

## ON ORTHOGONALLY EXPONENTIAL AND ORTHOGONALLY ADDITIVE MAPPINGS

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ABSTRACT. Let  $E$  be a real inner product space,  $(F, +)$  an abelian  $\sigma$ -bounded topological group, and  $K$  a discrete subgroup of  $F$ . It is proved that (under suitable assumptions on  $E$ ) the Christensen and Baire measurable orthogonally additive functions  $g: E \rightarrow F/K$  have particular selections. In consequence, descriptions of measurable orthogonally exponential complex functionals on  $E$  are obtained.

### 1. INTRODUCTION

Assume the following two hypotheses:

(H<sub>1</sub>)  $E$  is a real inner product space with  $\dim E > 1$ ,

(H<sub>2</sub>)  $(F, +)$  is an abelian topological group and  $K$  is a discrete subgroup of  $F$

(discrete means that there is a neighbourhood  $U \subset F$  of 0 with  $K \cap U = \{0\}$ ). We study orthogonally additive functions mapping  $E$  into the factor group  $F/K$ , i.e. functions  $g$  satisfying the condition

$$(1) \quad g(x + y) = g(x) + g(y) \quad \text{for orthogonal } x, y \in E.$$

We show that if such a function is continuous at a point, or Christensen or Baire measurable, then, under suitable assumptions, there are continuous additive functions  $a: R \rightarrow F$  and  $A: E \rightarrow F$  such that  $a(\|x\|^2) + A(x) \in g(x)$  for  $x \in E$ . In consequence, we obtain analogues, for the Baire and Christensen measurable functions, of the following theorem of K. Baron and J. Rätz.

**Theorem A** (see [3], p. 15). *Assume (H<sub>1</sub>) and (H<sub>2</sub>). Let  $F$  be continuously divisible by 2 (i.e. the mapping  $x \rightarrow 2x$  is a homeomorphism of  $F$  onto  $F$ ) and  $f: E \rightarrow F$  be continuous at the origin and satisfying*

$$(2) \quad f(x + y) - f(x) - f(y) \in K \quad \text{for orthogonal } x, y \in E.$$

*Then there are continuous additive functions  $a: R \rightarrow F$  and  $A: E \rightarrow F$  such that*

$$(3) \quad f(x) - a(\|x\|^2) - A(x) \in K \quad \text{for } x \in E.$$

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We also generalize Theorem A by showing that  $f$  can be supposed continuous at any point and that the assumption of continuous divisibility by 2 can be replaced by the following weaker one:

$$(H_3) \quad 2x := x + x \neq 0 \quad \text{for } x \in F \setminus \{0\}.$$

Finally, we characterize the Baire and Christensen measurable orthogonally exponential functionals  $g : E \rightarrow C$ , i.e. solutions of the conditional equation

$$(4) \quad g(x + y) = g(x)g(y) \quad \text{for orthogonal } x, y \in E.$$

The orthogonally exponential functionals  $g : E \rightarrow C$  which are continuous at the origin or measurable on rays (i.e. for every  $x \in E$  the function  $t \rightarrow g(tx)$ ,  $t \in R$ , is Baire or Lebesgue measurable) have been investigated in [1] and [3]. For the information and bibliography concerning the orthogonally additive functions refer e.g. to [10] and [11].

Throughout the paper  $N, Z, Q, R$ , and  $C$  denote, as usual, the sets of positive integers, integers, rationals, reals, and complex numbers, respectively.

Given  $F$  and  $K$  satisfying  $(H_2)$ , in the factor group  $F/K$  we take the factor topology, i.e. a set  $U \subset F/K$  is open if the set  $p^{-1}(U)$  is open in  $F$ , where  $p : F \rightarrow F/K$  is the natural projection. If  $F/K$  is endowed with this topology, then it is a topological group and  $p$  is open and continuous.

