PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 7, July 1997, Pages 2029–2032
S 0002-9939(97)03808-2

HÖLDER CONTINUITY PROPERTY OF FILLED-IN JULIA SETS IN $\mathbb{C}^n$

MARTA KOSEK

(Communicated by Eric Bedford)

ABSTRACT. It is proved that the pluricomplex Green function of the filled-in Julia set J associated with a polynomial mapping in $\mathbb{C}^n$ is Hölder continuous. This yields in particular that J preserves Markov's inequality.

1. INTRODUCTION

Different classes of Julia sets have been already studied by several authors (see [CG] and the references given there). In this paper following M.Klimek [K1] we will consider the filled-in Julia sets defined as follows:

Let $P : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial mapping satisfying

$$\liminf_{|z|\to\infty} \frac{|P(z)|}{|z|^\delta} > 0 \tag{1.1}$$

with some $\delta > 1$. We define the *filled-in Julia set* associated with P to be the set

$$J_P := \{z \in \mathbb{C}^n : \{P^j(z)\}_{j\in\mathbb{N}_0} \text{ is bounded}\},$$

where P^j denotes the j-th iteration of the polynomial mapping P.

We are interested in potential-theoretic properties of these sets. Let us recall the definition of the L-extremal function in $\mathbb{C}^n$. For a compact subset E of $\mathbb{C}^n$ the function

$$V_E := \sup\{u : u(z) \le \beta + log^+|z| \text{ for } z \in \mathbb{C}^n \text{and } u \le 0 \text{ on } E\}$$

is called the *L-extremal function corresponding to E*. A compact set E in $\mathbb{C}^n$ is said to be *L-regular* if V_E is continuous. It is well-known that V_E is a multidimensional counterpart of the classical Green function. For background material and thorough bibliography we refer the reader to [S2] or [K2].

In [K3] M.Klimek proved the following theorem, which answers the question about the continuity of the L-extremal function corresponding to a filled-in Julia set:

Theorem 1.1 ([K3, Corollary 6]). *Let $P : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial mapping satisfying (1.1) with some $\delta > 1$. Then the filled-in Julia set J_P associated with P is compact, polynomially convex and L-regular. Moreover*

$$P^{-1}(J_P) = J_P = P(J_P). \tag{1.2}$$

Received by the editors September 27, 1995 and, in revised form, January 23, 1996.
1991 *Mathematics Subject Classification.* Primary 32F05, 31C10.
This research was supported by KBN Grant No. 956/P03/95/08.

In this paper we show that it actually is Hölder regular, i.e. the following theorem holds:

Theorem 1.2. *Let $P : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial mapping satisfying (1.1) with some $\delta > 1$. Then there exist constants $C > 0$ and $\alpha \in (0,1]$ such that*

$$\text{(HCP)} \qquad V_{J_P}(z) \le C[\operatorname{dist}(z, J_P)]^{\alpha}, \qquad \text{if } \operatorname{dist}(z, J_P) \le 1.$$

In other words, Theorem 1.2 shows that the pluricomplex Green function V_{J_P} of the filled-in Julia set J_P has a Hölder continuity property.

The corresponding theorem in the case $n = 1$ is due to N.Sibony (1981). For the proof of that fact for quadratic polynomials we refer the reader to [CG, Theorem 8.3.2]. Our proof is based on a similar idea. This result was also already known for certain polynomial mappings on $\mathbb{C}^2$. J.E.Fornaess and N.Sibony studied properties of Julia sets associated with complex Hénon mappings (see [FS]). They obtained Hölder continuity of the L-extremal function corresponding to a set from this class (see [FS, Theorem 1.2]) and they mentioned that the proof easily extends to some bigger family of generalized Hénon mappings.

Recently A.Edigarian [E] found a nice generalization of Theorem 1.2; namely he proved that every L-regular compact set E in $\mathbb{C}^n$ has the Hölder continuity property, if there exist a C^{∞}-mapping F and a $\delta > 1$ such that $V_E(F(z)) \ge \delta V_E(z)$ for $z \in \mathbb{C}^n$.

