PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 7, July 1997, Pages 2103–2107
S 0002-9939(97)03800-8

INEQUALITIES BASED ON A GENERALIZATION OF CONCAVITY

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(Communicated by Hal L. Smith)

ABSTRACT. The concept of concavity is generalized to functions, y, satisfying nth order differential inequalities, $(-1)^{n-k}y^{(n)}(t) \geq 0, 0 \leq t \leq 1$, and homogeneous two-point boundary conditions, $y(0) = \ldots = y^{(k-1)}(0) = 0, y(1) = \ldots = y^{(n-k-1)}(1) = 0$, for some $k \in \{1, \ldots, n-1\}$. A piecewise polynomial, which bounds the function, y, below, is constructed, and then is employed to obtain that $y(t) \geq ||y||/4^m, 1/4 \leq t \leq 3/4$, where $m = \max\{k, n-k\}$ and $||\cdot||$ denotes the supremum norm. An analogous inequality for a related Green's function is also obtained. These inequalities are useful in applications of certain cone theoretic fixed point theorems.

In recent applications of cone theoretic fixed point theorems to boundary value problems (BVPs), inequalities that provide lower bounds for positive functions as a function of the supremum norm have been applied. This type of inequality has been useful in applications to both regular two-point BVPs ([5], [4]) on annular like regions, and singular two-point BVPs ([6], [3]). The particular inequality to which we refer is as follows: if $y''(t) \leq 0, 0 \leq t \leq 1$, and $y(t) \geq 0, 0 \leq t \leq 1$, then for $1/4 \leq t \leq 3/4$,

$$y(t) \geq ||y||/4, \tag{1}$$

where $||y|| = sup_{0 \leq t \leq 1}|y(t)|$. An analogous inequality for a Green's function has been employed for regular two-point BVPs [5], [4].

The purpose of this short paper is to obtain generalizations of (1) and the analogous inequalities for Green's functions. These inequalities will play analogous roles in the study of BVPs for nth order ordinary differential equations. In particular, we shall show that if $n \geq 2$ is an integer, $k \in \{1, \ldots, n-1\}$, and if

$$(-1)^{(n-k)}y^{(n)} \geq 0, 0 \leq t \leq 1, \tag{2}$$

$$y^{(j)}(0) = 0, j = 0, \ldots, k-1, y^{(j)}(1) = 0, j = 0, \ldots, n-k-1, \tag{3}$$

then for $1/4 \leq t \leq 3/4$,

$$y(t) \geq ||y||/4^m, \tag{4}$$

where $m = \max\{k, n-k\}$.

Received by the editors July 12, 1995 and, in revised form, February 6, 1996.
1991 *Mathematics Subject Classification.* Primary 34A40; Secondary 34B27.
Key words and phrases. Differential inequalities.

Inequality (1) can be obtained as follows: Assume for simplicity that $||y|| = y(t_1)$, and $0 < t_1 < 1$. By the concavity of y, $y(t) \geq p(t)$ where p is the piecewise polynomial,

$$p(t) = \begin{cases} (||y||/t_1)t, 0 < t < t_1, \\ (||y||/(t_1 - 1))(t-1), t_1 < t < 1. \end{cases}$$

In this paper, we shall obtain (4) by generalizing the argument given here for the second order problem.

Theorem 1. *Assume $y \in C^n[0,1]$ and satisfies* (2) *and* (3). *Then y satisfies* (4).

Proof. We first obtain (4) in the case that $(-1)^{(n-k)}y^{(n)}(t) > 0, 0 < t < 1$. Note that

$$y(t) = \int_0^1 G(t,s)y^{(n)}(s)ds,$$

where $G(t,s)$ is the Green's function of the boundary value problem, $y^{(n)}(t) = 0, 0 \leq t \leq 1$, (3). It is well-known ([2]) that $(-1)^{n-k}G(t,s) > 0$ on $(0,1) \times (0,1)$. Hence, $y(t) > 0, 0 < t < 1$. Also note that y has precisely one extreme point at, say, $t_1 \in (0,1)$. For if y' vanishes twice in $(0,1)$ then by employing the boundary conditions, (3), and by applying Rolle's theorem repeatedly, one obtains the contradiction that $y^{(n)}$ vanishes in $(0,1)$.

