ON ORTHOGONALLY EXPONENTIAL AND ORTHOGONALLY ADDITIVE MAPPINGS

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Abstract. Let $E$ be a real inner product space, $(F, +)$ an abelian $\sigma$-bounded topological group, and $K$ a discrete subgroup of $F$. It is proved that (under suitable assumptions on $E$) the Christensen and Baire measurable orthogonally additive functions $g: E \to F/K$ have particular selections. In consequence, descriptions of measurable orthogonally exponential complex functionals on $E$ are obtained.

1. Introduction

Assume the following two hypotheses:

$$(H_1) \quad E \text{ is a real inner product space with } \dim E > 1,$$

$$(H_2) \quad (F, +) \text{ is an abelian topological group and } K \text{ is a discrete subgroup of } F$$

(discrete means that there is a neighbourhood $U \subset F$ of 0 with $K \cap U = \{0\}$). We study orthogonally additive functions mapping $E$ into the factor group $F/K$, i.e. functions $g$ satisfying the condition

$$g(x + y) = g(x) + g(y) \quad \text{for orthogonal } x, y \in E. \quad (1)$$

We show that if such a function is continuous at a point, or Christensen or Baire measurable, then, under suitable assumptions, there are continuous additive functions $a: \mathbb{R} \to F$ and $A: E \to F$ such that $a(\|x\|^2) + A(x) \in g(x)$ for $x \in E$. In consequence, we obtain analogues, for the Baire and Christensen measurable functions, of the following theorem of K. Baron and J. Rätz.

Theorem A (see [3], p. 15). Assume $(H_1)$ and $(H_2)$. Let $F$ be continuously divisible by 2 (i.e. the mapping $x \to 2x$ is a homeomorphism of $F$ onto $F$) and $f: E \to F$ be continuous at the origin and satisfying

$$f(x + y) - f(x) - f(y) \in K \quad \text{for orthogonal } x, y \in E. \quad (2)$$

Then there are continuous additive functions $a: \mathbb{R} \to F$ and $A: E \to F$ such that

$$f(x) - a(\|x\|^2) - A(x) \in K \quad \text{for } x \in E. \quad (3)$$

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We also generalize Theorem A by showing that $f$ can be supposed continuous at any point and that the assumption of continuous divisibility by 2 can be replaced by the following weaker one:

$$(H_3) \quad 2x := x + x \neq 0 \quad \text{for} \quad x \in F \setminus \{0\}.$$ 

Finally, we characterize the Baire and Christensen measurable orthogonally exponential functionals $g : E \to C$, i.e. solutions of the conditional equation

$$g(x + y) = g(x)g(y) \quad \text{for orthogonal} \quad x, y \in E. \quad (4)$$ 

The orthogonally exponential functionals $g : E \to C$ which are continuous at the origin or measurable on rays (i.e. for every $x \in E$ the function $t \to g(tx)$, $t \in R$, is Baire or Lebesgue measurable) have been investigated in [1] and [3]. For the information and bibliography concerning the orthogonally additive functions refer e.g. to [10] and [11].

Throughout the paper $N, Z, Q, R, \text{and} \ C$ denote, as usual, the sets of positive integers, integers, rationals, reals, and complex numbers, respectively.

Given $F$ and $K$ satisfying $(H_2)$, in the factor group $F/K$ we take the factor topology, i.e. a set $U \subset F/K$ is open if the set $p^{-1}(U)$ so p e ni n $F$, where $p : F \to F/K$ is the natural projection. If $F/K$ is endowed with this topology, then it is a topological group and $p$ is open and continuous.

In the sequel Chr($X$) denotes the family of all Christensen measurable subsets of a Polish linear space $X$ which are not Christensen zero sets (for details concerning Christensen measurability refer to [5] and [6]). Analogously, if $X$ is a topological space, Bai($X$) stands for the family of all subsets of $X$ which are of the second category and with the Baire property (see e.g. [8], p. 92, and [9]). Let us recall that a function mapping a topological space $X$ into a topological space $Y$ is Baire measurable provided, for every open set $U \subset Y$, the set $f^{-1}(U)$ has the Baire property in $X$.

2. THE MAIN THEOREM

Let us start with the following definition and lemma.

**Definition 1.** We say that a topological group $(G, +)$ is $\sigma$-bounded provided, for every open neighbourhood $U \subset G$ of 0, there is a sequence $(x_n : n \in N) \subset G$ with

$$H = \bigcup \{U + x_n : n \in N\}.$$ 

For instance, every topological group $(G, +)$ possessing a dense countable subset is $\sigma$-bounded.

