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PRODUCTS WITH LINEAR AND COUNTABLE TYPE FACTORS

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ABSTRACT. The basic theorem presented shows that the product of a linearly ordered space and a countable (regular) space is normal. We prove that the *countable space* can be replaced by any of a rather large class of countably tight spaces. Examples are given to prove that *monotone normality* cannot replace *linearly ordered* in the base theorem. However, it is shown that the product of a monotonically normal space and a monotonically normal countable space is normal.

INTRODUCTION

By a *space* we mean a Hausdorff, regular topological space.

The origin of this paper was a theorem of S. Purisch that the product of a space embeddable in a linearly ordered (called *suborderable*) space and a countable space is normal. A number of possible generalizations arose.

The proof itself can be generalized to replace the countable space by any space in the class $\mathcal{N}$ of countably tight spaces whose product with all paracompact suborderable spaces is normal. For instance a countably tight σ-compact space would do, or a countably tight, paracompact, σ-locally compact space. This latter generalization of the original proof was independently observed by Peter Nyikos. To see that such a space is in $\mathcal{N}$ recall:

Lemma 1.1 ([M]). *The product of a paracompact space and a σ-locally compact, paracompact space is paracompact.*

In Theorem 1 we prove that the product of a suborderable space with any space from class $\mathcal{N}$ is normal. Linearly ordered spaces are monotonically normal and monotonically normal spaces have many of the properties of linearly ordered spaces. However, Theorem 2 provides a monotonically normal space and a countable space whose product is not normal. Nonetheless, Theorem 3 gives a proof that the product of a monotonically normal space with a monotonically normal countable space is always normal.

When referring to a suborderable space we will assume a fixed linear order compatible with its topology.

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Theorem 1. *The product of a suborderable space with a countably tight space whose product with all paracompact suborderable spaces is normal, is itself normal.*

Let X be a suborderable space. A *left gap* in X is a nonempty clopen initial segment in X that has no maximum. A left gap is a left Q-gap if there is no order preserving copy of a stationary set (i.e., an order preserving homeomorphism onto a stationary subset of an uncountable regular cardinal with the order topology) in the gap which is cofinal in the gap. (For example, see [P].) A right gap and right Q-gap are defined analogously. In [GH] it is shown that a linearly ordered space is paracompact if and only if all of its gaps are Q-gaps.

The *greatest ordered compactification* ([F]) bX of X is the ordered compactification of X whose order extends the order of X and whose growth is order-isomorphic to the set of gaps of X. Hence, to each left (right) gap A in X corresponds the *gap point* p_A in $(bX - X)$, which is the sup (inf) in bX of A.

Let X^* denote $X \cup \{p \in (bX - X) | p \text{ is a non-}Q\text{-gap point of } X\}$ and let $\leq$ denote the order relation on both X and X^*.

In [P] it is shown that X^* is paracompact. Note that if $p \in (X^* - X)$ is a left gap point, then $\{a \in X^* | a \leq p\}$ is clopen in X^*. If p is a right gap point, then $\{a \in X^* | p \leq a\}$ is clopen in X^*.

Through Lemma 1.4 let Y be a countably tight normal space and C a closed subset of $(X \times Y)$ contained in an open subset O. Let C' be the closure of C in $(X^* \times Y)$.

Interval notation refers to intervals of X.

Lemma 1.2. *Let $p \in (X^* - X)$; say p is a left non-Q-gap point where $A_p = (-\infty, p) = \{x \in X | x < p\}$. If $\langle p, y_0 \rangle \in C'$ for some $y_0 \in Y$, then $\pi_X(C \cap (A_p \times \{y_0\}))$ is cofinal in the non-Q-gap A_p, where $\pi_X : (X \times Y) \to X$ is the projection map.*

Proof. Let $\langle p, y_0 \rangle \in C'$. Assume $\pi_X(C \cap (A_p \times \{y_0\}))$ is *not* cofinal in A_p. Then there exists $x_0 \in A_p$ such that $C \cap ((x_0, p) \times \{y_0\}) = \emptyset$. (Recall that $(x_0, p) = \{x \in X | x_0 < x < p\}$.)

Now, by induction, we construct an increasing sequence $\{x_\alpha | \alpha < \kappa\}$ cofinal in A_p for some ordinal κ as follows.

