HEREDITARY NOETHERIAN CATEGORIES WITH A TILTING COMPLEX

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In memory of Maurice Auslander

Abstract. We are characterizing the categories of coherent sheaves on a weighted projective line as the small hereditary noetherian categories without projectives and admitting a tilting complex. The paper is related to recent work with de la Peña (Math. Z., to appear) characterizing finite dimensional algebras with a sincere separating tubular family, and further gives a partial answer to a question of Happel, Reiten, Smalø (Mem. Amer. Math. Soc. 120 (1996), no. 575) regarding the characterization of hereditary categories with a tilting object.

A characterization of weighted projective lines

We are going to characterize the categories coh(X) of coherent sheaves on a weighted projective line X = [3, 4].

Theorem 1. Let k be an algebraically closed field. For a small connected abelian k-category H with finite dimensional morphism and extension spaces the following assertions are equivalent:

(i) H is equivalent to the category of coherent sheaves on a weighted projective line.

(ii) Each object of H is noetherian. H is hereditary, has no non-zero projectives, and admits a tilting complex.

(iii) Each object of H is noetherian, moreover

(a) There exists an equivalence τ: H → H (Auslander-Reiten translation) such that Serre duality DExt^1(A, B) ∼= Hom(B, τA) holds functorially in A, B ∈ H.

(b) The Grothendieck group K_0(H) is finitely generated free, and the Euler form ⟨−, −⟩: K_0(H) × K_0(H) → Z given on classes of objects of H by ⟨[X], [Y]⟩ = dim_k Hom(X, Y) − dim_k Ext^1(X, Y) is non-degenerate of determinant ±1.

(c) H has an object without self-extensions which is not of finite length.
The result includes a characterization of the category $\text{coh}(\mathbb{P}_1(k))$ of coherent sheaves on the projective line adding the requirement: If $F$ is not of finite length and $S$ is simple in $\mathcal{H}$, then $\text{Hom}(F, S) \neq 0$. The theorem also gives a partial answer to a question raised by Happel, Reiten, Smalø [6] about the hereditary abelian $k$-categories admitting a tilting object.

For a finite dimensional $k$-algebra $\Sigma$, $\text{mod}(\Sigma)$ denotes the category of finite dimensional right modules. For a $k$-vectorspace $X$ we denote its $k$-dual $\text{Hom}_k(X, k)$ by $DX$. For an abelian category $\mathcal{H}$ its derived category of bounded complexes is denoted $D^b(\mathcal{H})$, and $X \mapsto X[n]$ denotes the $n$th iterate of the translation functor for $D^b(\mathcal{H})$. An object $X$ in $\mathcal{H}$ is called noetherian if each ascending chain of subobjects of $X$ is stationary. We say that $E \in \mathcal{H}$ is without self-extensions if $\text{Ext}^n(E, E) = 0$ holds for each $n \neq 0$. A $k$-category $\mathcal{H}$ is connected if a representation of $\mathcal{H}$ as a coproduct $\mathcal{H} \cong \mathcal{A} \coprod \mathcal{B}$ implies $\mathcal{A} = 0$ or $\mathcal{B} = 0$. We say that $\mathcal{H}$ is hereditary if $\text{Ext}^2_{\mathcal{H}}(-, -) = 0$.

A tilting object $\Sigma$ in $\mathcal{H}$ is defined by the fact that the right derived functor of $\text{Hom}(\Sigma, -): \mathcal{H} \rightarrow \text{mod}(\text{End}(\Sigma))$ induces an equivalence of triangulated categories $D^b(\mathcal{H}) \rightarrow D^b(\text{mod}(\text{End}(\Sigma)))$. More generally, we say that $\mathcal{H}$ admits a tilting complex if for some finite dimensional algebra $\Sigma$ there exists an equivalence $\Phi: D^b(\text{mod}(\Sigma)) \rightarrow D^b(\mathcal{H})$ of triangulated categories. Here, $\Sigma$ has an interpretation as the endomorphism ring of the tilting complex $\Phi(\Sigma)$ of $D^b(\mathcal{H})$. Whenever convenient, we prefer to view a tilting complex as a full subcategory of $D^b(\mathcal{H})$ with finitely many objects, thus avoiding the use of endomorphism rings.

This work has substantially profited from the collaboration with José Antonio de la Peña on finite dimensional algebras with a sincere separating tubular family [9]; we further thank Hagen Meltzer for critical comments on a first draft of this paper.

