THE UNIVERSAL NILPOTENT GROUP COMPACTIFICATION
OF A SEMIGROUP

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Abstract. The purpose of this paper is to introduce an algebra of functions
on a semitopological semigroup and to study these functions from the point of
view of universal semigroup compactification. We show that the corresponding
semigroup compactification of this algebra is universal with respect to the
property of being a nilpotent group.

The general approach to the theory of semigroup compactification is based on
the Gelfand-Naimark theory of $C^*$-algebras of functions. There are many papers
which deal with the characterization of certain universal semigroup compactifica-
tions in terms of function algebras. Seminal work in this context was done by
K. de Leeuw and I. Glicksberg [3]. The universal group compactification is given
by Junghenn [6], in terms of some types of distal functions. This paper deals with
the construction of a function algebra on a semitopological semigroup, used to char-
acterize the universal nilpotent group compactification of it, and investigating some
of its properties.

First we recall some preliminaries. Throughout this paper, $S$ shall be a semi-
topological semigroup, unless otherwise mentioned. For notation and terminology
we shall follow Berglund et al. [2] as far as possible. Thus a semigroup compact-
ification of $S$ is a pair $(\psi, X)$, where $X$ is a compact, Hausdorff, right topological
semigroup and $\psi: S \to X$ is a continuous homomorphism with dense image such
that for each $s \in S$, the mapping $x \mapsto \psi(s)x: X \to X$ is continuous.

The $C^*$-algebra of all continuous bounded complex-valued functions on a topo-
logical space $Y$ is denoted by $C(Y)$. For $C(S)$ left and right translations $L_s$ and
$R_t$ are defined for all $s, t \in S$ and $f \in C(S)$ by $(L_s f)(t) = f(st) = (R_t f)(s)$. A
left translation invariant $C^*$-subalgebra $F$ of $C(S)$ (i.e. $L_s f \in F$ for all $s \in S$
and $f \in F$) containing the constant functions is called $m$-admissible if the function
$s \mapsto (T_\mu f)(s) = \mu(L_s f)$ is in $F$ for all $f \in F$ and $\mu \in S^F$ (= the spectrum of $F$); then
the product of $\mu, \nu \in S^F$ can be defined by $\mu \nu = \mu \circ T_\nu$ and the Gelfand topology
on $S^F$ makes $(\epsilon, S^F)$ a semigroup compactification (called the $F$-compactification)
of $S$, where $\epsilon: S \to S^F$ is the evaluation mapping. The reader is referred to sec-
tions 3.1 and 3.3 of [2] for the one-to-one correspondence between compactifications
of $S$ and $m$-admissible subalgebras of $C(S)$, and also for a discussion of universal
$P$-compactifications, whose existence for a wide variety of properties $P$ is given in
terms of subdirect products.
Still following [2] and also [6], the \( \mathfrak{m} \)-admissible algebras of weakly almost periodic, strongly almost periodic, minimal distal and strongly distal functions on \( S \) are denoted by \( \mathcal{WAP}, \mathcal{SAP}, \mathcal{MD} \) and \( \mathcal{SD} \), respectively. We write \( \mathcal{GP} \) for \( \mathcal{MD} \cap \mathcal{SD} \); then the \( \mathcal{GP} \)-compactification of \( S \) is universal with respect to the property of being a group, [6, theorem 3.4]. Using Ellis’s (joint continuity) theorem [4], one can show that \( \mathcal{GP} \cap \mathcal{WAP} = \mathcal{SAP} \). The left invariant probability measure on the right topological group \( S^{\mathcal{GP}} \) (see [7], or [2, Appendix C, C5]) gives a left invariant mean on \( \mathcal{GP} \). In contrast to the situation for \( \mathcal{SAP} \), it follows from a remark at the end of [1] that, in general, the left invariant mean on \( \mathcal{GP} \) is not necessarily unique. The existing examples support the conjecture that the left invariant mean on \( \mathcal{GP} \) is unique if and only if \( \mathcal{GP} \) is left introverted (i.e. \( T_\mu f \in \mathcal{GP} \) for all \( f \in \mathcal{GP} \) and all \( \mu \) in the dual of \( \mathcal{GP} \)). Notice that the function \( f(n) = e^{in\zeta} \) on \( (\mathbf{Z}, +) \), which is in \( \mathcal{GP} \), has the indicator function of \( \{0\} \) in the pointwise closed convex hull of its set of translates; hence \( \mathcal{GP}(\mathbf{Z}, +) \) is not left introverted (see also the example at the end of [7]).

