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HARMONIC 2-SPHERES WITH r PAIRS OF EXTRA EIGENFUNCTIONS

MOTOKO KOTANI

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Abstract. In the present paper, deformations of harmonic 2-spheres in the unit n-sphere S^n respecting the degree are studied. The limit maps of such deformations are characterized as harmonic maps with extra eigenfunctions. The space $Harm_d(S^2, S^n)$ of harmonic 2-spheres in S^n with fixed degree d is described in terms of such deformations and the limit maps.

§0. Introduction

Recently there has been much advance in the study of the space of all harmonic maps from a compact Riemann surface Σ into a Riemann manifold M as well as in study on individual harmonic maps. The space $Harm_d(S^2, S^n)$ of harmonic maps from a compact Riemann surface S^2 of genus zero into the unit n-sphere S^n with fixed degree d has been shown to be path-connected if $n \geq 3$ ([V], [L] for $n = 4$, [K] for general $n \geq 3$) and its fundamental group is computed in [FGKO]. Similar results have been obtained for harmonic maps from S^2 to many other target manifolds. The space $Harm_{d,e}(S^2, \mathbf{C}P^N)$ of harmonic maps into $\mathbf{C}P^N$ with fixed energy and degree for $N \geq 2$ ([Cr], [GO]), the spaces $Harm_e(S^2, \mathbf{H}P^N)^{st.isot.}$ and $Harm_e(S^2, \mathbf{H}P^N)^{quat}$ of strongly isotropic harmonic maps and quaternionic mixed pairs , respectively, into $\mathbf{H}P^N$with fixed energy ([M2], [GMO]), the space $Harm_{d,q}(S^2, Q_n(\mathbf{C}))^{st.isot}$ of strongly isotropic harmonic maps into $Q_n(\mathbf{C})$ with fixed degree and energy ([M1], [GMO]), are all path-connected and the fundamental groups of some of them have been computed. Nevertheless, it seems that the structure of the space $Harm_d(S^2, S^n)$, which is a fundamental object, still is not sufficiently well-understood. The present paper is intended to be a step toward a better understanding of the space.

It is easy to observe that the proofs of the above-mentioned known results concerning the spaces of harmonic maps share the following common technique: for an arbitrary harmonic map as above, there exists a smooth deformation given as the gradient flow of a Morse-Bott function over the corresponding twistor space. The deformation reduces the codimension in the target space while it respects the degree and the energy. By using the deformation, important information,such as the connectedness or the fundamental group, of the space $Harm_d(S^2, M^{n_1})$ of harmonic maps is obtained out of the information of the spaces $Harm_d(S^2, M^{n_2})$ $(n_1 \geq n_2)$

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of harmonic maps of lower codimension. In this way, an induction on codimension proceeds and we get the results mentioned above. We, however, need more detailed information on each step of deformation in order to understand the space $Harm_d(S^2, M^{n_1})$ better. In other words, we should know exactly which elements of $Harm_d(S^2, M^{n_2})$ come from the above deformation. Equivalently, we have to explicitly describe the subset of $Harm_d(S^2, M^{n_2})$ which consists of those deformed maps. In this paper we will study such subsets in the case of $Harm_d(S^2, S^n)$.

Let L_r be the space of harmonic maps $x \in Harm_d(S^2, S^n)$ which admit smooth deformations $x_t : S^2 \longrightarrow S^{n+2r}$ of full harmonic maps of degree d, with $\lim_{t\to\infty} x_t = x$. We will show the space L_r is equal to the space P_r of harmonic maps in $Harm_d(S^2, S^n)$ which have exactly r pairs of extra eigenfunctions (see §1 for the definition). This is conjectured by Ejiri and has been proved in the case of $r = 1$ [E1], [E2]. The space P_r was studied in [EK1], [EK2] through the study of the Gauss maps of the complete totally isotropic minimal surfaces in the Euclidean space with only planar ends. It was shown, among others, that $P = \bigcup_r P_r$ has an algebraic structure and that it has codimension greater than one. In this paper, we will reconstruct the space $Harm_d(S^2, S^{n+2r})$ by the above deformations and the set $L_r = P_r$ of their limit maps.

