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A NOTE ON THE CHERN-SIMONS INVARIANT OF HYPERBOLIC 3-MANIFOLDS

MINGQING OUYANG

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Abstract. In this note we study how the Chern-Simons invariant behaves when two hyperbolic 3-manifolds are glued together along incompressible thrice-punctured spheres. Such an operation produces many hyperbolic 3-manifolds with different numbers of cusps sharing the same volume and the same Chern-Simons invariant. The results in this note, combined with those of Meyerhoff and Ruberman, give an algorithm for determining the unknown constant in Neumann's simplicial formula for the Chern-Simons invariant of hyperbolic 3-manifolds.

Introduction

Let M_1 and M_2 be oriented complete finite-volume hyperbolic 3-manifolds containing incompressible thrice-punctured spheres S_1 and S_2 respectively. Let $M_i' = M_i - N(S_i)$ where $N(S_i)$ is a neighborhood of S_i in M_i. Denote by S_i^0 and S_i^1 the two copies of S_i in $\partial M_i'$. Let $\phi_0 : S_1^0 \to S_2^0$ and $\phi_1 : S_1^1 \to S_2^1$ be two homeomorphisms which are either both orientation-preserving or both orientation-reversing. Then one can glue M_1 and M_2 together to form a new 3-manifold M by identifying S_1^0 and S_2^0 using ϕ_0 and identifying S_1^1 and S_2^1 using ϕ_1. As shown in [A1], M is hyperbolic with volume equal to the sum of the volumes of M_1 and M_2.

Denote by $CS(M)$ the Chern-Simons invariant of a complete finite-volume hyperbolic 3-manifold M.

The main result of this note is the following

Theorem 1. *If M is obtained by gluing M_1 and M_2 together along copies of S_1 and S_2 via the same identification (i.e., $\phi_0 = \phi_1$), then $CS(M) = CS(M_1) + CS(M_2)$.*

In particular, let L_1 and L_2 be hyperbolic links in S^3 with projections as shown in Figure 1(a) and (b).

The hyperbolicity of L_1 and L_2 implies that the twice-punctured disks bounded by the trivial components are incompressible. By cutting $S^3 - L_1$ and $S^3 - L_2$ open and gluing together along the disks, one gets the complement of a hyperbolic link L whose projection is shown in Figure 2.

Following [A1], we call L a belted sum of L_1 and L_2. In this note we denote $L = L_1 \oplus L_2$ (◯ represents the trivial component).

Figure 3 shows that the Borromean rings form a belted sum of the Whitehead links of the opposite-handed clasps.

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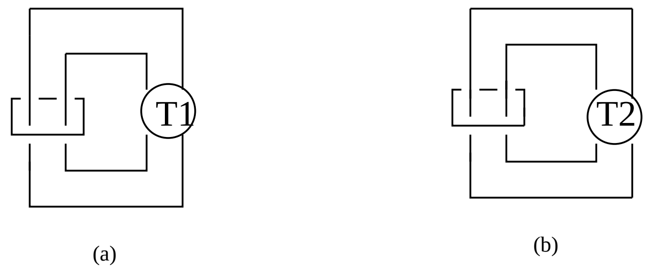

FIGURE 1

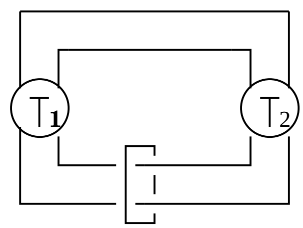

FIGURE 2

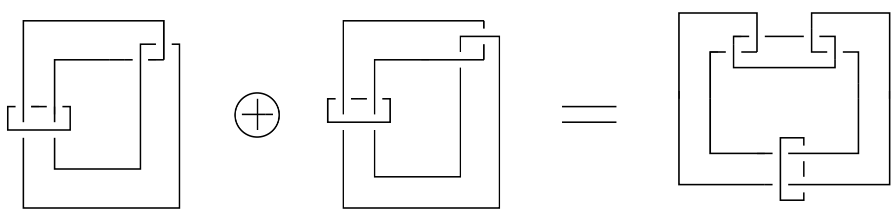

FIGURE 3

By the Mostow-Prasad Rigidity Theorem, $CS(S^3 - L)$ is actually a link invariant of a hyperbolic knot or link L. One has the following

Corollary 2. $CS(S^3 - L) = CS(S^3 - L_1) + CS(S^3 - L_2)$.