**Proof of the characterization**

Let $\mathcal{H}_0$ be the full subcategory consisting of all objects of $\mathcal{H}$ having finite length.

**Lemma 1.** Assume that $\mathcal{H}$ satisfies condition (iii)(a). Then $\mathcal{H}_0$ is uniserial, i.e. each object $U \in \mathcal{H}_0$ has a unique finite composition series. Accordingly, for some index set $X$, the category $\mathcal{H}_0$ decomposes into a coproduct $\mathcal{H}_0 = \coprod_{x \in X} \mathcal{U}_x$ of connected uniserial categories $\mathcal{U}_x$, whose indecomposables form stable tubes.

If moreover $K_0(\mathcal{H})$ is finitely generated, then for each $x \in X$ the simple objects in $\mathcal{U}_x$ form an Auslander-Reiten orbit of finite cardinality $u(x)$. Moreover $u(x) = 1$ for all but finitely many $x \in X$.

**Proof.** In view of (iii)(a) the category $\mathcal{H}$ is hereditary. Hence $\mathcal{H}_0$ is a hereditary abelian category with almost-split sequences where each object has finite length, and the Auslander-Reiten translation $\tau: \mathcal{H}_0 \rightarrow \mathcal{H}_0$ is an equivalence. For simple objects $S_i, S_j$, this implies that $\text{Ext}^1(S_j, S_i) \neq 0$ if and only if $S_i \cong \tau S_j$, moreover that in this case the Ext-space is one-dimensional. Uniseriality of $\mathcal{H}_0$ now follows easily [2].

For each $x \in X$ we fix a simple object $S_x$ from $\mathcal{U}_x$, and collect a representative system $(S_{\alpha}, \alpha \in I)$ of the remaining simple objects. $I$ admits a linear ordering such that $\langle [S_{\alpha}], [S_{\beta}] \rangle = 1$ for $\alpha = \beta$, but is zero for $\alpha > \beta$. This implies that the classes $[S_{\alpha}], \alpha \in I$, are linearly independent in $K_0(\mathcal{H})$, and proves the claim on $u$. 

\[\square\]
Proof of (ii) $\Rightarrow$ (iii). In view of $D^b(\mathcal{H}) \cong D^b(\text{mod}(\Sigma))$ the group $K_0(\mathcal{H})$ is isomorphic to $K_0(\text{mod}(\Sigma))$, which is finitely generated free. As a tilting complex for a hereditary category, the algebra $\Sigma$ has finite global dimension which implies the assertion on the Euler form. Happel’s theorem [5] shows that $D^b(\mathcal{H}) = D^b(\text{mod}(\Sigma))$ has Auslander-Reiten triangles, and that further the Auslander-Reiten translation $\tau$ is an equivalence for $D^b(\mathcal{H})$. Next we prove that $\tau(\mathcal{H})$ is contained in $\mathcal{H}$. Let $X$ be indecomposable in $\mathcal{H}$, we get $\tau X = Y[n]$ for some $Y \in \mathcal{H}$ and $n \in \mathbb{Z}$. Because $D \text{Ext}_{\mathcal{H}}^{n+1}(X,Y) \cong \text{Hom}_{D^b(\mathcal{H})}(Y[n+1], \tau X[1]) \neq 0$, $n \geq 1$ is impossible for a hereditary category. Assuming $n \leq -1$ implies

$$D \text{Ext}_{\mathcal{H}}^1(X,Z) \cong \text{Hom}_{D^b(\mathcal{H})}(Z,Y[n]) = 0$$

for each $Z \in \mathcal{H}$, so leads to a non-zero projective $X$ in $\mathcal{H}$. This proves the assertion on Serre duality for $\mathcal{H}$, which in turn implies the existence of almost-split sequences in $\mathcal{H}$ (cf. [3]).

Finally assume that condition (iii)(c) is violated. Then $D^b(\mathcal{H})$ has a tilting complex lying in $D^b(\mathcal{H}_0)$, which involving Lemma 1 implies that $\mathcal{H} = \mathcal{H}_0$ is connected uniserial, hence the existence of a simple object $S$ of $\tau$-period $p$ such that the classes $[S], [\tau S], \ldots, [\tau^{p-1} S]$ form a $\mathbb{Z}$-basis of $K_0(\mathcal{H})$. Then $a = \sum_{j=0}^{p-1} [\tau^j S]$ is a non-zero element of $K_0(\mathcal{H})$ with $\langle a, - \rangle = 0$, contradiction.

