

## ON ZSIGMONDY PRIMES

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ABSTRACT. We present simple proofs of Walter Feit's results on large Zsigmondy primes.

We present simple proofs of known results related to Zsigmondy primes. We recall that if  $a, n$  are integers greater than 1, then a prime  $p$  is called a *Zsigmondy prime* for  $\langle a, n \rangle$  if  $p \nmid a$  and the order of  $a \pmod{p}$  equals  $n$  (see [2], [4, §5], and Theorem 3 below). If  $p$  is a Zsigmondy prime for  $\langle a, n \rangle$ , then  $n \mid p - 1$ ; thus  $p \geq n + 1$ . A Zsigmondy prime  $p$  for  $\langle a, n \rangle$  is called a *large Zsigmondy prime* for  $\langle a, n \rangle$  if  $p > n + 1$  or  $p^2$  divides  $a^n - 1$  (equivalently, a prime  $p$  is a large Zsigmondy prime for  $\langle a, n \rangle$  iff  $p$  is a Zsigmondy prime for  $\langle a, n \rangle$  satisfying  $|a^n - 1|_p > n + 1$ . See [2]). Zsigmondy primes are used in finite group theory (see, e.g., [1]). For applications of large Zsigmondy primes to finite groups see [3] and [4].

The main results that we reprove here are Theorem 3 (Zsigmondy's Theorem) and Theorem 10 (due to Walter Feit).

We now recall some basic properties of cyclotomic polynomials, which we will use below (see, e.g., [6, Preliminaries, sec. 1]). For  $n \geq 1$  the cyclotomic polynomial  $\Phi_n(X)$  is defined as

$$\Phi_n(X) = \prod_{i=1}^{\varphi(n)} (X - \epsilon_i),$$

where  $\epsilon_1, \dots, \epsilon_{\varphi(n)}$  are the primitive roots of unity of order  $n$  and  $\varphi(n)$  is Euler's totient function. If  $n > 1$ , then  $\Phi_n(a) > 0$  for all real  $a$  since  $\Phi_2(X) = X + 1$ , and for  $n > 2$ ,  $\Phi_n(X)$  is a monic polynomial over the integers with no real roots. Moreover, if  $n > 1$  and  $a > 1$  are integers, then  $\Phi_n(a) > 1$  since  $\Phi_n(a) = \prod_{i=1}^{\varphi(n)} |a - \epsilon_i| > (a - 1)^{\varphi(n)} \geq 1$ .

Except for such basic facts, the paper is self-contained. Some of the proofs included here are the usual ones.

**Lemma 1.** *Let  $a > 1$  and  $n = q^i r$  be integers, where  $q$  is a prime,  $i \geq 1$  and  $r$  is a positive integer not divisible by  $q$ . Let  $b = a^{q^{i-1}}$ . Then*

$$\Phi_n(a) > (b^{q-2}(b-1))^{\varphi(r)}.$$

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*Proof.* We have

$$\Phi_n(a) = \frac{\Phi_r(a^{q^i})}{\Phi_r(a^{q^{i-1}})} = \frac{\Phi_r(b^q)}{\Phi_r(b)} = \frac{\prod_{i=1}^{\varphi(r)}(b^q - \epsilon_i)}{\prod_{i=1}^{\varphi(r)}(b - \epsilon_i)} > \left(\frac{b^q - 1}{b + 1}\right)^{\varphi(r)},$$

where the  $\epsilon_i$ 's are roots of unity. Since  $b^q - 1 \geq b^{q-2}(b^2 - 1)$  the lemma follows.  $\square$

*Remark.* In the notation of the previous lemma we have the inequality

$$\Phi_n(a) \geq \left(\frac{b^q + 1}{b + 1}\right)^{\varphi(r)}.$$

If  $q$  is a Zsigmondy prime for the pair  $\langle a, n \rangle$ , then  $q$  divides  $a^n - 1 = \prod_{d|n} \Phi_d(a)$ ; thus  $q$  divides  $\Phi_n(a)$ . The next proposition characterizes the Zsigmondy primes among the prime factors of  $\Phi_n(a)$ .