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§1. Preliminaries

Let $x : S^2 \longrightarrow S^n$ be a full harmonic map from a compact Riemann surface S^2 of genus 0 to the unit sphere S^n of degree d. Due to Calabi [Ca], the dimension n of the target manifold S^n is always even for a full harmonic map . From now on, we only consider a harmonic map $x : S^2 \longrightarrow S^{2m}$. Then x, as a $(2m+1)$-vector valued function, satisfies

$$\Delta x + 2x = 0,$$

where Δ is the Laplacian with respect to the metric induced by x. It implies that the $2m+1$ coordinate functions of x are eigenfunctions of Δ with eigenvalue 2. In this paper, eigenfunctions always mean eigenfunctions of Δ with eigenvalue 2. An eigenfunction is called *an extra eigenfunction* if it is not a linear combination of coordinate functions. In [EK2] it was proved that x rarely has extra eigenfunctions. More precisely, we have

Theorem 1.1 ([EK2]). *The set P of all full harmonic maps $x : S^2 \longrightarrow S^{2m}$ which admits extra eigenfunctions has $codim_{\mathbf{R}} \geqq 2$ in $Harm_d(S^2, S^{2m})$ and has a certain algebraic structure.*

We will quickly review in this section the results in [EK2] which we will need later.

Assume x admits an extra eigenfunction f. Then we can construct a complete isotropic branched minimal surface X in $\mathbf{R}^{2m+1}$ with the total curvature $4\pi d$, whose Gauss map is x and $f = \langle X, x\rangle$. We can also show that X has only planar ends. Actually there is a one-one correspondence between the set P and the set of all complete isotropic branched minimal surfaces of genus zero with only planar ends. We obtain a constant map $X = A$ in $\mathbf{R}^{2m+1}$ if we choose f to be a coordinate function instead of an extra eigenfunction in the above construction.

Proposition 1.2 ([EK2])**.** *Let X be a complete isotropic branched minimal surface in $\mathbf{R}^{2m+1}$. Take an isothermal coordinate z around a planar end $z = 0$. Then X_z has a Laurent expansion around the end $z = 0$ such that*

$$X_z = \frac{1}{z^k}V_{-k} + \cdots + \frac{1}{z}V_{-1} + \textit{holomorphic part},$$

(1) $V_{-1} = 0$,
(2) $\langle V_{-i}, V_{-j}\rangle = 0$ *for* $2 \leq i, j \leq k$,
(3) $\langle V_{-i}, x\rangle = 0$ *for* $2 \leq i \leq k$,

where $\langle\ ,\ \rangle$ is the complex bi-linear form induced by the standard metric $\langle\ ,\ \rangle$ of $\mathbf{R}^{2m+1}$.

The conjugate minimal surface $Y = Re[\int \sqrt{-1}X_z\, dz]$ is well-defined, up to a constant, because of (1). Notice that Y also has x as its Gauss map and its support function $\langle Y, x\rangle$ satisfies

$$\Delta\langle Y, x\rangle + 2\langle Y, x\rangle = 0.$$

As Y is not a constant map, $\langle Y, x\rangle$ is not a linear combination of the coordinate functions of x. Hence, if x has an extra eigenfunction $f = \langle X, x\rangle$, then it has one more extra eigenfunction $\langle Y, x\rangle$. We call them *a pair of extra eigenfunctions.*

Let X be the full complete isotropic branched minimal surfaces in $\mathbf{R}^{2m+1}$ constructed as above, Y be its conjugate minimal surface and z be a local isothermal coordinate. It is known that the trivial bundle $\underline{\mathbf{C}^{2m+1}} = S^2 \times \mathbf{C}^{2m+1}$ decomposes orthogonally as

$$\underline{\mathbf{C}^{2m+1}} = W_1 + \cdots + W_m + \overline{W}_1 + \cdots + \overline{W}_m + \{x\},$$

where each $W_1 + \cdots + W_k$ is the holomorphic subbundle of rank k locally spanned by $\{\frac{\partial X}{\partial z}, \ldots, \frac{\partial^k X}{\partial z^k}\}$ for $1 \leq k \leq m$. It should be remarked that the above decomposition does not depend on X but on only x. Let E_k be the local section of unit length generating W_k (see [EK2] for details). Note that

$$\{\frac{\partial Y}{\partial z}, \ldots, \frac{\partial^k Y}{\partial z^k}\} \cap \{\frac{\partial Y}{\partial z}, \ldots, \frac{\partial^{k-1} Y}{\partial z^{k-1}}\}^{\perp}, \quad 1 \leq k \leq m,$$

is also generated by the same E_k.

Lemma 1.3. *Let E_k be as above. Then*

(1) $\langle E_i, E_j\rangle = 0$ *for* $1 \leq i, j \leq m$,
(2) $\langle E_i, \overline{E}_j\rangle = \delta_{i,j}$ *for* $1 \leq i, j \leq m$,
(3) $\langle E_i, x\rangle = 0$ *for* $1 \leq i \leq m$.