For $0 < \beta$ assume $\{x_\alpha | \alpha < \beta\}$ has been chosen. If $\{x_\alpha | \alpha < \beta\}$ is cofinal in A_p, define $\kappa = \beta$ and cease the construction.

Otherwise, for limit β just choose $x_\beta \in A_p$ with $x_\beta > x_\alpha$ for all $\alpha < \beta$.

If β is a nonlimit ordinal, $\beta = \alpha + 1$ for some $\alpha < \beta$. Let $C_\alpha = C \cap ((x_\alpha, p) \times Y)$. Then $\langle p, y_0 \rangle \in C'$ implies $y_0 \in \overline{\pi_Y(C_\alpha)}$. So, by the countable tightness of Y, there exists a countable $Y_\alpha \subset \pi_Y(C_\alpha)$ such that $y_0 \in \overline{Y_\alpha}$. Since Y_α is countable and A_p is a non-Q-gap and therefore does not have countable cofinality, we can choose $x_{\alpha+1} \in A_p$ such that for every $y \in Y_\alpha$ there is an $x \in (x_\alpha, x_{\alpha+1})$ with $\langle x, y \rangle \in C_\alpha$.

Since A_p is a non-Q-gap, $\{x_\alpha | \alpha < \kappa\}$ has a cluster point $q \in A_p$. Since $q \in (x_0, p)$, by our definition of x_0, $\langle q, y_0 \rangle \notin C$. So $\langle q, y_0 \rangle$ has a basic neighborhood $(U \times V)$ in $(X \times Y)$ missing C. However, there is some $\alpha < \kappa$ with $(x_\alpha, x_{\alpha+1}) \subset U$. Since $y_0 \in \overline{Y_\alpha}$, there is $y \in Y_\alpha \cap V$, and, by our choice of $x_{\alpha+1}$, there is $x \in (x_\alpha, x_{\alpha+1})$ with $\langle x, y \rangle \in C$. But $\langle x, y \rangle \in C \cap (U \times V)$ is a contradiction to our choice of $(U \times V)$ and Lemma 1.2 is proved. □

Lemma 1.3. *Let $D = (X \times Y) - 0$. Then $C' \cap D' = \emptyset$.*

Proof. Recall that O is an open set in $(X \times Y)$ containing C. Observe that $C \cap D = \emptyset$.

Let $p \in X^* - X$, say p is a left non-Q-gap point. Let $\langle p, y_0\rangle \in C'$ for some $y_0 \in Y_0$. By Lemma 1.2, $\pi_X(C \cap (A_p \times \{y_0\}))$ is cofinal (as well as closed) in A_p. If $\langle p, y_0\rangle \in D'$, then, replacing C by D in Lemma 1.2, $\pi_X(D \cap (A_p \times \{y_0\}))$ is cofinal and closed in A_p. Since A_p is a non-Q-gap there are no two disjoint closed cofinal subsets of A_p. Thus $\langle p, y_0\rangle \notin D'$. That is, there is an $x \in A_p$ and an open neighborhood V of y_0 in Y such that $((x, p) \times V) \subset O$, as required. □

Lemma 1.4. *Assume that Y is in class $\mathcal{N}$. That is, Y is countably tight and the product of Y with any paracompact suborderable space is normal. Then there is an open U in $(X \times Y)$ such that $U \subset \overline{U} \subset O$.*

Proof. Let $O' = \bigcup\{R \text{ open in } (X^* \times Y) | R \cap (X \times Y) \subset O\}$. Obviously O' is open in $(X^* \times Y)$ and $O' \cap (X \times Y) = O$.

To see that $C' \subset O'$, recall that $C \subset O$ and, for $\langle p, y\rangle \in (C' - C), p \in X^* - X$. Again assuming that p is a left non-Q-gap point, by Lemma 1.3, for some $x \in A_p$ and open neighborhood V of y where $\langle p, y\rangle \in C'$, $((x, p) \times V) \subset O$. It follows that $\langle p, y\rangle \in O'$.