In the sequel  $\text{Chr}(X)$  denotes the family of all Christensen measurable subsets of a Polish linear space  $X$  which are not Christensen zero sets (for details concerning Christensen measurability refer to [5] and [6]). Analogously, if  $X$  is a topological space,  $\text{Bai}(X)$  stands for the family of all subsets of  $X$  which are of the second category and with the Baire property (see e.g. [8], p. 92, and [9]). Let us recall that a function mapping a topological space  $X$  into a topological space  $Y$  is Baire measurable provided, for every open set  $U \subset Y$ , the set  $f^{-1}(U)$  has the Baire property in  $X$ .

## 2. THE MAIN THEOREM

Let us start with the following definition and lemma.

**Definition 1.** We say that a topological group  $(G, +)$  is  $\sigma$ -bounded provided, for every open neighbourhood  $U \subset G$  of 0, there is a sequence  $(x_n : n \in N) \subset G$  with

$$H = \bigcup \{U + x_n : n \in N\}.$$

For instance, every topological group  $(G, +)$  possessing a dense countable subset is  $\sigma$ -bounded.

**Lemma 1.** Let  $E$  be a real inner product space,  $D \subset E$ , and  $D_0 = \{\|x\|^2 : x \in D\}$ . The following two conditions hold.

- (i) If  $E$  is a Polish linear space and  $D \in \text{Chr}(E)$ , then  $D_0$  contains a subset of positive Lebesgue measure in  $R$ .
- (ii) If  $D \in \text{Bai}(E)$ , there is  $T \in \text{Bai}(R)$  with  $T \subset D_0$ .

*Proof.* Take  $e \in E$  with  $\|e\| = 1$  and put  $Y = \{z \in E : z \perp e\}$ . Then  $Y$  is a linear subspace of  $E$  and  $Re \oplus Y = E$ .

First assume that  $E$  is a Polish space and  $D \in \text{Chr}(E)$ . Then  $D$  has a universally measurable subset  $D_1$  which is not a Haar zero set. Let  $m$  be the Lebesgue measure

in  $R$ ,  $r: R \rightarrow E$  be given by  $r(c) = ce$  for  $c \in R$ , and  $L_k = \{c \in R: k - 1 \leq |c| < k\}$  for  $k \in N$ . Define a Borel measure  $u$  on  $E$  by the formula

$$u(T) = \sum_{k=1}^{\infty} 2^{-k} [m(L_k)]^{-1} m(r^{-1}(T) \cap L_k)$$

for every Borel set  $T \subset E$ . It is easily seen that  $u$  extended to the family of all universally measurable subsets of  $E$  is a probability measure on  $E$ , which means that there are  $b \in R, y \in Y$  with  $u(D_1 + be + y) > 0$  (cf. [6]). Thus there is a Borel set  $D_2 \subset D_1$  with  $u(D_2 + be + y) > 0$ . Hence  $m(r^{-1}(D_2 + y)) > 0$ . Further, we have  $D_3 := \{c^2: ce - y \in D\} \supset \{c^2: c \in r^{-1}(D_2 + y)\}$  and

$$(5) \quad c^2 + \|y\|^2 = c^2 \|e\|^2 + \|y\|^2 = \|ce - y\|^2 \quad \text{for } c \in R.$$

Consequently,  $D_3$  contains a subset of positive Lebesgue measure and  $D_3 + \|y\|^2 \subset D_0$ , which implies the statement (i).

Now, suppose  $D \in \text{Bai}(E)$ . Define a continuous functional  $j: E \rightarrow R$  by  $j(x) = \langle x, e \rangle$  for  $x \in E$ , where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $E$ . Then  $j(ce) = c$  for  $c \in R$  and  $Y = \text{Ker } j$ . Let  $g: E \rightarrow R \times Y$  and  $h: R \times Y \rightarrow E$  be functions given by