Example. As shown in [MR], the Chern-Simons invariant of the Whitehead link is $\pm 1/8$ depending on the handedness. Therefore the invariant for the Borromean rings is equal to 0. This can also be seen from the amphicheirality of the Borromean rings ([MO]).

Unlike the volume, if a new manifold M is obtained by gluing together M_1 and M_2 along copies of S_1 and S_2 via different identifications, then $CS(M) \neq CS(M_1) + CS(M_2)$ in general. For example, denote by μ a mutation along either S_1 or S_2. If identifications ϕ and $\phi \circ \mu$ are used to glue M_1 and M_2 together to get M, then one would have $CS(M) = CS(M_1) + CS(M_2) + 1/4$. This follows from the fact that a mutation along an incompressible thrice-punctured sphere changes the Chern-Simons invariant by $1/4$ ([MR]).

Given a hyperbolic knot or link L. Suppose that L is in a particular regular alternating projection without any unnecessary crossings. Denote by P the projection plane. Let $C_1, \cdots, C_n$ be n nonisotopic embedded circles in the complement of

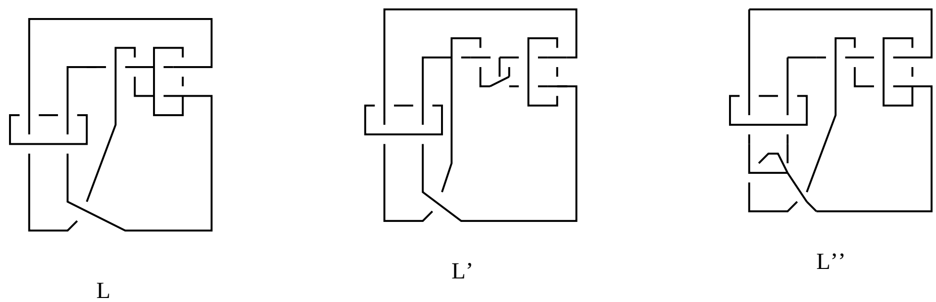

FIGURE 4

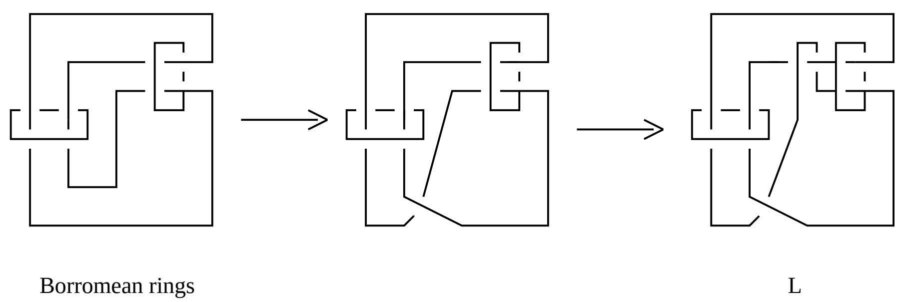

FIGURE 5

L such that each C_i intersects P in exactly two points in different regions. Denote by D_i the disk bounded by C_i for $1 \leq i \leq n$. Assume further that $D_i \cap D_j = \emptyset$ for $i \neq j$ and each D_i intersects L in exactly two points. As shown in [A2], the new link $L' = L \cup (\cup_{i=1}^n C_i)$, called an augmented alternating link, is also hyperbolic.

By performing different belt sums of two hyperbolic links, one may obtain different links whose complements have the same Chern-Simons invariant and the same volume. The following is an example.

