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A NON-METRIZABLE SPACE WHOSE COUNTABLE POWER IS σ-METRIZABLE

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ABSTRACT. We answer a question of A.V. Arhangel'skii by finding a non-metrizable space X such that X^ω is the countable union of metrizable spaces.

A space is σ*-metrizable* if it is the countable union of metrizable subspaces. A.V. Arhangel'skii (see [2] and [3]) asked if a space X must be metrizable if X^ω is σ-metrizable. In [2], M.G. Tkacenko showed that the answer is "yes" if X is separable or countably compact. In [3], V.V. Tkachuk proved that X must be metrizable if X^ω is the union of finitely many metrizable spaces. But here we shall show that in general the answer to Arhangel'skii's question is "no".

Let ω_1 denote the set of countable ordinals, and L the limit ordinals in ω_1. For each $\alpha \in L$, choose a sequence $\alpha_0, \alpha_1, ...$ of non-limits converging to α. Let X be the space whose set is ω_1, with points of $\omega_1 \setminus L$ isolated, and the k^{th} basic neighborhood of $\alpha \in L$ having the form $\{\alpha_n : k < n < \omega\} \cup \{\alpha\}$. Note that L is a closed discrete subset of X (i.e., every subset of L is closed in X).

Such an X is an example of what is often called a *ladder space* on ω_1. It is known that X is not metrizable or even paracompact. For the benefit of the reader, we give an elementary proof of this fact here. Note that X has an open cover each member of which contains at most one point of L. If X were paracompact, then there would be neighborhoods $U(\alpha)$ of α for $\alpha \in L$ such that the collection $\{U(\alpha) : \alpha \in L\}$ is locally finite. For each $\delta \in \omega_1$, $[0, \delta]$ would meet at most countably many $U(\alpha)$'s, so we could find $\delta_0 < \delta_1 < ...$ such that $[0, \delta_n] \cap U(\alpha) \neq \emptyset$ implies $\alpha < \delta_{n+1}$. But we arrive at a contradiction by noting that if $\delta_\omega = sup_{n<\omega}\delta_n$, then $U(\delta_\omega) \cap [0, \delta_n] \neq \emptyset$ for some n.

In [1], we answered a question of Tkachuk by showing that every finite power of X is the union of two metrizable spaces. Here we will show that if X has the additional property that L is a G_δ-set in X (which can be ensured by, e.g., always choosing α_n to have the form $\beta+n$ for some limit β), then X^ω will be the countable union of metrizable spaces.

Theorem. *Let X be a ladder space on ω_1 in which the set L of limit ordinals is a G_δ-set in X. Then X^ω is σ-metrizable.*

Proof. In this proof, letters i, j, k, l, m and n will always denote natural numbers.

Of course, X itself is the union of two metrizable (discrete) subspaces, L and $\omega_1 \setminus L$. So for each n, $X^n \times L^\omega$ is the union of finitely many metrizable spaces. It

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follows that

$$Y = \{f \in X^\omega : f(n) \in L \text{ for sufficiently large } n\}$$

is the countable union of metrizable spaces.

Let $D = \omega_1 \setminus L$. Since L is G_δ, $D = \bigcup_{n\in\omega} D_n$ where each D_n is closed in X. So each D_n is a closed discrete set of isolated points. Next we shall show that the subspace

$$Z = \{f \in X^\omega : \forall k \exists n > k(f(n) \in D \text{ and } max_{i<n} f(i) \in D)\}$$

is metrizable.

We shall consider $U = \prod_{i\in\omega} U_i$ to be a basic open set in X^ω if there is n, called the *length* of U, such that

(i) $U_i = X$ iff $i \geq n$;
(ii) if $i < n$, then U_i is a basic open neighborhood of its "top" point $max(U_i)$.

Fix $m < n$, and let $\alpha \in D$. Let U^α_{mni}, $i \in \omega$, index the countably many basic open sets U of length n satisfying $U_m = \{\alpha\}$ and $\alpha = max\{max(U_j) : j < n\}$. Then let

$$\mathcal{B}(mnikj) = \{U^\alpha_{mni} \cap \pi_n^{-1}(\{d\}) : \alpha \in D_j, d \in D_k\},$$

where $\pi_n : X^\omega \to X$ is the projection onto the n^{th} coordinate.