$$g(z) = (j(z), s(z)) \quad \text{for } z \in E,$$

$$h(c, y) = ce + y \quad \text{for } c \in R, y \in Y,$$

where  $s: E \rightarrow Y$  and  $s(ce + y) = y$  for  $c \in R, y \in Y$ . Next, suppose that  $Y$  is equipped with the restriction of the inner product from  $E$  and  $R \times Y$  is endowed with the product topology. Then it is easily seen that  $R \times Y$  is a real topological linear space and  $g$  and  $h$  are continuous. Thus  $g$  is a homeomorphism, because  $g = h^{-1}$ . Hence  $g(D) \in \text{Bai}(R \times Y)$ . Consequently there is  $y \in Y$  such that

$$D_y := \{c \in R: (c, y) \in g(D)\} \in \text{Bai}(R)$$

(see [9], p. 57) and therefore  $D_1 := \{c^2 \in R: c \in D_y\} \in \text{Bai}(R)$ . Further, since (5) is valid and  $\|y\| = \|-y\|$ ,  $D_1 + \|y\|^2 \subset D_0$ . This completes the proof.  $\square$

Now, we are in a position to formulate and prove the following

**Theorem 1.** *Suppose that hypotheses (H<sub>1</sub>)–(H<sub>3</sub>) are valid and  $g: E \rightarrow F/K$  is a function satisfying (1). Further, assume that one of the following three conditions holds:*

- (i)  $E$  is a Polish space,  $F$  is  $\sigma$ -bounded, and  $g$  is Christensen measurable;
- (ii)  $E$  is a Baire space (i.e. it is of the second category),  $F$  is  $\sigma$ -bounded, and  $g$  is Baire measurable;
- (iii)  $g$  is continuous at a point  $x_0 \in E$ .

Then there are continuous additive functions  $a: R \rightarrow F$  and  $A: E \rightarrow F$  such that

$$(6) \quad a(\|x\|^2) + A(x) \in g(x) \quad \text{for } x \in E.$$

*Proof.* Define functions  $g_0, g_1, g_2: E \rightarrow F/K$  by  $g_0(x) = g(-x)$ ,  $g_1(x) = g(x) - g(-x) = g(x) - g_0(x)$ , and  $g_2(x) = g(x) + g(-x) = g(x) + g_0(x)$  for  $x \in E$ . It is easily seen that  $g_1$  is odd and  $g_2$  is even, and they are solutions of (1). Thus, by Theorems 5 and 9 in [10],  $g_1$  is additive and there is an additive function  $h: R \rightarrow F/K$  such that

$$g_2(x) = h(\|x\|^2) \quad \text{for } x \in E.$$

We will show that  $g_1$  and  $h$  are continuous at the origins in  $E$  and  $R$ , respectively.

First consider the case (iii) where  $g$  is continuous at a point  $x_0$ . Then, in view of the definitions,  $g_1$  and  $g_2$  are continuous at  $x_0$ , too. Thus  $g_1$  is continuous at 0, because it is additive. Take a neighbourhood  $W \subset F/K$  of 0. There are neighbourhoods  $U \subset F/K$  and  $V \subset E$  of the respective origins such that  $U - U \subset W$  and

$$g_2(V + x_0) \subset U + g_2(x_0).$$

Put  $S = \{\|x\|^2 : x \in V + x_0\}$ . It is easily seen that  $\text{int}(S) \neq \emptyset$  in  $R$ . Hence  $S - S$  is a neighbourhood of 0 in  $R$ . To complete the proof of continuity of  $h$  at 0 it suffices to observe that

$$h(S - S) = h(S) - h(S) = g_2(V + x_0) - g_2(V + x_0) \subset U - U \subset W.$$

Now, assume that condition (i) ((ii), respectively) holds. Fix a neighbourhood  $W \subset F/K$  of 0. There are open neighbourhoods  $V, U \subset F/K$  of 0 such that  $V = -V$ ,  $V + V \subset U$ , and  $U - U \subset W$ . Furthermore, since  $F$  is  $\sigma$ -bounded,  $F/K$  is  $\sigma$ -bounded, too, and consequently there exists a sequence  $(x_n : n \in N) \subset F/K$  such that

$$F/K = \bigcup \{V + x_n : n \in N\}.$$

Note that

$$E = g^{-1}(F/K) \cap g_0^{-1}(F/K) = \bigcup \{g^{-1}(V + x_n) \cap g_0^{-1}(V + x_k) : n, k \in N\}.$$