We call $\{E_1, \ldots, E_m\}$ the *standard isotropic basis* corresponding to x. Let $G = X + \sqrt{-1}Y$. Then $G : S^2 \longrightarrow \mathbf{C}^{2m+1}$ is a meromorphic null curve, i.e.,

$$\overline{\partial}G = 0,$$

$$\langle \frac{\partial^k G}{\partial z^k}, \frac{\partial^k G}{\partial z^k}\rangle = 0, \quad 1 \leq k \leq m.$$

We call $\langle G, x\rangle = \langle X, x\rangle + \sqrt{-1}\langle Y, x\rangle$ *a pair of extra eigenfunctions* as well.

Note

$$\{\frac{\partial G}{\partial z}, \ldots, \frac{\partial^k G}{\partial z^k}\} = W_1 + \cdots + W_k, \quad 1 \leq k \leq m.$$

Define $G_+ \in \{W_1, \dots, W_m\}$ and $G_- \in \{\overline{W}_1, \dots, \overline{W}_m\}$ such that $G = G_+ + \langle G, x\rangle x + G_-$.

Lemma 1.4.

$$\partial G_+ + h_z x \parallel E_1,$$

$$\partial G_- + h\partial x = 0,$$

$$\overline{\partial} G_+ + h\overline{\partial} x = 0,$$

$$\overline{\partial} G_- + h_{\overline{z}} x = 0,$$

where $h = \langle G, x\rangle$.

§2. Deformations

Let $x : S^2 \longrightarrow S^{2m}$ be a full harmonic map from S^2 to S^{2m} of degree d with extra eigenfunctions $f_1, \dots, f_r$. As in §1, we construct a complete isotropic branched minimal surface X_i in $\mathbf{R}^{2m+1}$, whose Gauss map is x and support function $\langle X_i, x\rangle = f_i$, for $1 \le i \le r$. Let Y_i be its conjugate and we define $G_i = X_i + \sqrt{-1}Y_i$ and $h_i = \langle G_i, x\rangle$. It is easy to see

Lemma 2.1.

$$\frac{\partial G_i}{\partial \overline{z}} = 0,$$

$$\langle \frac{\partial^k G_i}{\partial z^k}, \frac{\partial^k G_i}{\partial z^k}\rangle = 0, \quad 1 \le k \le m.$$

We will construct a smooth deformation $x_t : S^2 \longrightarrow S^{2(m+r)}$ of harmonic map of degree d such that

(1) $\lim_{t\to\infty} x_t = x$,
(2) x_t is full in $S^{2(m+r)}$ for $t \neq \infty$,

when no linear combination of the h_i's is equal to a linear combination of coordinate functions.

Let $e_1, \dots, e_{2r}$ be the standard orthonormal basis of $\mathbf{R}^{2r}$ and let

$$E_{m+i} = \frac{e_{2i-1} - \sqrt{-1}e_{2i}}{\sqrt{2}}.$$

We may assume that the image of x is contained in $S^{2(m+r)} \cap \{e_1, \dots, e_{2r}\}^\perp = S^{2m}$. Take the standard isotropic basis (defined in §1) $E_1, \dots, E_m$ corresponding to x. Note that $\frac{\partial^k G_i}{\partial z^k} \in \operatorname{span}\{E_1, \dots, E_k\}$ for all $1 \le i \le r$ and $1 \le k \le m$.

Lemma 2.2. *$c_{i,j} = \int \langle dG_i, G_j\rangle$ is well-defined for all $1 \le i, j \le m$.*

Proof.

$$\langle dG_i, G_j\rangle = \langle \partial G_i, G_j\rangle dz = \langle \partial G_i, G_{j-}\rangle dz$$

$$= \partial\langle G_i, G_{j-}\rangle dz - \langle G_i, \partial G_{j-}\rangle dz$$

$$= d\langle G_i, G_{j-}\rangle - \overline{\partial}\langle G_i, G_{j-}\rangle d\overline{z} + \langle G_i, h_j \partial x\rangle dz$$

$$= d\langle G_i, G_{j-}\rangle + h_{iz} h_j dz + h_i h_{j\overline{z}} d\overline{z}.$$

Therefore

$$c_{i,j} = \langle G_i, G_{j-} \rangle + \int \omega,$$

where $\omega = h_{iz} h_j dz + h_i h_{j\bar{z}} d\bar{z}$ is an exact form.

As both h_i and h_j are smooth functions on S^2, the second term on the right hand side of the last equation is well-defined and hence so is the left hand side. □

Put $\tilde{G}_i = G_i + A_i$, where A_i is a constant vector in $\mathbf{C}^{2m+1}$. The integral $\tilde{c}_{i,j} = \int \langle d\tilde{G}_i, \tilde{G}_j \rangle$ is also well-defined. Define $F_i = \overline{E}_{m+i} + \tilde{G}_i - \sum_{j=1}^{r} \tilde{c}_{i,j} E_{m+j}$.