We say that a set E in $\mathbb{C}^n$ preserves *Markov's inequality* if for each polynomial p

$$\text{(M)} \qquad |\operatorname{grad} p(z)| \le M(\deg p)^m \sup_E |p|, \qquad \text{for } z \in E,$$

where M and m are positive constants depending only on E. Since A. A. Markov proved (M) for $E = [-1,1] \subset \mathbb{C}$ in 1889 (see [M]), it has played an important role in the developing of the constructive theory of functions and has become the object of extensive research (see [RS] in the one-dimensional case, [PP] for $\mathbb{R}^n$ and [P] for $\mathbb{C}^n$).

It is well-known that the (HCP) property is sufficient for E to preserve Markov's inequality ([S1, Remark after Lemma 1]). Hence by Theorem 1.2 we immediately get

Corollary 1.3. *Let $P : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial mapping satisfying (1.1) with some $\delta > 1$. Then the filled-in Julia set J_P associated with P satisfies Markov's inequality.*

2. Preliminaries

From now on, P denotes a polynomial mapping of degree d satisfying (1.1) with some $\delta > 1$.

Most of the material in this paper relies on the following transformation rule for the L-extremal functions (see [K2, Theorem 5.3.1]):

Theorem 2.1 (M. Klimek). *Let E be a subset of $\mathbb{C}^n$. Then*

$$\text{(2.1)} \qquad \frac{1}{d}(V_E \circ P) \le V_{P^{-1}(E)} \le \frac{1}{\delta}(V_E \circ P).$$

Let us mention the following trivial consequences of (2.1):

$$\text{(2.2)} \qquad \frac{1}{d^j}(V_E \circ P^j) \le V_{P^{-j}(E)} \le \frac{1}{\delta^j}(V_E \circ P^j) \qquad \text{for } j \in \mathbb{N}_0,$$

and of (1.2):

$$P^{-j}(J_P) = J_P = P^j(J_P) \qquad \text{for } j \in \mathbb{N}_0. \tag{2.3}$$

Here $P^{-j}(E)$ denotes $(P^j)^{-1}(E)$.

The following consequence of (1.1) will be needed in Section 3.

Lemma 2.2 ([K1, Corollary 4])**.**

$$J_P = \mathbb{C}^n \setminus \{z \in \mathbb{C}^n : \lim_{j\to\infty} |P^j(z)| = \infty\}.$$

3. Proof of the main result

Proof of Theorem 1.2. In view of Theorem 1.1 J_P is compact, thus we can fix such an $r > 1$ that $J_P \subset B_r := \{z \in \mathbb{C}^n : |z| \le r\}$. Then by (2.3) we get

$$P^j(J_P) \subset B_r, \qquad \text{if } j \in \mathbb{N}_0. \tag{3.1}$$

We define $A \ge \delta$ and $\alpha \in (0,1]$ as follows:

$$A := \max\{\delta, \max_{|z|\le 2r} |P'(z)|\} \quad \text{and} \quad \alpha := \log_A \delta.$$

If z is in J_P, we have $V_{J_P}(z) = \operatorname{dist}(z, J_P) = 0$, and inequality (HCP) is obvious.

Now let z be in $\mathbb{C}^n \setminus J_P$ with $|z| < 2r$. Since J_P is compact, there exists a $z_0 \in J_P$ such that $|z - z_0| = \operatorname{dist}(z, J_P) > 0$. If $w \in [z_0, z] := \{(1-t)z_0 + tz : t \in [0,1]\}$ and $w \ne z_0$ then w does not belong to J_P.

We pick $k = k(z)$ in $\mathbb{N}$ to satisfy the following conditions:

$$|P^j(w)| \le 2r \qquad \text{for each } w \in [z_0, z] \text{ and } 0 \le j \le k-1, \tag{3.2}$$

and

$$\text{there exists a } z_1 \in [z_0, z] \quad \text{with} \quad |P^k(z_1)| > 2r. \tag{3.3}$$

Note that we can find such a point according to Lemma 2.2.