Thus there are  $n, k \in N$  such that the set

$$D := g^{-1}(V + x_n) \cap g_0^{-1}(V + x_k)$$

belongs to  $\text{Chr}(E)$  ( $\text{Bai}(E)$ , resp.) and, by Lemma 1, the set  $D_0$  contains a subset of positive Lebesgue measure in  $R$  (a subset from  $\text{Bai}(R)$ , resp.). Hence, on account of Theorem 2 in [5] (the Difference Theorem in [8], p. 92, resp.),  $0 \in \text{int}(D - D)$  (in  $E$ ) and  $0 \in \text{int}(D_0 - D_0)$  (in  $R$ ). Since

$$\begin{aligned} g_1(D - D) &= g_1(D) - g_1(D) \subset (g(D) - g_0(D)) - (g(D) - g_0(D)) \\ &\subset [(V + x_n) - (V + x_k)] - [(V + x_n) - (V + x_k)] \subset U - U \subset W \end{aligned}$$

and

$$\begin{aligned} h(D_0 - D_0) &= h(D_0) - h(D_0) = g_2(D) - g_2(D) \\ &\subset (g(D) + g_0(D)) - (g(D) + g_0(D)) \subset U - U \subset W, \end{aligned}$$

this ends the proof of continuity of  $g_1$  and  $h$  at the origins.

It results from Lemma 1 in [4] that there are functions  $s_1 : E \rightarrow F$  and  $s_2 : R \rightarrow F$  continuous at the origins with  $s_1(x) \in g_1(x)$  for  $x \in E$  and  $s_2(c) \in h(c)$  for  $c \in R$ . Moreover,  $s_1(x+y) - s_1(x) - s_1(y) \in K$  for  $x, y \in E$  and  $s_2(c+d) - s_2(c) - s_2(d) \in K$  for  $c, d \in R$ , because  $g_1$  and  $h$  are additive. Consequently, in view of Theorem 3 in [2], there are additive and continuous functions  $A_0 : E \rightarrow F$  and  $a_0 : R \rightarrow F$  such that  $A_0(x) \in g_1(x)$  for  $x \in E$  and  $a_0(c) \in h(c)$  for  $c \in R$ . Let  $A : E \rightarrow F$  and  $a : R \rightarrow F$  be given by:

$$A(x) = A_0\left(\frac{1}{2}x\right) \quad \text{for } x \in E,$$

$$a(c) = a_0\left(\frac{1}{2}c\right) \quad \text{for } c \in R.$$

Then they are continuous and additive. It remains to show that (6) holds.

To this end take a function  $f: E \rightarrow F$  with  $f(x) \in g(x)$  for  $x \in E$ , which means that  $f$  satisfies (2). For every  $x \in E$  put  $f_1(x) = f(x) - f(-x)$  and  $f_2(x) = f(x) + f(-x)$ . Then  $f_i(x) \in g_i(x)$  for  $x \in E$ ,  $i = 1, 2$ , and consequently

$$f_1(x) - A_0(x) \in K \quad \text{for } x \in E,$$

$$f_2(x) - a_0(\|x\|^2) \in K \quad \text{for } x \in E.$$

Let  $k: E \rightarrow K$  be a function defined by the formula:

$$k(x) = f_1(x) - A_0(x) + f_2(x) - a_0(\|x\|^2) = 2f(x) - A_0(x) - a_0(\|x\|^2) \quad \text{for } x \in E.$$