**Lemma 1.** Let $E$ be a real inner product space, $D \subset E$, and $D_0 = \{\|x\|^2 : x \in D\}$. The following two conditions hold.

(i) If $E$ is a Polish linear space and $D \in \text{Chr}(E)$, then $D_0$ contains a subset of positive Lebesgue measure in $R$.

(ii) If $D \in \text{Bai}(E)$, there is $T \in \text{Bai}(R)$ with $T \subset D_0$.

**Proof.** Take $e \in E$ with $\|e\| = 1$ and put $Y = \{z \in E : z \perp e\}$. Then $Y$ is a linear subspace of $E$ and $Re \oplus Y = E$.

First assume that $E$ is a Polish space and $D \in \text{Chr}(E)$. Then $D$ has a universally measurable subset $D_1$ which is not a Haar zero set. Let $m$ be the Lebesgue measure
in $R$, $r: R \to E$ be given by $r(c) = ce$ for $c \in R$, and $L_k = \{ c \in R : k - 1 \leq |c| < k \}$ for $k \in \mathbb{N}$. Define a Borel measure $u$ on $E$ by the formula
\[
u(T) = \sum_{k=1}^{\infty} 2^{-k}[m(L_k)]^{-1}m(r^{-1}(T) \cap L_k)
\]
for every Borel set $T \subset E$. It is easily seen that $u$ extended to the family of all universally measurable subsets of $E$ is a probability measure on $E$, which means that there are $b \in R$, $y \in Y$ with $u(D_1 + be + y) > 0$ (cf. [6]). Thus there is a Borel set $D_2 \subset D_1$ with $u(D_2 + be + y) > 0$. Hence $m(r^{-1}(D_2 + y)) > 0$. Further, we have $D_3 := \{ c^2 : ce - y \in D \} \supset \{ c^2 : c \in r^{-1}(D_2 + y) \}$ and
\[
\|c^2 + \|y\|^2 = c^2\|e\|^2 + \|y\|^2 = \|ce - y\|^2 \quad \text{for } c \in R.
\]
Consequently, $D_3$ contains a subset of positive Lebesgue measure and $D_3 + \|y\|^2 \subset D_0$, which implies the statement (i).

Now, suppose $D \in \text{Bai}(E)$. Define a continuous functional $j: E \to R$ by $j(x) = \langle x, e \rangle$ for $x \in E$, where $\langle \ , \ \rangle$ denotes the inner product in $E$. Then $j(ce) = c$ for $c \in R$ and $Y = \text{Ker} j$. Let $g: E \to R \times Y$ and $h: R \times Y \to E$ be functions given by
\[
g(z) = (j(z), s(z)) \quad \text{for } z \in E,
\]
\[
h(c, y) = ce + y \quad \text{for } c \in R, y \in Y,
\]
where $s: E \to Y$ and $s(ce + y) = y$ for $c \in R$, $y \in Y$. Next, suppose that $Y$ is equipped with the restriction of the inner product from $E$ and $R \times Y$ is endowed with the product topology. Then it is easily seen that $R \times Y$ is a real topological linear space and $g$ and $h$ are continuous. Thus $g$ is a homeomorphism, because $g = h^{-1}$. Hence $g(D) \in \text{Bai}(R \times Y)$. Consequently there is $y \in Y$ such that
\[
D_y := \{ c \in R : (c, y) \in g(D) \} \in \text{Bai}(R)
\]
(see [9], p. 57) and therefore $D_1 := \{ c^2 \in R : c \in D_y \} \in \text{Bai}(R)$. Further, since (5) is valid and $\|y\| = \| - y\|$, $D_1 + \|y\|^2 \subset D_0$. This completes the proof.

Now, we are in a position to formulate and prove the following

**Theorem 1.** Suppose that hypotheses (H$_1$)–(H$_3$) are valid and $g: E \to F/K$ is a function satisfying (1). Further, assume that one of the following three conditions holds:

(i) $E$ is a Polish space, $F$ is $\sigma$-bounded, and $g$ is Christensen measurable;

(ii) $E$ is a Baire space (i.e. it is of the second category), $F$ is $\sigma$-bounded, and $g$ is Baire measurable;

(iii) $g$ is continuous at a point $x_0 \in E$.