It is easy to see that if U is any neighborhood of any point $z \in Z$, then there are m, n, i, k, j and $B \in \mathcal{B}(mnikj)$ with $z \in B \subset U$. Thus if we show each $\mathcal{B}(mnikj)$ is a discrete collection of sets in X^ω, it will follow that Z has a σ-discrete base, hence is metrizable. To this end, suppose every neighborhood of $f \in X^\omega$ meets a member of $\mathcal{B}(mnikj)$. As D_j and D_k are closed discrete sets of isolated points in X, it must be the case that there are $\alpha' \in D_j$ and $d' \in D_k$ with $f(m) = \alpha'$ and $f(n) = d'$. Now if U is the set of all points g agreeing with f on coördinates m and n, then the only member of $\mathcal{B}(mnikj)$ that U meets is the one with $\alpha = \alpha'$ and $d = d'$. So $\mathcal{B}(mnikj)$ is discrete, and Z is metrizable.

Now let W be the set of all points $f \in X^\omega$ satisfying:

(iii) $\forall k \exists n > k(f(n) \in D)$;
(iv) $\exists k \forall n > k(f(n) \in D \Rightarrow max_{i<n} f(i) \in L)$.

Since $X^\omega = Y \cup Z \cup W$, to complete the proof it remains to prove that W is σ-metrizable. Let $W_k = \{f \in W : \forall n > k(f(n) \in D \Rightarrow max_{i<n} f(i) \in L)\}$; then $W = \bigcup_{k\in\omega} W_k$.

Fix k; we shall prove that W_k is metrizable. Fix n, a non-empty subset F of n, and $\alpha \in L$. Let U^α_{nFi}, $i \in \omega$, index all basic open sets U of length n satisfying:

(v) $\alpha = max\{max(U_j) : j < n\}$;
(vi) $max(U_j) = \alpha \iff j \in F$;
(vii) $j \in F, j' \in n \setminus F \Rightarrow max(U_{j'}) < min(U_j)$.

Then let

$$\mathcal{B}(nmFi) = \{U^\alpha_{nFi} \cap \pi_n^{-1}(\{d\}) : \alpha \in L, d \in D_m\}.$$

It is easy to see that for k fixed, $\bigcup\{\mathcal{B}(nmFi) : m, i \in \omega, k < n \in \omega, \emptyset \neq F \subset n\}$ contains a base at all points of W_k. So it remains to prove that the trace of each $\mathcal{B}(nmFi)$ on W_k is a discrete collection in W_k. To this end, suppose every neighborhood of $f \in W_k$ meets a member of $\{B \cap W_k : B \in \mathcal{B}(nmFi)\}$, where $n > k$.

Then there is $d' \in D_m$ with $f(n) = d'$, and by the definition of W_k, there is $\alpha' \in L$ with $max_{j<n} f(j) = \alpha'$. Let $F' = \{j < n : f(j) = \alpha'\}$, and let U be a basic open set containing f satisfying conditions (v)-(vii) above with $\alpha = \alpha'$ and $F = F'$. Suppose

$$g \in W_k \cap (U \cap \pi_n^{-1}(\{d'\})) \cap (U^{\alpha}_{nFi} \cap \pi_n^{-1}(\{d\})).$$

Clearly $d = d'$, and then since $g \in W_k$ we have $max_{i<n} g(i) \in L$. Now $g \in U$ implies, by (vii), that $max_{i<n} g(i) \in U_j$ for some $j \in F'$. But the only limit ordinal in U_j is α', so $max_{i<n} g(i) = \alpha'$. Similarly, $g \in U^{\alpha}_{nFi}$ implies $max_{i<n} g(i) = \alpha$. So $\alpha = \alpha'$. It follows that the only member of $\mathcal{B}(nmFi)$ that meets $W_k \cap U \cap \pi_n^{-1}(\{d'\})$ is the one with $\alpha = \alpha'$ and $d = d'$. So $\{B \cap W_k : B \in \mathcal{B}(nmFi)\}$ is discrete in W_k, and this completes the proof. □

References

[1] Z. Balogh, G. Gruenhage, and V.V. Tkachuk, *Additivity of metrizability and related properties*, Topology and Appl., to appear.

[2] M.G. Tkacenko, *Ob odnom svoistve bicompactov (On a property of compact spaces)*, Seminar po obshchei topologii (A seminar on general topology), Moscow State Univ. P.H., Moscow, 1981, pp. 149-156.

[3] V.V. Tkachuk, *Finite and countable additivity of topological properties in nice spaces*, Trans. Amer. Math. Soc. **341** (2) (1994), 585-601. MR **94d:**54014

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