Since X^* is a paracompact suborderable space ([P]), by our choice of Y, $(X^* \times Y)$ is normal. Therefore, there is an open W in $(X^* \times Y)$ such that $C' \subset W \subset \overline{W} \subset O'$ where $\overline{W}$ is the closure of W in $(X^* \times Y)$. Hence

$$\begin{aligned} C &= (C' \cap (X \times Y)) \subset (W \cap (X \times Y)) \\ &= U \subset \overline{U} \subset (\overline{W} \cap (X \times Y)) \subset (O' \cap (X \times Y) = O, \end{aligned}$$

as desired. □

Theorem 2. *There is a monotonically normal space X and a countable (regular Hausdorff) space Y such that $(X \times Y)$ is not normal.*

Proof. We begin with some elementary but rather messy combinatorics on subsets of ω which make it possible to define our topologies.

We inductively choose for each $f \in 2^{<\omega}$ an infinite subset M_f of ω as follows:

For $f \in 2^0$, define $M_f = \omega$.

If $n \in \omega$, $f \in 2^n$, M_f has been chosen, and g and g' are the two terms of 2^{n+1} extending f, then choose M_g and $M_{g'}$ to be two infinite, disjoint sets whose union is M_f which separate the two smallest members of M_f.

Let $\mathcal{M} = \{M_f | f \in 2^{<\omega}\}$.

For $i \in \omega$, let $\mathcal{M}_i = \{M \in \mathcal{M} | i \in M\}$.

By induction choose $f_n \in 2^n$ for each $n \in \omega$ so that, for $n > 0$, f_n extends f_{n-1} and M_{f_n} does *not* contain the minimal element in $M_{f_{n-1}}$. Let $\mathcal{M}_\omega = \{M_{f_n} | n \in \omega\}$.

Let $\mathcal{A}$ be the set of all nonempty finite subsets of ω_1.

We next define for each $\alpha \in \omega_1$, disjoint infinite sets $I_{\alpha 0}$ and $I_{\alpha 1}$ whose union is ω such that for all disjoint A and B in $\mathcal{A}$ and $M \in \mathcal{M}$, $\left[\left(\bigcap_{\alpha \in A} I_{\alpha 0}\right) \cap \left(\bigcap_{\alpha \in B} I_{\alpha 1}\right) \cap M\right]$ is infinite.

We prove there are such sets by induction on ω_1.

If $\beta = 0$, for each $n \in \omega$ and $f \in 2^n$ choose distinct i_{f0} and i_{f1} from $M_f - \{i \in \omega | i$ is i_{g0} or i_{g1} for some $g \in 2^{<n}\}$. Since the M_f's are disjoint for f's in 2^n for a fixed n, the chosen points at this n are all distinct. Define $I_{\beta 0} = \{i_{f0} | f \in 2^{<\omega}\}$ and $I_{\beta 1} = (\omega - I_{\beta 0})$. Since $\{i_{f1} | f \in 2^{<\omega}\} \subset I_{\beta 1}$, both sets intersect every $M \in \mathcal{M}$ in an infinite set.

Assume $\beta \in \omega_1, \beta \neq 0$, and $I_{\alpha 0}$ and $I_{\alpha 1}$ have been defined for all $\alpha < \beta$ having all the desired properties for A and B contained in β. Index $\beta = \{\alpha_n | n \in \omega\}$.

Then, as in the case for $\beta = 0$, by induction for each $n \in \omega$ and $f \in 2^n$ choose distinct i_{f0} and i_{f1} from:

$$\left[\left(\bigcap_{m<n} I_{\alpha_m f(m)}\right) \cap M_f\right] - \{i \in \omega | i \quad \text{is} \quad i_{g0} \quad \text{or} \quad i_{g1} \quad \text{for some} \quad g \in 2^{<n}\}.$$

Since the first set is infinite by hypothesis and the second is finite because $2^{<n}$ is finite, we have no trouble choosing two distinct elements from this set. Define $I_{\beta 0} = \{i_{f0} | f \in 2^{<\omega}\}$ and $I_{\beta 1} = (\omega - I_{\beta 0})$. Again the fact that $\{i_{f1} | f \in 2^{<\omega}\} \subset I_{\beta 1}$ ensures the desired intersection property for our induction hypothesis.