Remark. We can choose all the A_i's to be zero to construct a smooth deformation satisfying the above conditions (1) and (2). In §3 we will see what elements in $Harm_d(S^2, S^{2m+2r})$ can be deformed to a given x in $Harm_d(S^2, S^{2m})$. For that we take arbitrary constant vectors as A_i's.

Lemma 2.3. *There are meromorphic functions* $b_1, \ldots, b_r$ *such that*

$$\Psi^{(k)} = \sum_{i=1}^{r} b_i^{(k)} F_i, \quad 0 \le k \le r-1,$$

where (k) *stands for the* k*-th derivative by* z*.*

Proof. It is enough to find meromorphic functions $b_1, \ldots, b_r$ such that

$$\sum_{i=1}^{r} b_i^{(k)} F_i' = \sum_{i=1}^{r} b_i^{(k)} \{G_i' - \sum_{j=1}^{r} \langle G_i', \tilde{G}_j \rangle E_{m+j}\} = 0, \quad 0 \le k \le r-2.$$

Recall that all the G_i''s are parallel and define functions μ_i by $G_i' = \mu_i E_1$. Let $\mathbf{b} = (b_1, b_2, \cdots, b_r)$, let $\mathbf{m} = (\mu_1, \mu_2, \cdots, \mu_r)$, and let $\mathbf{M} = (\mathbf{M}_1, {}^t\mathbf{m}^{(r-1)})$ $= ({}^t\mathbf{m}, {}^t\mathbf{m}', \ldots, {}^t\mathbf{m}^{(r-1)})$ be the Wronskian matrix of $\mu_1, \mu_2, \cdots, \mu_r$. Take $\mathbf{b}$ as a solution of the equation $\mathbf{b}\mathbf{M}_1 = 0$ and let $\psi = \mathbf{b}\,{}^t\mathbf{m}^{(r-1)}$. Then b_i's satisfy $\sum_{i=1}^{r} b_i \mu_i^{(k)} = 0$ for $0 \le k \le r-2$. Since the conditions $\sum_{i=1}^{r} b_i^{(k)} \mu_i = 0$ for $0 \le k \le r-2$ are equivalent to the conditions $\sum_{i=1}^{r} b_i \mu_i^{(k)} = 0$ for $0 \le k \le r-2$, it is easy to see that those b_i's satisfy $\sum_{i=1}^{r} b_i^{(k)} G_i' = \sum_{i=1}^{r} b_i^{(k)} \mu_i E_1 = 0$ for $0 \le k \le r-2$. That proves that $\sum_{i=1}^{r} b_i^{(k)} F_i' = 0$ for those b_i's. Since $\overline{\partial} log \mu_1 = \overline{\partial} log \mu_2 = \cdots = \overline{\partial} log \mu_r$ ($= \overline{\alpha}_1$ in Lemma 1.3 in [EK2]), $\mathbf{b}\overline{\partial}\mathbf{M} \parallel \mathbf{b}$. We can choose ψ so that b_i's are meromorphic functions. □

Remark. The $\Psi = \sum_{i=1}^{r} b_i F_i$ may not be full if we take $b_1, \cdots, b_r$, whose existence is assured in Lemma 2.3, at random. Later (in Lemma 2.5), we will see when $b_1, \cdots, b_r$ are independent.

Lemma 2.4. $\Psi = \sum_{i=1}^{r} b_i F_i : S^2 \longrightarrow \mathbf{C}^{2(m+r)+1}$ *is a totally isotropic meromorphic curve, i.e.,*

$$\langle \frac{\partial^k \Psi}{\partial z^k}, \frac{\partial^k \Psi}{\partial z^k} \rangle = 0, \quad 0 \le k \le m+r-1.$$

Proof. From

$$\langle F_i, F_j \rangle = -\tilde{c}_{i,j} - \tilde{c}_{j,i} + \langle \tilde{G}_i, \tilde{G}_j \rangle = 0,$$

it follows that $\langle \Psi^{(k)}, \Psi^{(k)} \rangle = 0$ for $0 \le k \le r-1$. $\langle \Psi^{(r+k)}, \Psi^{(r+k)} \rangle = 0$ for $0 \le k \le m-1$ because $F_i^{(k)} = G_i^{(k)} - \sum_{j=1}^{r} \langle G_i^{(k)}, G_j \rangle E_{m+j}$ is a null vector and $\langle F_i^{(k)}, F_j^{(l)} \rangle = 0$ for $1 \le k \le m$, $1 \le l \le m$. □