Define a piecewise polynomial, p, by

$$p(t) = \begin{cases} (||y||/(t_1^k))t^k, 0 < t < t_1, \\ (||y||/(t_1 - 1)^{n-k})(t-1)^{n-k}, t_1 < t < 1. \end{cases} \tag{5}$$

We shall show, in Lemma 2, that $y(t) \geq p(t), 0 \leq t \leq 1$. Inequality (4) will then follow in the case $(-1)^{n-k}y^{(n)}(t) > 0, 0 < t < 1$. To see this, first assume $t_1 \in [1/4, 3/4]$. For $1/4 \leq t \leq t_1$,

$$y(t) \geq p(t) \geq p(1/4) \geq ||y||/4^k.$$

For $t_1 \leq t \leq 3/4$,

$$y(t) \geq p(t) \geq p(3/4) \geq ||y||/4^{n-k}.$$

Thus, (4) holds in this case. In the case $t_1 < 1/4$, then $y(t) \geq p(3/4)$ for $1/4 \leq t \leq 3/4$, and in the case $3/4 < t_1$, then $y(t) \geq p(1/4)$ for $1/4 \leq t \leq 3/4$. Thus, once Lemma 2 is proved, (4) holds in the case $(-1)^{n-k}y^{(n)}(t) > 0, 0 < t < 1$.

Now consider the case where (2) holds. For $\epsilon > 0$, define $y(\epsilon, t) = y(t) + \epsilon t^k(1-t)^{n-k}$. By the preceding case, (4) holds for $\epsilon > 0$ and, by continuity in ϵ, (4) holds for $\epsilon = 0$. The proof of Theorem 1 is complete, once Lemma 2 has been proved. □

Lemma 2. *Assume $y \in C^n[0,1]$ and assume $(-1)^{n-k}y^{(n)}(t) > 0, 0 < t < 1$. Moreover, assume that y satisfies the homogeneous boundary conditions,* (3). *Then $y(t) \geq p(t), 0 \leq t \leq 1$, where $p(t)$ is defined by* (5).

Proof. First, assume that $k \in \{2, \dots, n-2\}$. Apply Rolle's theorem repeatedly to $y(t)$. In the proof of Theorem 1, we have argued that y' vanishes precisely once at $t_1 \in (0,1)$. We also note that since $(-1)^{n-k}y^{(n)}(t) > 0, 0 < t < 1$, y' has a simple zero at t_1. Similarly, y'' has two simple zeros in $(0,1)$. We shall provide two sets of labels for the zeros of y''. We shall label these zeros by $t_{21} < t_{22}$ or by $\tau_{22} < \tau_{21}$.

So, $t_{21} = \tau_{22}$ and $t_{22} = \tau_{21}$. Similarly, y''' has at least two simple zeros in $(0,1)$ and we label these zeros by $t_{31} < t_{32} < \dots$ or by $\dots < \tau_{32} < \tau_{31}$. Inductively, for $i = 2, \dots, n-2$, $y^{(i)}$ has at least two simple zeros in $(0,1)$ and we label these zeros by $t_{i1} < t_{i2} < \dots$ or by $\dots < \tau_{i2} < \tau_{i1}$. Note that

$$t_{k1} < t_{k+1,1} < \dots < t_{n-1,1} = \tau_{n-1,1} < \dots < \tau_{n-k+1,1} < \tau_{n-k,1}.$$