For each $A \in \mathcal{A}$ define $I_{A0} = \bigcap_{\alpha \in A} I_{\alpha 0}$ and $I_{A1} = \bigcap_{\alpha \in A} I_{\alpha 1}$.

For $\alpha \in \omega_1$, define $\mathcal{A}_\alpha = \{A \in \mathcal{A} | A \subset (\omega_1 - \alpha)\}$.

We now define our spaces X and Y.

Let $X = \omega_1 \times (\omega + 1)$ topologized as follows:

(1) Every singleton from $\omega_1 \times \omega$ is a basic open set in X.
(2) For $\alpha < \omega_1, \lambda < \alpha$ (where $\lambda = -1$ is allowed to get something less than 0), $M \in \mathcal{M}_\omega$, and $A \in \mathcal{A}_\alpha$, a basic open set (for $\langle \alpha, \omega \rangle$) is $U_{MA\lambda}(\alpha) = \{\langle \beta, \omega \rangle \in X | \lambda < \beta \leq \alpha\} \cup \{\langle \beta, i \rangle \in X | \lambda < \beta \leq \alpha \text{ and } i \in (I_{A0} \cap M)\} = (\lambda, \alpha] \times ((I_{A0} \cap M) \cup \{\omega\})$.

Two members of $\mathcal{M}_\omega$ intersect in a member of $\mathcal{M}_\omega$ and $I_{A0} \cap I_{B0} = I_{(A \cup B)0}$. If $\lambda < \alpha \leq \beta < \omega_1, A \in \mathcal{A}_\alpha$, and $B \in \mathcal{A}_\beta$, then $(A \cup B) \in \mathcal{A}_\alpha$ and $U_{MA\lambda}(\alpha) \cap U_{MB\lambda}(\beta) = U_{M(A\cup B)\lambda}(\alpha)$. The intersection of two "basic open sets" is clearly another one, so this is indeed a topology. For each distinct $i \in \omega$ and $j \leq \omega$, there are disjoint $M \in \mathcal{M}_i$ and $N \in \mathcal{M}_j$. This together with the linearity of ω_1 ensures that the topology is T_1.

To prove X is monotonically normal it suffices [B] to define an open $H(x, W)$ for each $x \in X$ and neighborhood W of x such that:

(a) $x \in H(x, W) \subset W$.
(b) If $W \subset Z$, then $H(x, W) \subset H(x, Z)$.
(c) If $H(x, W) \cap H(y, Z) \neq \emptyset$, either $x \in Z$ or $y \in W$.

Such a function is called a *monotone normality operator* for X.

For each $x \in X$ and neighborhood W of x, define $H(x, W)$ to be the union of all "basic open sets for x" which are contained in W. Certainly (a) and (b) hold. Since $H(x, W) = \{x\}$ for $x \in (\omega_1 \times \omega)$, to prove (c) it suffices to assume that $x = \langle \alpha, \omega \rangle, y = \langle \beta, \omega \rangle, \alpha < \beta < \omega_1$, W is a neighborhood of x and Z a neighborhood of y. If $x \notin Z$, then $H(y, Z) \subset [((\beta + 1) - (\alpha + 1)) \times (\omega + 1)]$. But $H(x, W) \subset [(\alpha + 1) \times (\omega + 1)]$. So $H(x, W) \cap H(y, Z) = \emptyset$ as desired and we have shown that X *is monotonically normal.*

If $i \in \omega$, define $\mathcal{A}_0(i) = \{A \in \mathcal{A} | i \in I_{A0}\}$ and $\mathcal{A}_1(i) = \{A \in \mathcal{A} | i \in I_{A1}\}$.

Let $Y = (\omega + 1)$ topologized as follows:

(1) If $M \in \mathcal{M}_\omega$ and $A \in \mathcal{A}, V_{MA} = (\{\omega\} \cup (I_{A1} \cap M))$ is a basic open set (for ω) in Y.
(2) If $i \in \omega$, $M \in \mathcal{M}_i, A \in \mathcal{A}_0(i)$, and $B \in \mathcal{A}_1(i)$, $V_{MAB}(i) = (I_{A0} \cap I_{B1} \cap M)$ is a basic open set (for i) in Y.