Before we construct the deformation, we briefly review Calabi's twistor construction [Ca]. Let $Z_n = \{W : n\text{-dimensional complex subspaces in } \mathbf{C}^{2n+1}, W \perp \bar{W}\} = SO(2n+1)/U(n)$ and call Z_n the *twisor space*. Define the projection $\pi : Z_n \longrightarrow S^{2n}$ by $\pi(W) = \{W \bigoplus \bar{W}\}^{\perp}_{\mathbf{R}}$. More precisely $x = \pi(W)$ is a unit vector $x \in S^{2n}$ such that the orientation of $\{W \bigoplus \bar{W}\}_{\mathbf{R}} \bigoplus \{x\}$ agrees with the natural orientation of $\mathbf{R}^{2n+1}$ and $\pi^{-1}(x) = \{\frac{\partial x}{\partial z} \wedge \cdots \wedge \frac{\partial^n x}{\partial z^n}\}$. There is the so-called Calabi 2:1 correspondence between the space $Harm_d^{full}(S^2, S^{2n})$ of full harmonic maps in $Harm_d(S^2, S^{2n})$ and the space $HH_d^{full}(S^2, Z_n)$ of full holomorphic maps from S^2 to Z_n, that are horizontal with respect to the projection π. This 2:1 correspondence arises because both x and $-x$ are harmonic maps in S^{2n}. We call x a harmonic map with natural orientation and $-x$ a harmonic map with reverse orientation and denote by $Harm_d^{\pm}(S^2, S^{2n}) \subset Harm_d(S^2, S^{2n})$ the space of harmonic maps with natural orientation and the space of harmonic maps with reverse orientation, respectively. $Harm_d(S^2, S^{2n}) = Harm_d^{+}(S^2, S^{2n}) \cup Harm_d^{-}(S^2, S^{2n})$, $Harm_d^{nonfull}(S^2, S^{2n}) = Harm_d^{+}(S^2, S^{2n}) \cap Harm_d^{-}(S^2, S^{2n})$ and $Harm_d^{+}(S^2, S^{2n}) \simeq Harm_d^{-}(S^2, S^{2n})$. (See [FGKO] for details.) In this paper, we mainly look at the space $Harm_d^{+}(S^2, S^{2n})$ rather than $Harm_d(S^2, S^{2n})$, but abuse the notation.

Now we consider an action $A_t \in SO(2m+2r+1, \mathbf{C})$ on Z_{m+r} defined by

$$\begin{cases} A_t E_{m+i} = e^{-t} E_{m+i}, \qquad 1 \le i \le r, \\ A_t \overline{E}_{m+i} = e^{t} \overline{E}_{m+i}, \qquad 1 \le i \le r, \\ \qquad A_t = \mathrm{id} \qquad \text{on the orthogonal complement.} \end{cases}$$

The action A_t induces a deformation in $HH_d(S^2, Z_{m+r})$, since it preserves holomorphicity and horizontality of maps. By the similar argument in [FGKO], we can see this deformation is a gradient flow of a Morse-Bott function over Z_{m+r} and its critical manifolds are

$$\begin{aligned} C_{k,l} = \{ & W_0 \oplus W_1 \oplus W_2 \in Z_{m+r} : W_0 \text{ is an } m\text{-dimensional subspace in} \\ & \{E_{m+1}, \cdots, E_{m+r}, \cdots \bar{E}_{m+1}, \cdots, \bar{E}_{m+r}\}^{\perp}, W_0 \perp \bar{W}_0, \\ & W_1 \text{ is a } k\text{-dimensional subspace in } \{E_{m+1}, \cdots, E_{m+r}\}, W_1 \perp \bar{W}_1, \\ & W_2 \text{ is an } l\text{-dimensional subspace in } \{\bar{E}_{m+1}, \cdots, \bar{E}_{m+r}\}, W_2 \perp \bar{W}_2\}, \end{aligned}$$

for $k + l = r$. Notice that $C_{0,r}, C_{r,0} \cong Z_m$ and

$$\pi(C_{k,l}) \subset S^{2m+2r} \cap \{E_{m+1} \wedge \cdots \wedge E_{m+r} \wedge \cdots \wedge \bar{E}_{m+1} \wedge \cdots \wedge \bar{E}_{m+r}\}^{\perp} = S^{2m}.$$

Now we have an element $\Omega = \Psi \wedge \Psi' \wedge \ldots \wedge \Psi^{(r+m-1)}$ in $HH_d(S^2, Z_{r+m})$ for the totally isotropic meromorphic curve Ψ in Lemma 2.4. We apply the above action A_t to Ω and get a 1-parameter smooth deformation

$$\Omega_t = \Psi_t \wedge \Psi_t' \wedge \ldots \wedge \Psi_t^{(r+m-1)},$$

where the meromorphic null map

$$\Psi_t = \sum_{i=1}^{r} b_i F_{i_t} = \sum_{i=1}^{r} b_i \{e^t \overline{E}_{m+i} + \tilde{G}_i - \sum_{j=1}^{r} \tilde{c}_{i,j} e^{-t} E_{m+j}\}.$$