In what follows, \( n \) will denote a fixed arbitrary positive integer and \([, \ldots, ,]\) is the commutator of weight \( n + 1 \), as defined in [8], in the group \( \{T_\mu : \mu \in S^{\mathcal{GP}}\} \); however in the proof of the next theorem it will be in the group \( S^{\mathcal{GP}} \). Now, we examine the properties of those functions \( f \in \mathcal{GP} \) for which

\[
\lim_{\alpha_1} R_{s_{\alpha_1}}, \lim_{\alpha_2} R_{s_{\alpha_2}}, \ldots, \lim_{\alpha_{n+1}} R_{s_{s_{\alpha_{n+1}}}}(f) = f
\]

for arbitrary nets \( \{s_{\alpha_k}\}, 1 \leq k \leq n + 1, \) in \( S \), where the involved limits are pointwise. We write \( N_nG(S) \) for the set of all such functions (but we usually suppress the letter \( S \)). Trivially a function \( f \in \mathcal{GP} \) is in \( N_nG \) iff \( [T_{\mu_1}, T_{\mu_2}, \ldots, T_{\mu_{n+1}}](f) = f \) (equivalently, \( [T_{\mu_1}, T_{\mu_2}, \ldots, T_{\mu_{n+1}}](T_\mu f) = T_\mu f \)) for arbitrary elements \( \mu_1, \mu_2, \ldots, \mu_{n+1}, \mu \) in \( S^{\mathcal{GP}} \); the latter is equivalent to the fact that the enveloping semigroup of the flow \((S, X_f)\) (which is obtained from the natural action \((s, g) \mapsto R_g s\)) is nilpotent. The following theorem states the main property of \( N_nG \).

**Theorem.** Let \( S \) be a semitopological semigroup. Then \( N_nG(S) \) is an \( \mathfrak{m} \)-admissible subalgebra of \( C(S) \), and \( N_nG \)-compactification of \( S \) is universal with respect to the property of being a nilpotent group of class \( n \).

**Proof.** The \( \mathfrak{m} \)-admissibility of \( N_nG \) is an immediate consequence of the definition and the discussions preceding the theorem. Let \( \mu_1, \mu_2, \ldots, \mu_{n+1} \) and \( \mu \) be arbitrary elements of \( S^{\mathcal{GP}} \). For each \( f \in N_nG \), \( [\mu_1, \mu_2, \ldots, \mu_{n+1}](f) = e(f) \), in which \( e \) is the identity element of \( S^{\mathcal{GP}} \); thus \( S^{N_nG} \) is a nilpotent group of class \( n \). To see that \((e, S^{N_nG})\) is universal with respect to this property, it is enough to show that, for any other such compactification \((\psi, X)\) of \( S \), \( \psi^*(C(S)) \subseteq N_nG \), where \( \psi^* \) is the dual mapping of \( \psi \). If \( g \in C(X) \) then \( \psi^*(g) \in \mathcal{GP} \), and

\[
(\mu|\mu_1, \mu_2, \ldots, \mu_{n+1})(\psi^*(g)) = g(\pi(\mu|\mu_1, \mu_2, \ldots, \mu_{n+1})) = g(\pi(\mu)) = \mu(\psi^*(g)),
\]

where \( \pi : (e, S^{\mathcal{GP}}) \to (\psi, X) \) is the canonical homomorphism whose existence is guaranteed by the universal property of \((e, S^{\mathcal{GP}})\); hence \( \psi^*(g) \in N_nG \), as claimed.