If M and M' are both in $\mathcal{M}_\omega$ or $\mathcal{M}_i$, then $M \cap M'$ is also. If A and A' are in $\mathcal{A}_0(i), (A \cup A') \in \mathcal{A}_0(i)$ and $(I_{A0} \cap I_{A'0}) = I_{(A\cup A')0})$. One can replace 0 by 1 here. The basis defined by (1) and (2) is clearly a topology for X. Since for $i \neq j$ in Y there are disjoint $M \in \mathcal{M}_i$ and $N \in \mathcal{M}_j$, the topology is Hausdorff. Also these separations as well as the separations by an $I_{\alpha 0}$ and $I_{\alpha 1} = (\omega - I_{\alpha 0})$ yield disjoint open sets, so the topology is 0-dimensional and regular.

It remains to prove *that* $(X \times Y)$ *is* not *normal.*

Define $F = \{\langle\langle\alpha,\omega\rangle,\omega\rangle \in (X \times Y) | \alpha \in \omega_1\}$. Since $\omega_1 \times \{\omega\}$ is closed in X and $\{\omega\}$ is closed in Y, F is closed in $(X \times Y)$.

Define $K = \{\langle\langle\alpha, i\rangle, i\rangle \in (X \times Y) | \alpha \in \omega_1, i \in \omega\}$. We prove K is closed by showing that $(X \times Y) - K$ is open.

If $\beta \in \omega_1$ and $M \in \mathcal{M}_\omega$, then $\langle\langle\beta,\omega\rangle,\omega\rangle \in \big(U_{M\{\beta\}(-1)}(\beta) \times V_{M\{\beta\}}\big)$ which does not intersect K since no $i \in \omega$ can be in both $I_{\{\beta\}0}$ and $I_{\{\beta\}1}$.

Suppose $j \leq \omega, k \leq \omega$, and $j \neq k$. If $j = \omega$ there are disjoint $M \in \mathcal{M}_\omega$ and $N \in \mathcal{M}_k$ and $\langle\langle\beta,\omega\rangle,k\rangle \in \big(V_{MA(-1)}(\beta) \times V_{NBC}(k)\big)$ for arbitrary choices of $A \in \mathcal{A}_\beta, B \in \mathcal{A}_0(k)$, and $C \in \mathcal{A}_1(k)$, and this set misses K since no $i \in \omega$ can be in both M and N. If $j < \omega$ and $\beta \in \omega_1$, $\langle\beta, j\rangle$ is isolated in X. There is $M \in \mathcal{M}_k$ not containing j and $\langle\langle\beta,j\rangle,k\rangle \in (\{\langle\beta,j\rangle\} \times V)$ where $V = V_{MA}$ for some $A \in \mathcal{A}$ if $k = \omega$, or $V = V_{MBC}(k)$ for some choice of $B \in \mathcal{A}_0(k)$ and $C \in \mathcal{A}_1(k)$ if $k < \omega$. In either case $(\{\langle\beta,j\rangle\} \times V) \cap K = \emptyset$ since $j \notin V$.

Thus F and K are closed and disjoint. Assume that G is an open set in $(X \times Y)$ containing F. We prove that $(X \times Y)$ is not normal by proving:

Claim. $\overline{G} \cap K \neq \emptyset$.

Proof of claim. For each $\alpha \in \omega_1, \langle\langle\alpha,\omega\rangle,\omega\rangle \in G$. Thus there are basic neighborhoods $U_\alpha = U_{M_\alpha A_\alpha \lambda_\alpha}(\alpha)$ for $\langle\alpha,\omega\rangle$ in X and $V_\alpha = V_{N_\alpha B_\alpha}$ for ω in Y such that $(U_\alpha \times V_\alpha) \subset G$ and $(B_\alpha - A_\alpha) \neq \emptyset$.

Since $(M_\alpha \cap N_\alpha) \in \mathcal{M}_\omega$ which is countable, there is an $M \in \mathcal{M}_\omega$ and a stationary subset S of ω_1 such that $(M_\alpha \cap N_\alpha) = M \in \mathcal{M}_\omega$ for all $\alpha \in S$. For all $\alpha \in S$, $\langle\alpha,\omega\rangle \in U_{MA_\alpha\lambda_\alpha}(\alpha) \subset U_\alpha$ and $\omega \in V_{MB_\alpha} \subset V_\alpha$ so we can assume $M_\alpha = N_\alpha = M$ for all $\alpha \in S$.