By an easy calculation, we get

Lemma 2.5.

$$\begin{aligned}\Omega_t &= \Psi_t \wedge \Psi_t' \wedge \ldots \wedge \Psi_t^{(r+m-1)} \\ &= B(\sum_k b_k^{(r-1)} \mu_k)^m F_1 \wedge F_2 \wedge \cdots \wedge F_r \wedge F_{r+1} \wedge \cdots \wedge F_{r+m},\end{aligned}$$

where

$$B = \det \begin{vmatrix} b_1 & b_2 & \ldots & b_r \\ b_1' & b_2' & \ldots & b_r' \\ \vdots & \vdots & \vdots & \vdots \\ b_1^{(r-1)} & b_2^{(r-1)} & \ldots & b_r^{(r-1)} \end{vmatrix},$$

$$\begin{aligned}F_{r+k} &= \beta_1 \ldots \beta_{k-1}(\overline{E}_{r+k} - \sum_{j=1}^{r} \langle E_{r+k}, G_j \rangle e^{-t} E_{m+j}) \\ &= \frac{F_i^{(k)}}{\mu_k} \cap \{F_i', \cdots, F_i^{(k-1)}\}^{\perp}.\end{aligned}$$

If and only if no linear combination of $h_1, \ldots h_r$ is a linear combination of coordinate functions, we can take the b_i's so that $B \neq 0$.

Proof. We prove only the last assertion. Let us use the same notation as in the proof of Lemma 2.3. It is known that meromorphic functions $b_1, \ldots, b_r$ are linearly dependent if and only if the Wronskian $B = 0$. Recall b_i's are determined by the equations $\sum_{i=1}^{r} \mu_i^{(k)} b_i = 0$ for $0 \leq k \leq r-2$, i.e., $\mathbf{b}$ is perpendicular to the vectors $\mathbf{m}_, \ldots, \mathbf{m}^{(r-2)}$. When $b_1, \ldots, b_r$ are linearly dependent, this means that a linear combination of $\mathbf{m}_, \ldots, \mathbf{m}^{(r-2)}$ is a non-zero constant vector. Thus $\mu_1, \ldots, \mu_r$ are solutions of a homogeneous linear O.D.E. of order $r-1$, i.e., there are constants c_i such that $\sum_{i=1}^{r} c_i \mu_i = 0$. On the other hand, if $\sum_{i=1}^{r} c_i \mu_i = 0$ for some constants $c_1, \ldots, c_r$, we can choose $\mathbf{b} = (c_1, c_2, \ldots, c_r)$. For this $\mathbf{b}$, the Wronskian $B = 0$. Thus we prove that $\sum_{i=1}^{r} c_i \mu_i = 0$ for some constants c_i if and only if $B = 0$ for some $\mathbf{b}$. When $\sum_{i=1}^{r} c_i \mu_i = 0$ for some constants c_i, $\sum_{i=1}^{r} c_i G_i$ is a constant vector A in $\mathbf{C}^{2m+1}$, because

$$\partial(\sum_{i=1}^{r} c_i G_i) = \sum_{i=1}^{r} c_i \partial G_i = (\sum_{i=1}^{r} c_i \mu_i) E_1 = 0,$$

$$\overline{\partial}(\sum_{i=1}^{r} c_i G_i) = \sum_{i=1}^{r} c_i \overline{\partial} G_i = 0.$$

We can see that $\sum_{i=1}^{r} c_i h_i = \langle \sum_{i=1}^{r} c_i G_i, x \rangle = \langle A, x \rangle$ is a linear combination of coordinate functions. □

We say that extra eigenfunctions $h_1, \ldots h_r$ are linearly independent when no linear combination of $h_1, \ldots h_r$ is a linear combination of coordinate functions.As we see in Lemma 2.5, $\Omega_t \in HH^{full}(S^2, Z_{r+m})$ for $t \neq \infty$ when $h_1, \ldots h_r$ are linearly independent and therefore we can define $x_t = \pi(\Omega_t) \in Harm(S^2, S^{2r+2m})$.

Proposition 2.6. *If $h_1, \ldots, h_r$ are linearly independent,then x_t constructed above satisfies*

(1) *For* $t \neq \infty$, $x_t = \{\Omega_t \wedge \overline{\Omega}_t\}^{\perp}$ *defines a full harmonic map from* S^2 *to* $S^{2(r+m)}$,
(2) $lim_{t\to\infty} x_t = x$,
(3) x_t *preserves degree* d.