In Figure 4, L is an augmented Whitehead link. L' and L'' are two belted sums of L and the Whitehead link of right-handed clasp. Note that L' has three components while L'' has four. As shown in Figure 5, L can be obtained from the Borromean rings by twisting one component a half twist and another component a full twist. The first operation changes the Chern-Simons invariant by 1/4 ([MR]) and the second one does not change the homeomorphism type of the complement. Thus one has $CS(S^3 - L) = 1/4 \pmod{1/2}$. It follows from the corollary that $CS(S^3 - L') = CS(S^3 - L'') = 1/4 - 1/8 = 1/8 \pmod{1/2}$.

Currently, there is a problematic aspect to calculating the Chern-Simons invariant of a hyperbolic 3-manifold in the "SnapPea" program of Weeks. The program is based on Neumann's simplicial formula ([N]) which involves an analytic function on hyperbolic Dehn surgery space that is only given up to an imaginary constant. Such a constant can be determined by a "bootstrapping" process, using surgery relations discovered among hyperbolic 3-manifolds. Theorem 1, together with the result of [MR] mentioned above, gives an algorithm to determine the constant for the hyperbolic 3-manifolds obtained by Dehn-surgeries on alternating hyperbolic knots and links.

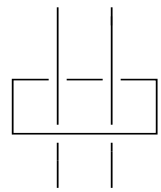

FIGURE 6

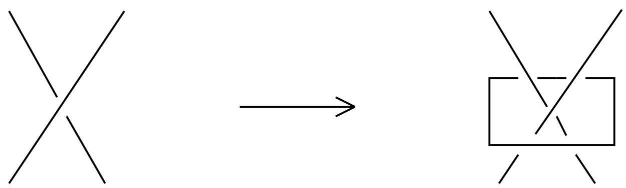

FIGURE 7

Recall that the simplicial formula for the Chern-Simons invariant ([N]) involves the same constant for the family of hyperbolic 3-manifolds obtained from a hyperbolic knot or link L by Dehn surgeries (including $S^3 - L$). Thus if one knows the Chern-Simons invariant of some manifold obtained from L by a hyperbolic Dehn surgery, then one can determine the constant in the simplicial formula. We denote such a constant by C_L.

Suppose that M is obtained from L by a hyperbolic Dehn surgery and $CS(M)$ is not known. If M is also the result of a hyperbolic Dehn surgery on another knot or link L' and one knows the Chern-Simons invariant of some manifold obtained from L' by a hyperbolic Dehn surgery, then one can calculate $C_{L'}$ and hence $CS(M)$. This is how the bootstrapping process goes.

Definition. Let L be a knot or link. Suppose L is in a particular regular projection without any unnecessary crossings. If L contains some portions as shown in Figure 6, then delete the trivial component in each of these portions to obtain a new knot or link L'; If L does not contain such a portion, take $L' = L$. The number of crossings in L' is called the *bootstrapping complexity* of L with respect to the projection.

The following is the algorithm that allows one to determine the unknown constants for all alternating hyperbolic knots and links.

Given an alternating hyperbolic knot or link L. Let n be the bootstrapping complexity of L with respect to some projection which does not contain any unnecessary crossings. Let L' be a link obtained by adding a trivial component to L around some crossing as shown in Figure 7.

Then L' is still hyperbolic ([A2]). Since L can be recovered from L' by doing $(1, 0)$-surgery along the added trivial component, C_L can be determined if one knows $CS(S^3 - L')$. If L' is a belt sum of two hyperbolic links, then one reduces the calculation for the Chern-Simons invariant to those for the two links with bootstrapping complexity less than n. Otherwise, by doing a half twist along the added trivial component in L', one can get rid of the crossing to obtain a new hyperbolic link L'' with bootstrapping complexity $n - 1$. Following this process, one can