Since $\lambda_\alpha < \alpha$ and S is stationary, there is a $\lambda \in \omega_1$ and a stationary subset S' of S such that $\lambda_\alpha = \lambda$ for all $\alpha \in S'$.

For $i \in \omega$, let $S_i = \{\alpha \in S' | i \in (I_{A_\alpha 0} \cap I_{(B_\alpha - A_\alpha)1} \cap M)\}$.

Fix some i for which S_i is uncountable.

Let $\delta = \min S_i$. For all $\alpha \in S_i, \lambda < \delta \leq \alpha, A_\alpha \in \mathcal{A}_\alpha$, and $i \in (I_{A_\alpha 0} \cap M)$; so $\langle\delta, i\rangle \in U_\alpha$. If $j \in (I_{B_\alpha 1} \cap M)$ for some $\alpha \in S_i$, then $j \in V_\alpha$ and $\langle\langle\delta,i\rangle,j\rangle \in (U_\alpha \times V_\alpha) \subset G$. Let $J = \{j \in \omega | j \in (I_{B_\alpha 1} \cap M)$ for some $\alpha \in S_i\}$. If $i \in$ (closure of J in Y), then $\langle\langle\delta,i\rangle,i\rangle \in K \cap$ (closure of G in $(X \times Y)$) proving our claim.

Otherwise there is a basic neighborhood $V_{NDE}(i)$ of i in Y not intersecting J. Since $i \in N, (N \cap M) \in \mathcal{M}_i$. Since for all $\alpha \in \omega_1, A_\alpha \in \mathcal{A}_\alpha, A_\alpha \subset (\omega_1 - \alpha)$. Since S_i is uncountable, there is some $\alpha \in S_i$ such that $A_\alpha \cap D = \emptyset$. Fix this α. Since $i \in I_{\beta 1}$ for all $\beta \in (B_\alpha - A_\alpha)$ and $i \in I_{\gamma 0}$ for all $\gamma \in D, (B_\alpha - A_\alpha) \cap D = \emptyset$. Also $(E \cap D) = \emptyset$ since $i \in V_{NDE}(i)$.

Hence we can choose j from the infinite set $[I_{D0} \cap I_{(B_\alpha \cup E)1} \cap (M \cap N)]$. But $j \in (V_{NDE}(i) \cap J)$ which contradicts our choice of $V_{NDE}(i)$. □

Comment. We want to thank the referee for pointing out three interesting facts:

(One) In [H], R. W. Heath essentially proved that *no countable dense subset of* 2^{ω_1} (with the usual product topology) *is monotonically normal.*

(Two) *The countable space* Y *constructed in the proof of Theorem 2 can be densely embedded in* 2^{ω_1}.

(Three) *If Y' is any countable dense subset of 2^{ω_1}, there is a monotonically normal space X' such that $X' \times Y'$ is not normal.*

Trivially, there is a natural homeomorphism $p : (2^\omega \times 2^{\omega_1}) \to 2^{\omega_1}$. Let $p_1 : (2^\omega \times 2^{\omega_1}) \to 2^\omega$ and $p_2 : (2^\omega \times 2^{\omega_1}) \to 2^{\omega_1}$ be the projections.

Proof of (One). We may assume the f_n's defined in the proof of Theorem 2 satisfy $f_n(j) = 1$ for all $j < n$. Define $g : Y\ (= \omega + 1) \to (2^\omega \times 2^{\omega_1})$ by having $k = p_1 \circ g$ and $h = p_2 \circ g$ where $k(\omega)(j) = 1$ for all $j \in \omega$ and $h(\omega)(\alpha) = 1$ for all $\alpha \in \omega_1$. For each $i \in \omega$, let $k(i) \in 2^\omega$ be the unique function such that $i \in M_{k(i)\restriction n}$ for each $n \in \omega$. Define $h \in 2^{\omega_1}$ by $h(i)(\alpha) = 0$ if and only if $i \in I_{\alpha 0}$. It is then easy to check that g is a dense embedding of Y into $2^\omega \times 2^{\omega_1}$. □