**Proposition 2** (cf. [5, Satz 1]). *Let  $a > 1$  and  $n > 1$  be integers. Let  $q$  be a prime factor of  $\Phi_n(a)$ . Then  $q$  is a non Zsigmondy prime for  $\langle a, n \rangle$  iff  $q$  divides  $n$ . In this case  $q$  is the largest prime factor of  $n$ , and  $n = q^i r$ , where  $r$  is a positive integer dividing  $q - 1$ ; moreover,  $q^2$  does not divide  $\Phi_n(a)$  unless  $q = n = 2$ .*

*Thus, if there are no Zsigmondy primes for  $\langle a, n \rangle$ , then  $\Phi_n(a)$  is a power of  $q$ ; if also  $n > 2$  then  $\Phi_n(a) = q$ .*

*Proof.* If  $q | n$ , then  $a^{\frac{n}{q}} \equiv a^n \equiv 1 \pmod{q}$ ; thus  $q$  is not a Zsigmondy prime for  $\langle a, n \rangle$  (actually, we have already proved this above).

Conversely, assume that  $q$  is not a Zsigmondy prime for  $\langle a, n \rangle$ . Then there exists a prime factor  $p$  of  $n$  such that  $a^{n/p} \equiv 1 \pmod{q}$ . Since  $\Phi_n(X)$  divides  $\frac{X^n - 1}{X^{n/p} - 1}$  we obtain, for  $c = a^{n/p}$ , that  $q$  divides  $\frac{c^p - 1}{c - 1} = \sum_{i=0}^{p-1} c^i \equiv p \pmod{q}$ . Thus  $q = p$  divides  $n$ . Moreover, since  $a^{n/p} \not\equiv 1 \pmod{q}$  for any prime factor  $p \neq q$  of  $n$ , it follows that the order of  $a \pmod{q}$  is of the form  $n/q^i$  for some  $i \geq 1$ , and so  $r := n/q^i$  divides  $q - 1$ . Thus  $q$  is the largest prime factor of  $n$ .

Let  $c = a^{n/q}$ .

If  $q > 2$  let  $d = c - 1$ ; thus  $q | d$ . We have

$$\frac{a^n - 1}{a^{\frac{n}{q}} - 1} = \frac{c^q - 1}{c - 1} = \frac{(1 + d)^q - 1}{d} = q + \sum_{i=2}^{q-1} \binom{q}{i} d^{i-1} \equiv q \pmod{q^2}.$$

Thus  $q^2 \nmid \Phi_n(a)$ .

If  $q = 2$  and  $n > 2$ , then  $n$  is a power of 2 and  $n \geq 4$ . We have  $\frac{c^2 - 1}{c - 1} = c + 1$ . Since  $a$  is odd and  $\frac{n}{2}$  is even, we obtain  $c = a^{n/2} \equiv 1 \pmod{4}$ ; thus  $c + 1 \equiv 2 \pmod{4}$ . Hence  $4 \nmid \Phi_n(a)$  as claimed.  $\square$

**Theorem 3** (Zsigmondy's Theorem). *Let  $a$  and  $n$  be integers greater than 1. There exists a prime divisor  $q$  of  $a^n - 1$  such that  $q$  does not divide  $a^j - 1$  for all  $j$ ,  $0 < j < n$ , except exactly in the following cases:*

- (1)  $n = 2$ ,  $a = 2^s - 1$ , where  $s \geq 2$ .
- (2)  $n = 6$ ,  $a = 2$ .

*Proof.* It is easy to verify that if one of the conditions (1) or (2) holds, then there is no  $q$  satisfying our requirements.

Assume that there is no prime  $q$  such that the order of  $a$  modulo  $q$  is  $n$ .

If  $n = 2$ , then by Proposition 2,  $\Phi_n(a) = a + 1 = 2^s$  for some integer  $s$ , and case (1) holds.

Assume that  $n > 2$ . By Proposition 2,  $\Phi_n(a) = q$ , where  $q$  is the largest prime factor of  $n$ .

If  $q = 2$ , then  $n = 2^s$  and  $\Phi_n(a) = a^{2^{s-1}} + 1 > 2$ , a contradiction.

Thus  $q \geq 3$ . Let  $n = q^i r$ , where  $r$  is an integer not divisible by  $q$ . Set  $b = a^{q^{i-1}}$ . By Lemma 1,  $b^{q-2} < q$ . Hence  $q = 3$  and  $b = 2, a = 2$  (indeed,  $2^{q-2} < q$  implies  $q = 3$ . Thus  $b < 3$ , that is,  $b = 2$ ). Since 7 is a Zsigmondy prime for  $\langle 2, 3 \rangle$ , we see that  $n = qr > 3$ . Since  $n$  divides  $q(q - 1) = 6$ , we conclude that  $n = 6$ .  $\square$

For a proof of Zsigmondy's Theorem in a stronger form see [6, (P1.7)]. For previous proofs see the references in [6], and especially [5]. For the present proof I have used a proof of Zsigmondy's Theorem based on [1] due to Yakov Berkovich and Gregory Freiman.

We now turn to the proof of Feit's results on Zsigmondy primes.