**Remarks.** 1. As mentioned for \( \mathcal{GP} \), by the Ellis’s theorem [4], we have

\[(* ) \quad N_nG \cap \mathcal{WAP} = N_nG \cap \mathcal{SAP}, \]
and it follows from the fact that $S^{SAP}$ is a topological group, that each side of (*) (and so $N_nG$, when $S$ is compact) consists precisely of those $f \in SAP$ such that for arbitrary elements $s_1, s_2, \ldots, s_{n+1}$ of $S, [R_{s_1}, R_{s_2}, \ldots, R_{s_{n+1}}](f) = f$; hence for a nilpotent group (of class $n$), each side of (*) is equal to $SAP$, and so for a compact nilpotent group $S$, $N_nG(S) = C(S)$.

2. (The case $n = 1$). By the theorem, in this case we obtain the universal abelian group compactification. Trivially $N_1G \subseteq SAP$; hence by the above characterization of (*) we have

\[ (** ) \quad N_1G = \{ f \in SAP : f(stu) = f(sut) \text{ for all } s, t, u \text{ in } S \}; \]

so for abelian semigroups, $N_1G = SAP$, and this can fail to hold if the abelian hypothesis is dropped, as the following examples demonstrate:

For $m \geq 3$, let $D_{2m} = \langle a, b \mid a^m = b^2 = (ab)^2 = 1 \rangle$ be the discrete dihedral group of degree $m$, [8]. A direct computation, using (**) shows that

\[ N_1G(D_6) = \{ f \in C(D_6) : f(1) = f(a) = f(a^2) \text{ and } f(b) = f(ab) = f(a^2b) \}, \]

and

\[ N_1G(D_8) = \{ f \in C(D_8) : f(1) = f(a^2), f(a) = f(a^3), f(b) = f(a^2b), \text{ and } f(ab) = f(a^3b) \}; \]

but $SAP(D_{2m}) = C(D_{2m})$.

Some other straightforward facts about $N_1G$ are as follows:

(i) Using (**), a function $f \in SAP$ is in $N_1G$ if and only if $f$ (each finite product of elements of $S = f$(each re-ordering of it);

(ii) $N_1G$ has a unique invariant mean;

(iii) $N_1G$ is the closed linear span of the continuous characters of $S$ (using the Peter-Weyl theorem [5, 22.17], for $S^{N_1G}$); and

(iv) The $N_1G$-compactification of a semidirect product of $S$ and a semitopological semigroup $T$ (in the sense of [2, sec. 5.2]) is isomorphic to a semidirect product of an $F$-compactification of $S$ and the $N_1G$-compactification of $T$, for which $F \subseteq N_1G(S)$ (with the equality holding in the direct product case).

3. Trivially $N_nG \subseteq N_{n+1}G$, and the inclusions can be proper. For example, $N_1G(D_8)$ is properly contained in $N_2G(D_8) = C(D_8)$ (recall that $D_8$ is nilpotent of class 2).

4. $N_nG$ is defined by right translates, analogously we can define a space $LN_nG$ in terms of left translates (used for constructing the universal left topological nilpotent group compactification); a natural question that arises is whether the equality $N_nG = LN_nG$ holds. Of course, for $n = 1$ this equality holds.

Some questions concerning the properties of $N_nG$, such as left introversion, right amenability, the equality $N_nG = LN_nG$ and the inclusion $N_nG \subseteq WAP$, for $n \geq 2$ are therefore left undecided in this paper; for example we don’t know anything about $N_2G \subseteq WAP$ in the familiar case of the discrete group of integers. It would be desirable to answer these, at least in the locally compact group setting.

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