Proof of (Three). Let Y' be a countable dense subset of $2^\omega \times 2^{\omega_1}$. Since y' is countable we can assume that $Y' = \omega + 1$ as a set and ω is the point of Y' whose coordinates are all 1. For each $f \in 2^{<\omega}$, define $M_f = \{i \in \omega (= Y' - \{\omega\}) \mid p_1(i)$ is an extension of $f\}$. Define $I_{\alpha 0} = \{i \in Y' - \{\omega\} \mid p_2(\alpha) = 0\}$. Again one easily checks that these M_f's and $I_{\alpha 0}$'s operate with Y' exactly as the ones defined for Y operated in the proof of Theorem 2. □

Theorem 3. *If X is a monotonically normal space and Y is a monotonically normal countable space, then $(X \times Y)$ is normal.*

Proof. Using the standard proof that (regular) Lindelöf spaces are normal one can prove:

Lemma 3.1 ([S]). *If X is normal and Y is countable, then $(X \times Y)$ is normal if and only if for all $y \in Y$ and closed K in $(X \times Y)$ with $K \cap (X \times \{y\}) = \emptyset$, $(X \times \{y\})$ and K can be separated.*

Fix such a y and K. Then for the purposes of this paper we define $Z \subset X$ to be *ok* if Z is closed in X and there is an open O in $(X \times Y)$ with $(Z \times \{y\}) \subset O$ and $\overline{O} \cap K = \emptyset$. Our aim, of course, is to prove that X is ok, thus by Lemma 3.1 proving Theorem 3.

Lemma 3.2. *Every union of a closed discrete family of ok sets is ok.*

Proof. Suppose $\mathcal{Z} = \{Z_\alpha | \alpha \in \kappa\}$ is a closed discrete family of ok sets and $Z = \bigcup \mathcal{Z}$. By definition Z is closed and, for each $\alpha \in \kappa$, there is an open U_α in X with $(Z_\alpha \times \{y\}) \subset U_\alpha$ while $\overline{U}_\alpha \cap K = \emptyset$. Since X is monotonically normal and thus collectionwise normal, we can assume that $\{\overline{U}_\alpha | \alpha \in \kappa\}$ is closed discrete. Then $O = \bigcup\{U_\alpha | \alpha \in \kappa\}$ testifies to Z being ok. □

Lemma 3.3. *Every closed subset of X which is homeomorphic to a subset of an ordinal is ok.*

Comment. This lemma is an obvious consequence of Theorem 1 for linearly ordered X and countable Y. But for monotonically normal X and countable Y, the lemma fails as the examples in Theorem 2 show. By adding monotone normality to countability for Y we again have the lemma (and normality for $(X \times Y)$).

Proof of Lemma 3.3. If the lemma fails, there is a minimal ordinal κ for which there is a non-ok closed Z in X homeomorphic to a subset of κ. We think of Z as a subspace of both X and κ with its ordinary $\leq$ order topology. By Lemma 3.2 and the minimality of κ, κ has uncountable cofinality and Z is stationary in κ.

For each $\alpha \in Z$ choose a basic open neighborhood $(U_\alpha \times V_\alpha)$ of $\langle \alpha, y\rangle$ such that $[(\overline{U}_\alpha \times \overline{V}_\alpha) \cap K] = \emptyset$ and let $\{y_n | n \in \omega\}$ be a one-to-one indexing of $(Y - \{y\})$.

Defining Z^* for the GO space Z as we defined X^* for X in Theorem 1, let p be the sup of Z in Z^*. Then p is a non-Q-gap point in Z^*. By Lemma 1.3, $\langle p, y\rangle$ is not in the closure of $K \cap (Z \times Y)$ in $(Z^* \times Y)$ which implies that for some $\gamma \in Z$ and open neighborhood V of y in Y, $[(Z - \gamma) \times \overline{V}] \cap K = \emptyset$. Since Z is the union of the two closed sets $(Z - \gamma)$ and $Z \cap (\gamma + 1)$ and, by the minimality of κ, $Z \cap (\gamma + 1)$ is ok, we can assume that $(Z \cap \gamma) = \emptyset$. Thus $(Z \times \overline{V}) \cap K = \emptyset$. Assume $V_\alpha \subset V$ for all $\alpha \in Z$.