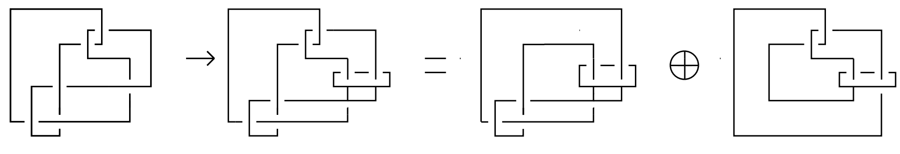

FIGURE 8

reduce bootstrapping complexity inductively to 0. It is not hard to see that any knot or link of bootstrapping complexity 0 with respect to a particular projection is amphicheiral. As shown in [MO], the Chern-Simons invariant of an ampheichiral hyperbolic knot or link is equal to 0.

Example. Figure 8 shows how the bootstrapping works for the knot 6_3. Notice that L_1 is an augmented Figure-8 knot and the Chern-Simons invariant of the Figure-8 knot is 0 because of its amphicheirality.

PROOF OF THEOREM 1

We first review the definition of the Chern-Simons invariant for a finite-volume cusped hyperbolic 3-manifold.

Let M be an oriented complete hyperbolic 3-manifold of finite volume. Let $\mathcal{L}$ be a link in M. An orthonormal frame field $\alpha = (e_1, e_2, e_3)$ on $M - \mathcal{L}$ is said to have a special singularity at $\mathcal{L}$ if it has the following local structure near each component K_i of $\mathcal{L}$:

(a) vector e_1 is tangent to K_i in the limit,

(b) vectors e_2, e_3 determine an index ± 1 singularity in a small disk transverse to K_i.

A singular frame field $\alpha = (e_1, e_2, e_3)$ on M is called linear if in every cusp the e_3-vectors are perpendicular to the horospheres and point outwards, and the e_1-, e_2-vector fields are parallel on each horosphere.

Denote by ω_{ij} the Riemannian connection form associated to the hyperbolic metric on M defined on the orthonormal frame bundle $F(M)$ of M. As in [Y], for every simple closed curve C in M and an orthonormal frame field β defined on a tubular neighborhood of C whose first component is tangent to C, the torsion of C with respect to β is defined by

$$\tau(C, \beta) = -\int_{s_\beta(C)} \omega_{23}$$

where : $s_\beta : C \to F(M)$ is the section defined by β.

If β' is another such frame field, then as shown in [Y],

$$\tau(C, \beta') - \tau(C, \beta) \in 2\pi\mathbf{Z}.$$

Thus one has a well-defined torsion $\tau(C)$ modulo 2π.

Let α be a linear singular frame field on M having a special singularity at some link $\mathcal{L}$. Denote by Q the Chern-Simons 3-form defined on $F(M)$. In [M], the

Chern-Simons invariant of M is defined by

$$CS(M) = \frac{1}{8\pi^2}\int_{s_\alpha(M-\mathcal{L})} Q - \frac{1}{4\pi}\sum_{i=1}^{n}\tau(K_i) \qquad (\bmod \frac{1}{2}) \tag{1}$$

where $s_\alpha : M - \mathcal{L} \to F(M)$ is the section defined by α and K_i $(1 \le i \le n)$ are the components of $\mathcal{L}$.

The important fact is that $CS(M)$ does not depend on the choice of $(\mathcal{L}, \alpha)$.

To prove the theorem, let S_1 and S_2 be two incompressible thrice-punctured spheres properly embedded in M_1 and M_2 respectively. Let $M_i' = M_i - N(S_i)$ where $N(S_i)$ is a neighborhood of S_i in M_i. Denote by S_i^0 and S_i^1 the two copies of S_i in $\partial M_i'$. Suppose that M is obtained from M_1 and M_2 by gluing S_1^0 to S_2^0 and S_1^1 to S_2^1 using the same homeomorphism ϕ.

