GROUP HOMOMORPHISMS INDUCING $\text{mod-}p$ COHOMOLOGY MONOMORPHISMS

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Abstract. Let $f: G \to K$ be a homomorphism of $p$-groups such that $f^{(n)}: H^n(K, \mathbb{Z}/p) \to H^n(G, \mathbb{Z}/p)$ is injective, for $1 \leq n \leq 2$. We prove that the non-bijectivity of $f$ implies the existence of a quotient $L$ of $G$ containing $K$ as a proper direct factor. This gives a refined proof of a result of Evens, which asserts that $f$ is bijective if $f^{(1)}$ is.

Let $p$ be a prime number and let $\mathbb{Z}/p$ be the prime field of $p$ elements. For every $p$-group $K$, let us denote by $H^\ast(K)$ the mod-$p$ cohomology of $K$.

Let $f: G \to K$ be a homomorphism of $p$-groups such that $f^{(n)}: H^n(K) \to H^n(G)$ is injective, for $1 \leq n \leq 2$. We shall give a refined proof of the following result of Evens.

Theorem A (Evens [1, Th. 7.2.4]). If $f^{(1)}$ is bijective, then so is $f$.

Further, we prove

Theorem B. If $f$ is not bijective, then there exists a quotient $L$ of $G$ containing $K$ as a proper direct factor (i.e., $L = J \times K$, with $J \neq \{1\}$).

Note that the maps $H^1(K) \xrightarrow{\text{Inf}_1(L,K)} H^1(L)$ and $H^1(L) \xrightarrow{\text{Inf}_1(G,L)} H^1(G)$ are injective, and $\text{Im Inf}_1(L,K)$ is a proper subgroup of $H^1(L)$ (since $J \neq \{1\}$), so, by Theorem B, the non-bijectivity of $f$ implies that
$$\dim_{\mathbb{Z}/p} H^1(K) < \dim_{\mathbb{Z}/p} H^1(L) \leq \dim_{\mathbb{Z}/p} H^1(G),$$

hence $f^{(1)}$ is not bijective. We obtain then an alternative proof of Theorem A.

Proof of Theorem B. We first prove that $f$ is surjective. Set $K' = \text{Im } f$. So $f$ factors through $G \xrightarrow{f} K' \xrightarrow{} K$. Since $f^{(1)} = (H^1(K) \xrightarrow{\text{Res}} H^1(K') \xrightarrow{f^{(1)}} H^1(G))$ is injective, it follows that $\text{Res}: H^1(K) \to H^1(K')$ is injective. This implies $K = K'$, so $f$ is surjective.

Consider the extension
$$1 \to \text{Ker } f \to G \xrightarrow{f} K \to 1.$$

Let $p^m$ be the order of $\text{Ker } f$ ($m \geq 1$, since $f$ is not bijective). The above extension...
can then be obtained by successive central extensions

$$(G_i) \quad 0 \to \mathbb{Z}/p \to G_i \xrightarrow{p_i} G_{i-1} \to 1,$$

$1 \leq i \leq m,$ with $G_0 = K$, $G_m = G$, $p_1 \circ \cdots \circ p_m = f$.

Let $z \in H^2(K)$ be the factor set of the extension $(G_1)$, viewed as a cohomology class of $H^2(K)$; then $z \in \text{Ker} \text{Inf}_2(G_1, K)$. Since $f^{(2)} = \text{Inf}_2(G, K) = \text{Inf}_2(G, G_1) \circ \text{Inf}_2(G_1, K)$, it follows from the injectivity of $f^{(2)}$ that $\text{Inf}_2(G_1, K)$ is also injective. Hence $z = 0$, so $G_1 = \mathbb{Z}/p \times K$. The theorem follows.

References


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