Let H and J be monotone normality operators for X and Y, respectively. (Such an operator is defined in the proof of Theorem 2.) Let $O = \bigcup\{H(\alpha, U_\alpha) \times J(y, V_\alpha) | \alpha \in Z\}$.

Suppose $\langle x, w\rangle \in K$; then $w \neq y$.

If $w \notin \overline{V}, [X \times (Y - \overline{V})]$ is an open neighborhood of $\langle x, w\rangle$ missing O.

Suppose $w \in \overline{V}$. Then $x \notin Z$ and we claim that in this case $[H(x, X - Z) \times J(w, Y - \{y\})]$ misses O. If this claim is correct, O is an open neighborhood of $(Z \times \{y\})$ in $(X \times Y)$ whose closure misses K. Thus Z is ok and Lemma 3.3 holds.

So let us prove our claim by supposing that there is an $\alpha \in Z$ and $\langle p, q\rangle \in ([H(x, X - Z) \times J(w, Y - \{y\})] \cap [H(\alpha, U_\alpha) \times J(y, V_\alpha)])$. Since H is a monotonic normality operator on X, $\alpha \in Z$, and $p \in (H(\alpha, U_\alpha) \cap H(x, X - Z)), x \in U_\alpha$. Since J is a monotonic normality operator on Y and $q \in (J(w, Y - \{y\}) \cap J(y, V_\alpha)], w \in V_\alpha$. But we have a contradiction since $K \cap (U_\alpha \times V_\alpha) = \emptyset$ and $\langle x, w\rangle \in K$. □

Lemma 3.4. *X is ok.* (Which will complete the proof of Theorem 3.)

Since K is closed in $(X \times Y)$ and misses $(X \times \{y\})$, for each $x \in X$ there is a basic neighborhood $(U_x \times V_x)$ of $\langle x, y\rangle$ whose closure misses K. Since $\mathcal{U} = \{U_x | x \in X\}$ is an open cover of the monotonically normal X, by Theorem (II) of [BR], there is a σ-disjoint partial refinement $\mathcal{W}$ of $\mathcal{U}$ such that $(X - \bigcup \mathcal{W})$ is the union of a closed discrete family $\mathcal{Z}$ of copies of subsets of ordinals.

By Lemma 3.3 each term of $\mathcal{Z}$ is ok, and by Lemma 3.2, $\bigcup \mathcal{Z}$ is ok. Let O denote the resulting open set in $(X \times Y)$ containing $((\bigcup \mathcal{Z}) \times \{y\})$ with $\overline{O} \cap K = \emptyset$.

By definition $\mathcal{W} = \bigcup\{\mathcal{W}_n|_{n\in\omega}\}$ where each $\mathcal{W}_n$ is a set of disjoint open sets each contained in a member of $\mathcal{U}$; let $W_n = \bigcup \mathcal{W}_n$. Also $(\bigcup \mathcal{Z}) \subset W_\omega = \{x \in X | \langle x, y\rangle \in O\}$.

Since the monotonically normal X is countably paracompact [BR], there is a locally finite refinement $\{G_n | n \leq \omega\}$ of $\{W_n | n \leq \omega\}$ with $G_n \subset W_n$ for all $n \leq \omega$.

Then $\mathcal{E} = \{(G_n \cap W) | n < \omega \text{ and } W \in \mathcal{W}_n\}$ is a locally finite open cover of $X - W_\omega$. For $E \in \mathcal{E}$, let $f(E)$ denote a point x with $E \subset U_x$. For $x \in X$, let $E_x = \bigcup\{E \in \mathcal{E} | f(E) = x\}$.

Finally let $\mathcal{M} = \{E_x \times V_x | x \in X\}$. Recall that $E_x \subset U_x$ and $(\overline{U}_x \times \overline{V}_x) \cap K = \emptyset$. Since $\mathcal{E}$ is locally finite, $((\overline{\bigcup \mathcal{M}}) \cap K) = \emptyset$. Hence $(O \cup (\bigcup \mathcal{M}))$ is an open set in $(X \times Y)$ containing $(X \times \{y\})$ whose closure misses K, as required for Lemma 3.4 (and Theorem 3). □

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