Let α_1 be a linear singular frame field on M_1 which has a special singularity at some link $\mathcal{L}_1 \subset M_1$. Denote $S_1 \cap \mathcal{L}_1 = \{p_1, \cdots, p_n\}$. Then α_1 restricts to a frame field on $S_1 - \{p_1, \cdots, p_n\}$. The identification ϕ induces a frame field on $S_2 - \{\phi(p_1), \cdots, \phi(p_n)\}$.

As remarked in [M], if one replaces a singular frame field used in the definition of $CS(M)$ by a so-called homotopically linear one at the cusps, then (1) still holds. Such a frame field merely requires that e_1 and e_2 be homotopic to parallel vector fields on each horosphere.

Using this remark, one can first extend the frame field on $S_2 - \{\phi(p_1), \cdots, \phi(p_n)\}$ to the cusps of M_2 so that it is homotopically linear there.

Choose n simple closed curves $\gamma_1, \cdots, \gamma_n$ in M_2 such that each γ_i intersects S_2 precisely in $\phi(p_i)$. The obstruction to extending the frame field to the rest of M_2 is represented by a link $\mathcal{L}_2'$ in the complement of $S_2 \cup (\cup_{i=1}^n \gamma_i)$. As in [M], one thus obtains a singular frame field α_2 having a special singularity at $\mathcal{L}_2 = \mathcal{L}_2' \cup (\cup_{i=1}^n \gamma_i)$.

Since S_1 (resp. S_2) does not separate M_1 (resp. M_2), $\mathcal{L}_1 \cup (\cup_{i=1}^n \gamma_i)$ forms a new link $\mathcal{L}_1'$ in M. The singular frame fields α_1 and α_2 are glued together along copies of S_1 and S_2 via the same identification ϕ to form a singular frame field α on M having a special singularity at $\mathcal{L} = \mathcal{L}_1' \cup \mathcal{L}_2'$. Also, to define $\tau(K_i)$ for each $K_i \subseteq \mathcal{L}_1 \subset M_1$ one needs an orthonormal frame field β_i on a neighborhood of each such K_i. Since the same identification ϕ is used to glue along copies of S_1 and S_2, one can simply choose an orthonormal frame field on a neighborhood of each γ_i such that its first component is tangent to γ_i and the second and the third component are constant along γ_i and agree with those of β_i on the copies of S_1 and S_2. Thus one obtains a frame field needed to define the torsion of each component of $\mathcal{L}_1'$ in M. It follows that

$$\begin{aligned}
CS(M) &= \frac{1}{8\pi^2}\int_{s_\alpha(M-\mathcal{L})} Q - \frac{1}{4\pi}\sum_{K_i\subset\mathcal{L}}\tau(K_i) \\
&= \frac{1}{8\pi^2}\int_{s_\alpha(M-\mathcal{L})} Q - \frac{1}{4\pi}\sum_{K_i\subset\mathcal{L}_1'}\tau(K_i) - \frac{1}{4\pi}\sum_{K_i\subset\mathcal{L}_2'}\tau(K_i) \\
&= \frac{1}{8\pi^2}\int_{s_\alpha(M-\mathcal{L})} Q - \frac{1}{4\pi}\sum_{K_i\subset\mathcal{L}_1}\tau(K_i) - \frac{1}{4\pi}\sum_{i=1}^{n}\tau(\gamma_i) - \frac{1}{4\pi}\sum_{K_i\subset\mathcal{L}_2'}\tau(K_i)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8\pi^2}\int_{s_{\alpha_1}(M_1-\mathcal{L}_1)} Q - \frac{1}{4\pi}\sum_{K_i\subset\mathcal{L}_1}\tau(K_i) \\
&\quad + \frac{1}{8\pi^2}\int_{s_{\alpha_2}(M_2-\mathcal{L}_2)} Q - \frac{1}{4\pi}\sum_{K_i\subset\mathcal{L}_2}\tau(K_i) \\
&= CS(M_1) + CS(M_2).
\end{aligned}
$$

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Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109
E-mail address: mouyang@math.lsa.umich.edu