Then there are continuous additive functions $a: R \to F$ and $A: E \to F$ such that
\[
a(\|x\|^2) + A(x) \in g(x) \quad \text{for } x \in E.
\]

**Proof.** Define functions $g_0, g_1, g_2: E \to F/K$ by $g_0(x) = g(-x)$, $g_1(x) = g(x) - g(-x) = g(x) - g_0(x)$, and $g_2(x) = g(x) + g(-x) = g(x) + g_0(x)$ for $x \in E$. It is easily seen that $g_1$ is odd and $g_2$ is even, and they are solutions of (1). Thus, by Theorems 5 and 9 in [10], $g_1$ is additive and there is an additive function $h: R \to F/K$ such that
\[
g_2(x) = h(\|x\|^2) \quad \text{for } x \in E.
\]

We will show that $g_1$ and $h$ are continuous at the origins in $E$ and $R$, respectively.
First consider the case (iii) where \( g \) is continuous at a point \( x_0 \). Then, in view of the definitions, \( g_1 \) and \( g_2 \) are continuous at \( x_0 \), too. Thus \( g_1 \) is continuous at 0, because it is additive. Take a neighbourhood \( W \subset F/K \) of 0. There are neighbourhoods \( U \subset F/K \) and \( V \subset E \) of the respective origins such that \( U - U \subset W \) and 

\[
g_2(V + x_0) \subset U + g_2(x_0).
\]

Put \( S = \{ ||x||^2 : x \in V + x_0 \} \). It is easily seen that \( \text{int}(S) \neq \emptyset \) in \( R \). Hence \( S - S \) is a neighbourhood of 0 in \( R \). To complete the proof of continuity of \( h \) at 0 it suffices to observe that 

\[
h(S - S) = h(S) - h(S) = g_2(V + x_0) - g_2(V + x_0) \subset U - U \subset W.
\]

Now, assume that condition (i) (or (ii), respectively) holds. Fix a neighbourhood \( W \subset F/K \) of 0. There are open neighbourhoods \( V, U \subset F/K \) of 0 such that \( V = -V \), \( V + V \subset U \), and \( U - U \subset W \). Furthermore, since \( F \) is \( \sigma \)-bounded, \( F/K \) is \( \sigma \)-bounded, too, and consequently there exists a sequence \( (x_n : n \in N) \subset F/K \) such that 

\[
F/K = \bigcup \{ V + x : n \in N \}.
\]

Note that 

\[
E = g^{-1}(F/K) \cap g_0^{-1}(F/K) = \bigcup \{ g^{-1}(V + x_n) \cap g_0^{-1}(V + x_k) : n, k \in N \}.
\]

Thus there are \( n, k \in N \) such that the set 

\[
D := g^{-1}(V + x_n) \cap g_0^{-1}(V + x_k)
\]

belongs to \( \text{Chr}(E) \) (\( \text{Bai}(E) \), resp.) and, by Lemma 1, the set \( D_0 \) contains a subset of positive Lebesgue measure in \( R \) (a subset from \( \text{Bai}(R) \), resp.). Hence, on account of Theorem 2 in [5] (the Difference Theorem in [8], p. 92, resp.), \( 0 \in \text{int}(D - D) \) (in \( E \)) and \( 0 \in \text{int}(D_0 - D_0) \) (in \( R \)). Since 

\[
g_1(D - D) = g_1(D) - g_1(D) \subset (g(D) - g_0(D)) - (g(D) - g_0(D))
\]

and

\[
h(D_0 - D_0) = h(D_0) - h(D_0) = g_2(D) - g_2(D)
\]

this ends the proof of continuity of \( g_1 \) and \( h \) at the origins.

It results from Lemma 1 in [4] that there are functions \( s_1 : E \to F \) and \( s_2 : R \to F \) continuous at the origins with \( s_1(x) \in g_1(x) \) for \( x \in E \) and \( s_2(c) \in h(c) \) for \( c \in R \). Moreover, \( s_1(x+y) - s_1(x) - s_1(y) \in K \) for \( x, y \in E \) and \( s_2(c+d) - s_2(c) - s_2(d) \in K \) for \( c, d \in R \), because \( g_1 \) and \( h \) are additive. Consequently, in view of Theorem 3 in [2], there are additive and continuous functions \( A_0 : E \to F \) and \( a_0 : R \to F \) such that \( A_0(x) \in g_1(x) \) for \( x \in E \) and \( a_0(c) \in h(c) \) for \( c \in R \). Let \( A : E \to F \) and \( a : R \to F \) be given by:

\[
A(x) = A_0 \left( \frac{1}{2} x \right) \quad \text{for} \ x \in E,
\]

\[
a(c) = a_0 \left( \frac{1}{2} c \right) \quad \text{for} \ c \in R.
\]

