tq)\|^2 \\ &\leq (1 - \beta_n)^2\|x_n - q\|^2 + 2\beta_n\langle Tx_n - Tq, J(y_n - q) \rangle \\ &\leq \|x_n - q\|^2 + M_1\beta_n \end{aligned}$$

for some constant $M_1 > 0$, since $\langle Tx_n - Tq, J(y_n - q) \rangle$ is bounded.

Substituting (13) in (12) yields

$$\begin{aligned}
 (14) \quad \|x_{n+1} - q\|^2 &\leq ((1 - \alpha_n)^2 + 2(1 - k)\alpha_n)\|x_n - q\|^2 \\
 &\quad + 2\alpha_n(a_n + (1 - k)M_1\beta_n) \\
 &= (1 - 2\alpha_n + \alpha_n^2 + 2\alpha_n - 2k\alpha_n)\|x_n - q\|^2 \\
 &\quad + 2\alpha_n(a_n + (1 - k)M_1\beta_n) \\
 &= (1 + \alpha_n^2 - 2k\alpha_n)\|x_n - q\|^2 + \sigma_n,
 \end{aligned}$$

where $\sigma_n = 2\alpha_n(a_n + (1 - k)M_1\beta_n)$.

We choose a large positive integer N such that for all $n > N$

$$\alpha_n < k.$$

Thus, the above inequality yields

$$\|x_{n+1} - q\|^2 \leq (1 - k\alpha_n)\|x_n - q\|^2 + \sigma_n.$$

Set $\rho_n = \|x_n - q\|^2$ and $\lambda_n = k\alpha_n$. Then

$$\rho_{n+1} \leq (1 - \lambda_n)\rho_n + \sigma_n,$$

with $\lambda_n \in [0, 1]$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sigma_n = o(\lambda_n)$. By Lemma 1.2 we see that $\rho_n \rightarrow 0$ as $n \rightarrow \infty$.

Now we prove the uniqueness. Suppose q_1 is another fixed point of T ; then by (10) we have

$$0 = \langle (q - Tq) - (q_1 - Tq_1), J(q - q_1) \rangle \geq k\|q - q_1\|^2,$$

which implies $q_1 = q$.

The proof of this theorem is completed. \square

Remark. Our theorem extends Theorem 2 of Chidume [4] to a larger class of continuous strongly pseudocontractive mappings; thus we have solved an open question put forth by Chidume in [4].

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