Since, for every  $x \in E$ ,

$$k(x) + k(-x) = 2[f(x) + f(-x) - a_0(\|x\|^2)] = 2[f_2(x) - a_0(\|x\|^2)] \in 2K,$$

according to Theorem 5 in [10], the function  $k_0: E \rightarrow K/2K$ , given by  $k_0(x) = k(x) + 2K$  for  $x \in E$ , is additive. Thus  $k_0(x) = 2k_0(\frac{1}{2}x) = 0$  for  $x \in E$  and therefore  $k(E) \subset 2K$ . Whence, for every  $x \in E$ ,

$$2[f(x) - A(x) - a(\|x\|^2)] = 2f(x) - A_0(x) - a_0(\|x\|^2) = k(x) \in 2K,$$

which jointly with  $(H_3)$  yields (6). This ends the proof. □

### 3. APPLICATIONS

Now, we present two theorems which result from Theorem 1. The first one is a generalization of Theorem 1 in [3] and contains a result concerning stability, of Hyers-Ulam type (see e.g. [7]), for orthogonally additive mappings; the second characterizes orthogonally exponential functionals.

**Theorem 2.** *Suppose  $(H_1)$ – $(H_3)$ . Let  $f: E \rightarrow F$  be a function satisfying (2). If one of conditions (i)–(iii) of Theorem 1 is valid with  $g = f$ , then there exist continuous additive functions  $a: R \rightarrow F$  and  $A: E \rightarrow F$  such that (3) holds.*

*Proof.* Put  $g = p \circ f$ , where  $p: F \rightarrow F/K$  is the natural projection. Then one of conditions (i)–(iii) of Theorem 1 is satisfied. Thus Theorem 1 implies the assertion. □

**Theorem 3.** *Let  $E$  be a real inner product space with  $\dim E > 1$  and  $h: E \rightarrow C$  be a function satisfying (4). Suppose that one of the following three conditions is valid:*

- (i)  $E$  is a Polish space and  $h$  is Christensen measurable;
- (ii)  $E$  is a Baire space and  $h$  is Baire measurable;
- (iii)  $h$  is continuous at a point.

*Then either  $h(x) = 0$  for  $x \in E$  or*

$$h(x) = \begin{cases} 0 & \text{if } x \in E \setminus \{0\}, \\ 1 & \text{if } x = 0, \end{cases}$$

*or there are  $c \in C$  and a continuous  $R$ -linear functional  $A: E \rightarrow C$  such that*

$$h(x) = \exp(c\|x\|^2 + A(x)) \quad \text{for } x \in E.$$

*Proof.* Suppose that  $h(x) \neq 0$  for some  $x \in E \setminus \{0\}$ . Then, according to Proposition 3 in [1],  $0 \notin h(E)$ . Let  $S = \{z \in \mathbb{C} : |z| = 1\}$  and  $h_0: E \rightarrow S$ ,  $f: E \rightarrow \mathbb{R}$ ,  $g: E \rightarrow \mathbb{R}/Z$ ,  $T: S \rightarrow \mathbb{R}/Z$  be functions given by  $f(x) = \log |h(x)|$  for  $x \in E$ ,

$$h_0(x) = \frac{h(x)}{|h(x)|} \quad \text{for } x \in E,$$

$$T(\exp 2\pi it) = t + Z \quad \text{for } t \in [0, 1),$$

and  $g = T \circ h_0$ . It is easily seen that  $g$  and  $f$  satisfy the assumptions of Theorems 1 and 2, respectively, with  $F = \mathbb{R}$  and  $K = Z$ , and, moreover,  $f$  satisfies (1), i.e. (2) with  $K = \{0\}$ . Thus there are  $c_1, c_2 \in \mathbb{R}$  and continuous linear functionals  $A_1, A_2: E \rightarrow \mathbb{R}$  with

$$f(x) = c_1 \|x\|^2 + A_1(x) \quad \text{for } x \in E,$$

$$c_2 \|x\|^2 + A_2(x) \in g(x) \quad \text{for } x \in E.$$

Since  $h(x) = h_0(x) \exp(f(x))$  for  $x \in E$ , setting  $c = c_1 + 2\pi i c_2$  and  $A = A_1 + 2\pi i A_2$  we obtain the statement.  $\square$

*Remark.* It results from Remark in [3] (on page 15) that the regularity assumptions made in Theorems 1–3 are essential.

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