Proof. Ω_t is spanned by $F_{1_t}, \dots, F_{r+m_t}$. We will find a better basis. Let $F^*_{r+i_t} = E_{r+i} - \sum_{j=1}^{r} \langle E_{r+i}, G_j \rangle e^{-t} E_{m+j}$ for $1 \leq i \leq m$ and $F^*_{i_t} = F_{i_t} - \sum_{j=1}^{m} \langle G_i, \overline{E}_j \rangle F^*_{r+j_t} = e^t \overline{E}_{m+i} + (G_i - G_{i+}) - \sum_{j=1}^{r} \int h_{iz} h_j \, dz e^{-t} E_{m+j}$ for $1 \leq i \leq r$. We see that $G_i - G_{i+} = G_{i-} + h_i g$ satisfies

$$\partial(G_{i-} + h_i g) = \partial h_i x,$$

$$\overline{\partial}(G_{i-} + h_i g) = h_i \overline{\partial} x.$$

Since x is a harmonic map on S^2 and $Re(h_i)$ and $Im(h_i)$ are eigenfunctions of Δ with respect to the metric induced by x, the above equations show that $G_i - G_{i+} = G_{i-} + h_i x$ is bounded. We use the basis $\{F^*_{1_t}, \dots, F^*_{r_t}, F^*_{r+1_t}, \dots, F^*_{r+m_t}\}$. It is easy to see that Ω_t does not intersect with any critical manifold $C_{k,l}$ for $k + l = r$, which proves that the deformation constructed preserves the degree. By an easy calculation, we get

$$x_t = \{\Omega_t \wedge \overline{\Omega}_t\}^{\perp}$$

$$= x + e^{-t} \sum_{i=1}^{r} (\overline{h}_i E_{m+i} + h_i \overline{E}_{m+i}) + o(e^{-2t}).$$

As t goes to ∞, x_t converges to x. Assume x_t, $t \neq \infty$, is not full. Then $\sum_{i=1}^{r} (\overline{h}_i E_{m+i} + h_i \overline{E}_{m+i})$ must lie in a hyperplane of $span\{E_{m+1}, \dots, E_{m+r}, \overline{E}_{m+1}, \dots, \overline{E}_{m+r}\}$. Because the eigenfunctions $h_1, \dots, h_r$ are linearly independent, this means that $\overline{h}_i E_{m+i} + h_i \overline{E}_{m+i}$ is parallel to a constant line for some i, which is impossible. Hence we have proved that x_t is full for all $t \neq \infty$. □

We have

Theorem A. *Suppose a full harmonic map* $x : S^2 \longrightarrow S^{2m}$ *of degree* d *has* r *linearly independent pairs of extra eigenfunctions. We can construct a smooth deformation of full harmonic maps* $x_t : S^2 \longrightarrow S^{2(r+m)}$, *preserving degree* d *and converging to* x *as* t *goes to* ∞.

Corollary. *The space* P_r *of full harmonic maps* $x : S^2 \longrightarrow S^{2m}$ *of degree* d *which admit* r *linearly independent pairs of extra eigenfunctions is equal to the space* L_r *of full harmonic maps* $x : S^2 \longrightarrow S^{2m}$ *of degree* d *which admit the smooth deformations as in Theorem* A.

Proof. We have seen that $x \in P_r$ admits such a deformation. On the other hand, suppose that $x = \lim_{t\to\infty} x_t$. Then x_t has at least $2(m + r) + 1$ eigenfunctions of eigenvalue 2 as its coordinate functions. Because of the continuity of Δ, x also must have at least $2(r + m) + 1$ eigenfunctions. However, x has only $2m + 1$ linearly independent coordinate functions and hence x has $2r$ linearly independent extra eigenfunctions. □

Corollary. *A harmonic map* $x : S^2 \longrightarrow S^{2m}$ *of degree* d *has at most* $\sqrt{8d+1} - 2m - 1$ *linearly independent extra eigenfunctions. In particular,* x *has no extra eigenfunction when* $d = \frac{m(m+1)}{2}$.