We shall now show that $y(t) - p(t) \geq 0, 0 \leq t \leq 1$, where p is given by (5). For $0 \leq t \leq t_1$ let $h(t) = y(t) - p(t)$. Note that $h(0) = \dots = h^{(k-1)}(0) = h(t_1) = 0$ and $h'(t_1) = -p'(t_1) < 0$. So, $h(t_1^-) > 0$. Assume, for the sake of contradiction, that $h(c) = 0$ for some $c \in (0, t_1)$. Apply Rolle's theorem and note that h' vanishes at $c_{1i}, i = 1, 2$, where $0 < c_{11} < c_{12} < t_1$. Apply Rolle's theorem again and note that h'' vanishes at $c_{2i}, i = 1, 2$, where $0 < c_{21} < c_{22} < c_{12}$. We now argue that, in fact, $0 < c_{21} < c_{22} < t_{21}$. We have noted that for each $j = 1, \dots, n-1$, each root of $y^{(j)}$ in $(0,1)$ is simple, so y'' changes sign at t_{21} and at t_{22}. Note that $y^{(k)}(0) > 0$. This follows since $y(t) = \int_0^1 G(t,s) y^{(n)}(s) ds$ and $(-1)^{n-k} (\partial^k / \partial t^k) G(0,s) > 0, 0 < s < 1$ [2]. So by Taylor's theorem, $y^{(j)}(t) > 0, 0 < t < t_{j1}, j = 2, \dots, k-1$, and $y^{(j)}(t) < 0, t_{j1} < t < t_{j2}, j = 2, \dots, k-1$. Now, $h''(c_{2i}) = 0, i = 1, 2$, implies $y''(c_{2i}) = p''(c_{2i}) > 0, i = 1, 2$. Since, $c_{22} < t_1 < t_{22}$, then $c_{2i} \in (0, t_{21}), i = 1, 2$. It now follows inductively for $j = 2, \dots, k$, that $h^{(j)}$ vanishes at $0 < c_{j1} < c_{j2} < t_{j1}$. To see this, it follows by Rolle's theorem that $0 < c_{j1} < c_{j2} < c_{j-1,2}$. $h^{(j)}(c_{ji}) = 0, i = 1, 2$, implies $y^{(j)}(c_{ji}) = p^{(j)}(c_{ji}) > 0, i = 1, 2$. By the simplicity of the interior roots of $y^{(j)}$ and the fact that $c_{j2} < t_{j-1,1} < t_{j2}$ it follows that $0 < c_{j1} < c_{j2} < t_{j1}, j = 2, \dots, k$. Now, apply Rolle's theorem to $h^{(k)}$ and note that $h^{(k+1)}$ vanishes at some c_{k+1} where $c_{k+1} < c_{k2} < t_{k1} < t_{k+1,1}$. This produces a contradiction since $h^{(k+1)} \equiv y^{(k+1)}$ and $t_{k+1,1}$ is the smallest positive root of $y^{(k+1)}$. Thus, h does not vanish on $(0, t_1)$ and $h(t) > 0, 0 < t < t_1$.

The argument to show $y(t) \geq p(t), t_1 \leq t \leq 1$, is similar. Let $h = y - p$ on $(t_1, 1)$ and note that $h(t_1) = h(1) = \dots = h^{(n-k-1)}(1) = 0$ and $h'(t_1) = -p'(t_1) > 0$. Thus, $h(t_1^+) > 0$. Now assume, for the sake of contradiction, that h vanishes at $c \in (t_1, 1)$. The analogous contradiction arises if for each $j = 2, \dots, n-k$, $h^{(j)}$ vanishes at $c_{ji}, i = 1, 2$, and $\tau_{j1} < c_{j1} < c_{j2} < 1$. Again, it follows that ([2]) $(-1)^{n-k} y^{(n-k)}(1) > 0$, and so, by Taylor's theorem and the simplicity of the interior roots, it follows that $(-1)^j y^{(j)}(t) > 0, \tau_{j1} < t < 1$, and $(-1)^j y^{(j)}(t) < 0, \tau_{j2} < t < \tau_{j1}$. If $h^{(j)}$ vanishes at $c_{ji}, i = 1, 2$, then $y^{(j)}(c_{ji}) = p^{(j)}(c_{ji})$ and $(-1)^j p^{(j)}(c_{ji}) > 0$. Thus, $\tau_{j1} < c_{j1} < c_{j2} < 1$ as in the inductive argument in the preceding paragraph. Again, as above, apply Rolle's theorem to $h^{(n-k)}$ and obtain that $h^{(n-k+1)}$ vanishes at some c_{k+1} where $\tau_{k+1,1} < c_{k+1}$. Again, this is a contradiction as $h^{(n-k+1)} \equiv y^{(n-k+1)}$ and this completes the proof of Lemma 2 in the case that $k \in \{2, \dots, n-2\}$.