Then they are continuous and additive. It remains to show that (6) holds.
To this end take a function \( f : E \to F \) with \( f(x) \in g(x) \) for \( x \in E \), which means that \( f \) satisfies (2). For every \( x \in E \) put \( f_1(x) = f(x) - f(-x) \) and \( f_2(x) = f(x) + f(-x) \). Then \( f_i(x) \in g_i(x) \) for \( x \in E, \ i = 1, 2, \) and consequently
\[
f_1(x) - A_0(x) \in K \quad \text{for} \ x \in E,
\]
\[
f_2(x) - a_0(\|x\|^2) \in K \quad \text{for} \ x \in E.
\]
Let \( k : E \to K \) be a function defined by the formula:
\[
k(x) = f_1(x) - A_0(x) + f_2(x) - a_0(\|x\|^2) = 2f(x) - A_0(x) - a_0(\|x\|^2) \quad \text{for} \ x \in E.
\]
Since, for every \( x \in E \),
\[
k(x) + k(-x) = 2[f(x) + f(-x) - a_0(\|x\|^2)] = 2[f_2(x) - a_0(\|x\|^2)] \in 2K,
\]
according to Theorem 5 in [10], the function \( k_0 : E \to K/2K \), given by \( k_0(x) = k(x) + 2K \) for \( x \in E \), is additive. Thus \( k_0(x) = 2k_0(\frac{1}{2}x) = 0 \) for \( x \in E \) and therefore \( k(E) \subset 2K \). Whence, for every \( x \in E \),
\[
2[f(x) - A(x) - a(\|x\|^2)] = 2f(x) - A_0(x) - a_0(\|x\|^2) = k(x) \in 2K,
\]
which jointly with \( (H_3) \) yields (6). This ends the proof.

\[\square\]

3. Applications

Now, we present two theorems which result from Theorem 1. The first one is a generalization of Theorem 1 in [3] and contains a result concerning stability, of Hyers-Ulam type (see e.g. [7]), for orthogonally additive mappings; the second characterizes orthogonally exponential functionals.

**Theorem 2.** Suppose \((H_1)-(H_3)\). Let \( f : E \to F \) be a function satisfying (2). If one of conditions (i)-(iii) of Theorem 1 is valid with \( g = f \), then there exist continuous additive functions \( a : R \to F \) and \( A : E \to F \) such that (3) holds.

**Proof.** Put \( g = p \circ f \), where \( p : F \to F/K \) is the natural projection. Then one of conditions (i)-(iii) of Theorem 1 is satisfied. Thus Theorem 1 implies the assertion. \[\square\]

**Theorem 3.** Let \( E \) be a real inner product space with \( \dim E > 1 \) and \( h : E \to C \) be a function satisfying (4). Suppose that one of the following three conditions is valid:

(i) \( E \) is a Polish space and \( h \) is Christensen measurable;

(ii) \( E \) is a Baire space and \( h \) is Baire measurable;

(iii) \( h \) is continuous at a point.

Then either \( h(x) = 0 \) for \( x \in E \) or
\[
h(x) = \begin{cases} 0 & \text{if} \ x \in E \setminus \{0\}, \\ 1 & \text{if} \ x = 0, \end{cases}
\]
or there are \( c \in C \) and a continuous \( R \)-linear functional \( A : E \to C \) such that
\[
h(x) = \exp(c\|x\|^2 + A(x)) \quad \text{for} \ x \in E.
\]
Proof. Suppose that $h(x) \neq 0$ for some $x \in E \setminus \{0\}$. Then, according to Proposition 3 in [1], $0 \not\in h(E)$. Let $S = \{z \in C : |z| = 1\}$ and $h_0 : E \to S$, $f : E \to R$, $g : E \to R/Z$, $T : S \to R/Z$ be functions given by $f(x) = \log|h(x)|$ for $x \in E$,

$$h_0(x) = \frac{h(x)}{|h(x)|} \quad \text{for } x \in E,$$

and $g = T \circ h_0$. It is easily seen that $g$ and $f$ satisfy the assumptions of Theorems 1 and 2, respectively, with $F = R$ and $K = Z$, and, moreover, $f$ satisfies (1), i.e. (2) with $K = \{0\}$. Thus there are $c_1, c_2 \in R$ and continuous linear functionals $A_1, A_2 : E \to R$ with

$$f(x) = c_1\|x\|^2 + A_1(x) \quad \text{for } x \in E,$$

$$c_2\|x\|^2 + A_2(x) \in g(x) \quad \text{for } x \in E.$$

Since $h(x) = h_0(x) \exp(f(x))$ for $x \in E$, setting $c = c_1 + 2\pi ic_2$ and $A = A_1 + 2\pi iA_2$ we obtain the statement. \hfill \square

Remark. It results from Remark in [3] (on page 15) that the regularity assumptions made in Theorems 1–3 are essential.

References

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