Proof. If x has r linearly independent pairs of extra eigenfunctions, then we construct a full harmonic maps $x_t : S^2 \longrightarrow S^{2m+2r}$ of degree d. Due to [B], $d \geq \frac{(m+r)(m+r+1)}{2}$. ☐

§3.DIMENSION

Lemma 3.1. *Let V_x be the set of harmonic maps in $Harm_d(S^2, S^{2m+2r})$ which can be deformed to a fixed element $x \in Harm_d(S^2, S^{2m})$. Then the set $\bigcup_{x \in L_r} V_x$ is open dense in $Harm_d(S^2, S^{2(m+r)})$.*

Proof. Take a harmonic map $y \in Harm_d(S^2, S^{2(m+r)})$ and $y_t = A_t y$. This y_t gives a deformation such that $\lim_{t \to \infty} y_t = t_\infty : S^2 \longrightarrow S^{2m}$. If the image of the corresponding holomorphic horizontal curve in Z_{m+r} of y does not intersect with any of the critical manifolds $C_{k,l}$ for $k + l = r$, the deformation preserves the degree and y admits a deformation y_t which converges to an element y_∞ of $Harm_d(S^2, S^{2m})$ as t goes to ∞. Notice that

$$\begin{aligned}
&\dim Z_{m+r} - \max_{k+l=r} \dim C_{k,l} - 2 - 1 \\
&\qquad = (m+r)(m+r+1) - \max_{k+l=r} \{2kl + m(m+1)\} - 3 \\
&\qquad\qquad \begin{cases} \geq \frac{r^2}{2} + r(2m+1) - 3 \geq 0, & 2 \leq r, \\ \geq 2m - 1 \geq 0, & r = 1, \end{cases}
\end{aligned}$$

which implies that a generic $y \in Harm_d(S^2, S^{2(m+r)})$ can be deformed to an element $y_\infty \in Harm_d(S^2, S^{2m})$. ☐

Lemma3.2. *For an arbitrary $x \in L_r \subset Harm_d^{full}(S^2, S^{2m})$, the dimension of the set V_x of harmonic maps in $Harm_d^{full}(S^2, S^{2m+2r})$ which can be deformed to x is $r^2 + r(2m+1)$.*

Proof. If there is such a deformation x_t with $x = \lim_{t \to \infty} x_t$, we can take the following basis of the twistor lift W_t of x_t:

$$W_t = span \begin{Bmatrix} e^t \overline{E}_{m+1} + G_1 + e^{-t} u_1, & \ldots, & e^t \overline{E}_{m+r} + G_r + e^{-t} u_r, \\ w_1 + e^{-t} u_{r+1}, & \ldots, & w_m + e^{-t} u_{r+m} \end{Bmatrix},$$

where all G_i's are meromorphic curves in $\mathbf{C}^{2m+1}$, all u_i's take value in the space spanned by $E_{m+1}, \ldots, E_{m+r}$, and $\{w_1, \ldots, w_m\}$ is a basis of the twistor lift W_∞ of x in $\mathbf{C}^{2m+1}$. From the holomorphic horizontal conditions for W_t, we can see that G_i's are meromorphic null curves, and $u_i = -\sum_{k=1}^r c_{i,k} E_{m+k}$ with $\partial c_{i,k} = \langle \partial G_i, G_k \rangle$ for $1 \leq i \leq r$ and $u_{r+j} = -\sum_{k=1}^r \langle w_j, G_k \rangle E_{m+k}$ for $1 \leq j \leq m$.

We can also see that $\langle G_i, x \rangle$'s are extra eigenfunctions if x_t is full. Thus if there is such a deformation x_t, it is given in the form we have constructed in Theorem A. G_i is uniquely determined up to constant multiple, a constant vector A_i and a choice of a unitary basis $E_{m+1}, \ldots, E_{m+r}$. The dimension of V_x is $r^2 + r(2m+1)$. ☐

Corollary. *We have*

$$\begin{aligned}
&\dim Harm_d(S^2, S^{2(m+r)}) \\
&\qquad = \dim Harm_d(S^2, S^{2m}) + (r - \mathrm{codim} L_r) + (r+m)^2 - m^2,
\end{aligned}$$

and codim $L_r - r$ *does not depend on* r.

Remark. If we prove $\mathrm{codim} L_r = r$ for some r, then we prove $\dim Harm_d(S^2, S^{2N}) = 2d + N^2$, which is a conjecture in [V] (See [BW1], [BW2] for more about the moduli space of harmonic 2-spheres.)

Proof. By Lemma 3.1, the set $\bigcup_{x \in L_r} V_x$ is open dense in $Harm_d(S^2, S^{2(m+r)})$. Thus we obtain

$$\dim Harm_d(S^2, S^{2(m+r)}) = \dim \bigcup_{x \in L_r} V_x = \dim L_r + \dim V_x$$
$$= \dim Harm_d(S^2, S^{2m}) - \mathrm{codim} L_r + r^2 + r(2m+1)$$
$$= \dim Harm_d(S^2, S^{2m}) + (r - \mathrm{codim} L_r) + (r+m)^2 - m^2.$$

□

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Department of Mathematics, Faculty of Sciences, Toho University, Funabashi, Chiba, 274, Japan

E-mail address: `kotani@tansei.cc.u-tokyo.ac.jp`