From Proposition 2 we obtain:

**Corollary 4.** *Let  $a, n$  be integers greater than 1. Let  $q$  be the largest prime factor of  $n$ . Assume that there are Zsigmondy primes for  $\langle a, n \rangle$ , but no large Zsigmondy primes. Then  $n + 1$  is the unique Zsigmondy prime for  $\langle a, n \rangle$ . If  $n > 2$ , then either  $\Phi_n(a) = n + 1$  or  $\Phi_n(a) = q(n + 1)$ .*

Corollary 4 shows that for  $a > 1$  and  $n > 1$ , the largest prime factor of  $a^n - 1$  is  $\geq n + 1$ . For a short review of far reaching generalizations of this remark see [7, Chapter 2, Section II.G]. Especially see [8-11].

**Lemma 5.** *For  $n \geq 1, n \neq 6$  we have  $2^{\varphi(n)} \geq n$ ; thus for all  $n \geq 1$  we have  $2^{\varphi(n)} \geq \frac{2}{3}n$ .*

*Proof.* Let  $n > 1$ . First let  $n$  be odd. Then  $2^{\varphi(n)} \equiv 1 \pmod n$ . Thus  $2^{\varphi(n)} = kn + 1$  for some  $k \geq 1$ . It follows that  $2^{\varphi(n)} > n$ . If  $k = 1$ , let  $s = \varphi(n)$ . The  $s$  numbers  $2^i$  for  $0 \leq i \leq s - 1$  are coprime with  $n = \varphi(2^s - 1)$  and are between 1 and  $n$ . Since  $1 \leq 2^s - 2 \leq n$  and  $(2^s - 2, n) = 1$ , we have  $2^s - 2 = 2^i$  for some  $0 \leq i \leq s - 1$ . Hence  $2^{s-1} - 1 = 2^{i-1}$ ; so,  $2^{s-1} - 1 = 1$  and  $n = 3$ . It follows that  $2^{\varphi(n)} > 3n$  holds for odd  $n > 3$ .

For an even  $n > 1$ , let  $n = 2^i m$ , where  $m$  is odd. If  $m = 1$ , then  $2^{\varphi(n)} = 2^{2^i - 1} \geq 2^i = n$ . If  $i = 1$  and  $m > 3$ , then  $2^{\varphi(n)} = 2^{\varphi(m)} > 2m = n$ . If  $i > 1$  and  $m \geq 3$ , then  $2^{\varphi(n)} = 2^{2^{i-1}\varphi(m)} \geq 2^{i\varphi(m)} \geq 2^{i+\varphi(m)} > 2^i m = n$  since a product of two integers greater than 1 is greater than or equal to their sum.  $\square$

As seen from the proof of the previous lemma, we have equality  $2^{\varphi(n)} = n$  iff  $n = 2, 4$ .

**Theorem 6** [2, Theorem A]. *If  $a$  and  $n$  are integers greater than 1, then there exists a large Zsigmondy prime for  $\langle a, n \rangle$  except exactly in the following cases:*

- (1)  $n = 2$  and  $a = 2^s 3^t - 1$  for some natural  $s \geq 0$  and  $t = 0, 1$  with  $s \geq 2$  if  $t = 0$ .
- (2)  $a = 2$  and  $n = 4, 6, 10, 12$  or 18.
- (3)  $a = 3$  and  $n = 4$  or 6.
- (4)  $\langle a, n \rangle = \langle 5, 6 \rangle$ .

*Proof.* It is easy to show that in each case there are no large Zsigmondy primes for  $\langle a, n \rangle$ .

For the converse, by Theorem 3 we may assume that there are Zsigmondy primes for  $\langle a, n \rangle$ , but no large Zsigmondy primes. First let  $n = 2$ . As in [2], since the greatest common divisor of  $a - 1$  and  $a + 1$  is at most 2 it follows that any odd prime factor of  $a + 1$  is a Zsigmondy prime and so it equals  $n + 1 = 3$ . Hence case (1) holds.

Now assume that  $n > 2$ . By Corollary 4,  $\Phi_n(a)$  equals either  $n + 1$  or  $q(n + 1)$ , and  $n + 1$  is prime. Thus  $n$  is even.

◦ Let  $n$  be a power of 2:  $n = 2^i, i \geq 2$ .

If  $\Phi_n(a) = n + 1$  then

$$a^{2^{i-1}} = 2^i.$$

Since  $2^i \leq a^i \leq a^{2^{i-1}} = 2^i$ , we obtain  $a = 2, 2^{i-1} = i$ , and  $i = 2$ , that is  $n = 4$ .