To handle the case $k = 1$ or $k = n-1$, we consider the case $k = n-1$. The case $k = 1$ is handled similarly or simply with a change of variable. Assume $-y^{(n)}(t) > 0, 0 < t < 1, y(0) = \dots = y^{(n-2)}(0) = 0, y(1) = 0$. It is clear by Rolle's theorem that y has precisely one extreme point at some $t_1 \in (0,1)$. Define p by (5) with $k = n-1$. On $[0, t_1]$, let $h = y - p$. Then $h^{(n)}(t) < 0, 0 < t < 1, h(0) = h'(0) = \dots = h^{(n-2)}(0) = 0, h(t_1) = 0$. Thus,

$$h(t) = \int_0^{t_1} G(t,s) h^{(n)}(s) ds$$

where G is the Green's function of the boundary value problem, $y^{(n)}(t) = 0, 0 < t < t_1, y(0) = \ldots = y^{(n-2)}(0) = 0, y(t_1) = 0$. It is well-known [2] that $G(t,s) < 0$ on $(0,t_1)\times(0,t_1)$ and so, $h > 0$ on $(0,t_1)$. To analyze $h = y-p$ on $[t_1,1]$, apply Rolle's theorem to y and obtain that for $j = 1,\ldots,n-2$, $y^{(j)}$ vanishes precisely once at some $t_j \in (0,1)$ and $0 < t_{j+1} < t_j < 1, j = 1,\ldots,n-1$. In particular, $t_2 < t_1$; $y''(t_1) < 0$ and so, y is concave down on $(t_1,1)$. Thus, $y(t) > p(t), t_1 < t < 1$. This completes the proof of Lemma 2. ☐

In the next theorem, we obtain the analogue of (4) for the Green's function of the BVP, $y^{(n)} = 0, 0 \le t \le 1$,(3). Erbe and Wang [5] have shown that such lower bounds are useful in certain cone theoretic fixed point theorems to construct appropriate annular like regions. In related works, ([4], [7]) analogous inequalities have been obtained for the Green's function related to the specific BVP being studied.

The proof of Theorem 3 is analogous to the proofs of Theorem 1 and Lemma 2. Recall [2] that $G(t,s)$ is C^{n-2} on $[0,1]\times[0,1]$. Moreover, for $0 < s < t, G(t,s)$ is C^{n-1} as a function of t and is, in fact, an $n-1$ order polynomial in t. Similarly, for $0 < t < s, G(t,s)$ is C^{n-1} as a function of t and is an $n-1$ order polynomial in t. Since [2] for each $s \in (0,1)$,

$$(-1)^{n-k}(\partial^k/\partial t^k)G(0,s) > 0, (\partial^{n-k}/\partial t^{n-k})G(1,s) > 0, \tag{6}$$

it follows by the boundary conditions, (3), that for each $s \in (0,1)$ $(\partial/\partial t)G(t,s)$ has precisely one root, say $t_1 = t_1(s)$.

Theorem 3. *For each $s \in (0,1)$, let $||G(\cdot,s)|| = \sup_{0\le t\le 1}|G(t,s)|$. Then for $1/4 \le t \le 3/4$,*

$$(-1)^{n-k}G(t,s) \ge ||G(\cdot,s)||/4^m, \tag{7}$$

where $m = max\{k, n-k\}$.

Proof. Let $s \in (0,1)$ be fixed throughout this argument. First, recall [2] that $(-1)^{n-k}G(t,s) > 0, 0 < t < 1$. Let t_1 denote the unique extreme point of $(-1)^{n-k}G(t,s) > 0, 0 < t < 1$. Define

$$p(t) = \begin{cases} (||G(\cdot,s)||/(t_1^k))t^k, 0 < t < t_1, \\ (||G(\cdot,s)||/(t_1-1)^{n-k})(t-1)^{n-k}, t_1 < t < 1. \end{cases} \tag{8}$$

We shall argue, as in Lemma 2, that $(-1)^{n-k}G(t,s) \ge p(t), 0 \le t \le 1$. Inequality (7) will then follow as in the proof of Theorem 1.