Let $w \in [z_0, z]$ be given. By the definition of A we conclude that

$$\begin{aligned} |(P^k)'(w)| &= |P'(P^{k-1}(w)) \circ \ldots \circ P'(P(w)) \circ P'(w)| \\ &\le |P'(P^{k-1}(w))| \ldots |P'(w)| \le A^k \end{aligned}$$

since $|P^{k-1}(w)|, \ldots, |P(z)|, |z| \le 2r$ from (3.2). Hence integration along $[z_0, z_1]$ gives $|P^k(z_0) - P^k(z_1)| \le A^k |z_0 - z_1|$. Combining this with (3.1) and (3.3) we obtain $2r < |P^k(z_1)| \le A^k|z_0 - z_1| + |P^k(z_0)| \le A^k \operatorname{dist}(z, J_P) + r$, since $|z_0 - z_1| \le |z_0 - z| = \operatorname{dist}(z, J_P)$. This gives $A^k \operatorname{dist}(z, J_P) \ge r > 1$ by the choice of r. Consequently by the definition of α we get

$$[\operatorname{dist}(z, J_P)]^\alpha > (A^{-k})^\alpha = \delta^{-k}. \tag{3.4}$$

In view of (2.3) and (2.2) we obtain

$$V_{J_P}(z) = V_{P^{-k}(J_P)}(z) \le \delta^{-k} V_{J_P}(P^k(z)). \tag{3.5}$$

We define now $C := \sup\{V_{J_P}(P(w)) : |w| \le 2r\}$. Note that C is a finite number, because the polynomial mapping P is continuous and by Theorem 1.1 the L-extremal function V_{J_P} is also continuous. From (3.2) we get $|P^{k-1}(z)| \le 2r$, whence $|V_{J_P}(P^k(z))| \le C$. Thus in view of (3.4) and (3.5)

$$V_{J_P}(z) \le C[\operatorname{dist}(z, J_P)]^\alpha.$$

Moreover, if $\operatorname{dist}(z, J_P) \le 1$, then $|z| \le \operatorname{dist}(z, J_P) + |z_0| \le 1 + r \le 2r$ and this completes the proof of our assertion.

Acknowledgments

The author wishes to express her thanks to Professor Wiesław Pleśniak for his advice and help.

References

[CG] L. Carleson, T. W. Gamelin, Complex Dynamics, *(Springer, New York, 1993).* MR **94h:**30033

[E] A.Edigarian, Remarks on Hölder Continuity Property of L-regular sets, *(personal communication).*

[FS] J. E. Fornaess, N. Sibony, Complex Hénon mappings in $\mathbb{C}^2$ and Fatou-Bieberbach domains, *Duke Math. Journal* **65** (1992), 345–380. MR **93d:**32040

[K1] M. Klimek, Pluricomplex Green functions for filled-in Julia sets in $\mathbb{C}^n$, *(preprint, University College Dublin, 1991).*

[K2] M. Klimek, Pluripotential Theory, *(Oxford University Press, 1993).*

[K3] M. Klimek, Metrics associated with extremal plurisubharmonic functions, *Proc. Amer. Math. Soc.* **123(9)** (1995), 2763–2770. MR **95k:**32015

[M] A. A. Markov, On a problem posed by D. I. Mendeleev, *Izv. Akad. Nauk. St-Petersbourg* **62** (1889), 1–24 (in Russian).

[PP] W. Pawłucki, W. Pleśniak, Markov inequality and C^∞ functions on sets with polynomial cusps, *Math. Ann.* **275(3)** (1986), 467–480. MR **87k:**32031

[P] W. Pleśniak, Compact subsets of $\mathbb{C}^n$ preserving Markov's inequality, *Mat. Vesnik* **40** (1988), 295–300. MR **91e:**32011

[RS] Q. I. Rahman, G. Schmeisser, Les inégalités de Markoff et de Bernstein, *(Les Presses de l'Université de Montréal, 1983).* MR **85f:**41009

[S1] J. Siciak, Degree of convergence of some sequences in the conformal mapping theory, *Colloq. Math.* **16** (1967), 49–59. MR **35:**4385

[S2] J. Siciak, Extremal plurisubharmonic functions in $\mathbb{C}^n$, *Ann. Polon. Math.* **39** (1981), 175–211. MR **83e:**32018

Institute of Mathematics, Jagiellonian University, ul.Reymonta 4, 30-059 Kraków, Poland

E-mail address: kosek@im.uj.edu.pl