If  $\Phi_n(a) = 2(n + 1)$ , then  $a^{2^{i-1}} + 1 = 2(2^i + 1)$ ; thus  $a^{2^{i-1}} = 2^{i+1} + 1$ , and  $a \geq 3$ . If  $i = 2$ , then  $a = 3$  and  $n = 4$ . If  $i > 2$ , then  $a^{2^{i-1}} \geq 3^{i+1} > 2^{i+1} + 1$ , a contradiction.

◦ Assume that  $n$  is not a power of 2. Let  $q$  be the largest prime factor of  $n$  and let  $n = q^i r$ , where  $r$  is an integer not divisible by  $q$ . Thus  $r$  is even. Set  $b = a^{q^{i-1}}$ .

• First assume that

$$\Phi_n(a) = q(n + 1).$$

By Lemma 1

$$(b^{q-2}(b - 1))^{\varphi(r)} < \Phi_n(a) = q(rq^i + 1) \leq q(rq(b - 1) + 1).$$

Divide by  $b - 1$  to obtain

$$(1) \quad b^{(q-2)\varphi(r)} < q(rq + 1).$$

By Proposition 2, we have  $q(rq + 1) \leq q((q - 1)q + 1) < q^3$ , and thus

$$(2) \quad b^{(q-2)\varphi(r)} < q^3.$$

If  $q = 3$ , then  $r = 2$  by Proposition 2, and  $b < 21$  by (1); thus  $i = 1, 2$ . Hence  $n = 6, 18$ . Since  $q(n + 1) = \Phi_n(a) = b^2 - b + 1$ , we obtain for  $n = 6$  that  $b = a = 5$  and for  $n = 18$  that  $b = 8, a = 2$ .

Now assume that  $q > 3$ .

If  $r > 2$ , then  $\varphi(r) \geq 2$ . By (2)

$$(3) \quad b^{2(q-2)} < q^3.$$

By induction we easily obtain that  $2^{2(q-2)} > q^3$  for any integer  $q \geq 6$ . Thus  $q = 5$ . Since the inequality (3) does not hold for  $b = 3$  and  $q = 5$ , it follows that  $b = 2, i = 1, n = 5r$ . Since  $r \mid q - 1$ , we obtain that  $r = 4$ . Thus  $n + 1 = 21$  is not prime, a contradiction.

Let  $r = 2$ .

We have

$$\Phi_n(a) = \Phi_{2q^i}(a) = \frac{b^q + 1}{b + 1}.$$

Hence

$$b + 1 \equiv b^q + 1 \equiv 0 \pmod{q}.$$

Thus  $b \geq q - 1$ . By (1)

$$(q - 1)^{q-2} < q(2q + 1).$$

But for  $q \geq 5$  we have  $(q - 1)^{q-2} \geq (q - 1)^3 > q(2q + 1)$ , since  $q^3 - 5q^2 + 2q - 1 > 0$ , a contradiction.

- Assume that  $\Phi_n(a) = n + 1$ .

Similarly to (1) we have

$$(4) \quad (b^{q-2})^{\varphi(r)} < rq + 1.$$

Divide by  $2^{\varphi(r)}$  and use Lemma 5 to obtain

$$(5) \quad \left(\frac{b^{q-2}}{2}\right)^{\varphi(r)} < \frac{3}{2}q + 1.$$

Hence

$$(6) \quad b^{q-2} < 3q + 2.$$

For any integer  $q \geq 7$  we show by induction that  $2^{q-2} \geq 3q + 2$ . Thus  $q = 3, 5$ .

Let  $q = 3$ .

Let  $n = 2^j 3^i$ . Since  $r \neq 6$ , by Lemma 5 we have  $b^{q-2} < 2q + 2$  (cf. (6)), that is,  $b < 8$ . Hence  $i = 1$ . We have

$$\Phi_n(a) = \frac{\Phi_{2^j}(a^3)}{\Phi_{2^j}(a)} = \frac{c^3 + 1}{c + 1},$$

where  $c = a^{2^{j-1}}$ . Thus  $c^2 - c + 1 = n + 1$ , that is,  $c(c - 1) = 2^j \cdot 3$ . So  $c = 3$  or  $c - 1 = 3$ . If  $c = 3$  then  $a = 3, n = 6$ . If  $c - 1 = 3$  then  $4 = c = 2^j, j = 2, a = 2, n = 12$ .

Let  $q = 5$ .

By (5) we obtain  $2^{2\varphi(r)} < \frac{3}{2} \cdot 5 + 1$ . Thus  $4^{\varphi(r)} < 9$ , which implies  $\varphi(r) = 1, r = 2$ . By (6), we have  $b^3 < 17$ , and so  $b = a = 2, i = 1$ , and  $n = 10$ . □

In the sequel we prove [2, Theorem B], but first some auxiliary results.