First, consider the case $k \in \{2,\ldots,n-2\}$. Employ the boundary conditions and Rolle's theorem to see that $(\partial^j/\partial t^j)G(t,s)$ has at least two simple roots in $(0,1), j = 2,\ldots,n-2$. Label these roots, as in the proof of Lemma 2, by $0 < t_{j1} < t_{j2} < \ldots < 1$, and by $0 < \ldots < \tau_{j2} < \tau_{j1} < 1$. Note that $t_{n-2,1} < s < t_{n-2,2}$ for if not, then by Rolle's theorem, $(\partial^{n-1}/\partial t^{n-1})G(t,s)$ vanishes on $(0,s)$ or $(s,1)$. Since, G is a polynomial of order $n-1$ on triangles $s < t, t < s$, then $G \equiv 0$ on one of these triangles. This contradicts (6).

Let $h(t) = (-1)^{n-k}G(t,s) - p(t), 0 \le t \le t_1$. Then $h(0) = \ldots = h^{k-1}(0) = h(t_1) = 0$ and $h'(t_1) < 0$. Assume for the sake of contradiction that h vanishes at $c \in (0,t_1)$. Apply Rolle's theorem and obtain that h' vanishes at $0 < c_{11} < c_{12} < t_1$ and h'' vanishes at $0 < c_{21} < c_{22} < c_{12}$. It follows by (6), precisely as in the proof of Lemma 2, that $0 < c_{21} < c_{22} < t_{21}$. It follows inductively that for each $j = 2,\ldots,k$ that $h^{(j)}$ vanishes at c_{j1} and c_{j2} where $0 < c_{j1} < c_{j2} < t_{j1}$. Thus, as in the proof of Lemma 2, there exists $c_{k+1} < t_{k+1,1}$ such that $(\partial^{k+1}/\partial t^{k+1})G(c_{k+1}, s) = 0$. But this contradicts the definition of $t_{k+1,1}$ and so, $(-1)^{n-k}G(t,s) \geq p(t), 0 \leq t \leq t_1$. A similar argument holds for $t_1 \leq t \leq 1$ and the proof of Theorem 3 is complete for the case $k \in \{2, \ldots, n-2\}$.

We shall now handle the case $k = n-1$. The case $k = 1$ will follow with a similar argument or with a change of variable. As in the proof of Lemma 2 for the case $k = n-1$ apply Rolle's theorem repeatedly and for each $j = 1, \ldots, n-2$, obtain a uniquely determined t_j such that $(\partial^j/\partial t^j)G(t_j, s) = 0$ and $0 < s < t_{n-2} < \ldots < t_1 < 1$. In particular, $t_2 < t_1$ and $-G(t,s)$ is concave down on $(t_1, 1)$. Define $p(t)$ by (8) with $k = n-1$. Let $h(t) = (-G(t,s) - p(t))$. On $[t_1, 1]$, $h(t) \geq 0$ since $-G$ is concave down.

To address the interval $[0, t_1]$, first recall [1, p.192] that four properties related to smoothness, a jump discontinuity in the $n-1$st derivative at $t = s$, the solvability of the homogeneous differential equation on the triangles, $t < s$ and $s < t$, and the solvability of the homogeneous boundary conditions uniquely determine the existence of a Green's function of a boundary value problem. We shall employ this characterization to argue that, in fact, $-h$ is a Green's function of some boundary value problem. On the interval, $[0, t_1]$, note that $h \in C^{n-2}[0, t_1]$,

$$h^{(n-1)}(s^+) - h^{(n-1)}(s^-) = (\partial^{n-1}/\partial t^{n-1})G(s^-, s) - (\partial^{n-1}/\partial t^{n-1})G(s^+, s) = -1,$$

h satisfies $h^{(n)}(t) = 0$ for $s < t$ and $t < s$, and $h(0) = \ldots = h^{(n-2)}(0) = h(t_1) = 0$. In particular, (now letting s range over $[0, t_1]$) $-h(t,s)$ is the Green's function for the BVP, $y^{(n)} = 0, 0 \leq t \leq t_1, y(0) = \ldots = y^{(n-2)}(0) = 0, y(t_1) = 0$. It is well-known [2] that $-h < 0$ on $(0, t_1) \times (0, t_1)$. This completes the proof of Theorem 3. □

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