**Lemma 7.** *Let  $a > 1$  and  $n > 2$  be integers. Then*

$$\Phi_n(a) > a^{\varphi(n)/2}.$$

*Proof.* If  $n$  is a power of 2,  $n = 2^m$ , then

$$\Phi_n(a) = a^{2^{m-1}} + 1 > a^{2^{m-1}} = a^{\varphi(n)}.$$

Otherwise, let  $q$  be a prime odd factor of  $n$ . Set  $n = q^i r$ , where  $r$  is an integer not divisible by  $q$ . Set  $b = a^{q^{i-1}}$ . By Lemma 1, we have

$$\Phi_n(a) > (b^{q-2})^{\varphi(r)} \geq \left(b^{\frac{q-1}{2}}\right)^{\varphi(r)} = a^{\varphi(n)/2}.$$

□

**Lemma 8** [2, Lemma 2.5]. *For  $n \geq 1$  we have  $\varphi(n) \geq \sqrt{n}/2$ .*

*Proof.* If  $p$  is an odd prime and  $m \geq 1$  then  $\varphi(p^m) = (p-1)p^{m-1} \geq \sqrt{p^m}$ . Also  $\varphi(2^m) = 2^{m-1} \geq \sqrt{2^m}/2$ . Since  $\varphi$  is a multiplicative function ( $\varphi(mn) = \varphi(m)\varphi(n)$  for  $m, n$  coprime) the lemma follows.  $\square$

As a result of Lemmas 7 and 8 we obtain:

**Corollary 9.** *For any integers  $a > 1, n > 2$  we have*

$$\Phi_n(a) > a^{\frac{\sqrt{n}}{4}}.$$

**Theorem 10** [2, Theorem B]. *Let  $N$  be a positive integer. Then for all but finitely many pairs of integers  $\langle a, n \rangle$  with  $a > 1$  and  $n > 2$ , there exists a Zsigmondy prime  $p$  with  $|a^n - 1|_p > nN + 1$ .*

*Proof.* Let  $a > 1$  and  $n > 2$  be positive integers such that there are no Zsigmondy primes  $p$  for  $\langle a, n \rangle$  with  $|a^n - 1|_p > nN + 1$ . Since any Zsigmondy prime  $p$  satisfies  $p \equiv 1 \pmod{n}$ , there are at most  $N$  Zsigmondy primes satisfying  $p \leq nN + 1$ . Let  $q$  be the largest prime factor of  $n$ . Hence, by Proposition 2

$$\Phi_n(a) \leq q(nN + 1)^N.$$

By Corollary 9 this implies  $a^{\frac{\sqrt{n}}{4}} < n(nN + 1)^N$ . The theorem follows.  $\square$

Finally, we recall that Zsigmondy's Theorem was used by Wedderburn in order to prove that any finite division ring is commutative [12]. We reproduce here one of the Wedderburn's proofs slightly revised. Let  $D$  be a finite division ring of dimension  $n$  over its prime subfield  $\mathbb{F}_p$ . First assume that there is a Zsigmondy prime  $q$  for  $\langle p, n \rangle$ . Let  $g \in D \setminus \{0\}$  be an element of order  $q$ . Let  $F$  be the subring of  $D$  generated by  $g$ . Let  $m = [F : \mathbb{F}_p]$ . Since  $g^{p^m - 1} = 1$  we have  $q \mid p^m - 1$ . Thus  $m = n$  and so  $D = F$  is commutative.

If there are no Zsigmondy primes for  $\langle p, n \rangle$ , then, by Zsigmondy's Theorem, either  $n = 2$ , or  $n = 6$  and  $p = 2$ . If  $n = 2$ , then  $D$  is commutative since it is the subring generated by any element in  $D \setminus \mathbb{F}_p$ . If  $n = 6$  and  $p = 2$ , then the order of 2 (mod 9) is 6. Since  $D \setminus \{0\}$  contains a subgroup of order 9 (which is abelian), we can use the previous argument to complete the proof.

To obtain a uniform formulation of the proof for  $n > 2$  note that Zsigmondy's Theorem implies that for any  $n > 2$  and  $a > 1$  there exists a prime  $p$  such that for  $m = p$  or  $m = p^2$ , the order of  $a$  modulo  $m$  equals  $n$ .

Of course, there are simpler proofs of Wedderburn's Theorem. However, it is interesting that Wedderburn's Theorem is a simple consequence of Zsigmondy's Theorem.

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