The proof of implication (iii) $\Rightarrow$ (i) will occupy the rest of the paper. Each object $X$ from $\mathcal{H}$ has a largest subobject $tX$ of finite length (its torsion part), and $X/tX$ has no simple submodules, i.e. is torsionfree. The full subcategory of torsionfree objects is denoted $\mathcal{H}_+$.  

Lemma 2. An indecomposable object $X$ of $\mathcal{H}$ is either torsion $(X \in \mathcal{H}_0)$ or torsionfree ($X \in \mathcal{H}_+$).

Proof. Since $D \text{Ext}^1(X/tX,tX) = \text{Hom}(\tau^-(tX),X/tX) = 0$ the sequence $0 \rightarrow tX \rightarrow X \rightarrow X/tX \rightarrow 0$ splits.

Lemma 3. There is a surjective linear mapping $\text{rk}: K_0(\mathcal{H}) \rightarrow \mathbb{Z}$, $[X] \mapsto \text{rk} X$, called rank function, such that $\text{rk} \tau X = \text{rk} X$ for each $X \in \mathcal{H}$, $\text{rk}$ vanishes on $\mathcal{H}_0$, and $\text{rk} F > 0$ for each $F \neq 0$.\footnote{de la Peña}. For $x \in X$ and $S_x \in \mathcal{U}_x$ simple, we define $w_x = \sum_{j=1}^{\mathcal{U}(x)} [\tau^j S_x]$ in $K_0(\mathcal{H})$. Note that each $w_x$ is stable under the mapping induced by $\tau$ on $K_0(\mathcal{H})$. We choose $x_1, \ldots, x_s$ from $X$ such that the subgroup of $K_0(\mathcal{H})$ generated by $w_{x_1}, \ldots, w_{x_s}$ contains all $w_x$ ($x \in X$) and put $w = w_{x_1} + \cdots + w_{x_s}$.

By noetherianness each non-zero $F \in \mathcal{H}_+$ has a simple quotient $S_x$, hence $\langle [F], w_x \rangle > 0$, which implies $\langle [F], w_{x_i} \rangle > 0$ for some $i = 1, \ldots, s$, and finally $\langle [F], w \rangle > 0$. If $\varepsilon$ denotes the index of $\langle K_0(\mathcal{H}), w \rangle$ in $\mathbb{Z}$ we may take $\text{rk} a = 1/\varepsilon \cdot \langle a, w \rangle$.

Lemma 4. Assume $B \in \mathcal{H}_+$ and $\text{Ext}^1(\mathcal{U}_x,B) \neq 0$. For each $n \geq 0$ there exists an exact sequence $0 \rightarrow B \rightarrow B_n \rightarrow U_n \rightarrow 0$ with $B_n \in \mathcal{H}_+, U_n \in \mathcal{U}_x$ and $[U_n] = n \cdot w_x$.

Proof. Let $S_x$ be simple in $\mathcal{U}_x$ with $\text{Ext}^1(S_x,B) \neq 0$. Choose non-split exact sequences $0 \rightarrow B'_i \rightarrow B'_{i+1} \rightarrow \tau^{-i} S_x \rightarrow 0$ ($i = 0, 1, 2, \ldots$), where $B'_0 = B$, and put $B_n = B'_{n \mathcal{U}(x)}$.
Lemma 5. For each non-zero $F$ in $\mathcal{H}_+$ and each $x \in X$ there is a non-zero homomorphism from $F$ into an object from $\mathcal{U}_x$. Accordingly, $[X] = 0$ in $K_0(\mathcal{H})$ implies $X = 0$ in $\mathcal{H}$.

Proof. We put

\[
\mathcal{Y} = \{ y \in X | \text{Hom}(F, \mathcal{U}_y) = 0 \},
\]

\[
\mathcal{A} = \{ A \in \mathcal{H} | \text{Hom}(A, \mathcal{U}_y) = 0 \text{ for all } y \in \mathcal{Y} \},
\]

\[
\mathcal{B} = \{ B \in \mathcal{H} | \text{Hom}(A, B) = 0 \}.
\]

Step 1. $\text{Hom}(\mathcal{B}, \mathcal{U}_x) = 0$ for all $x \in X \setminus \mathcal{Y}$.

Since $\tau \mathcal{U}_x = \mathcal{U}_x$ it is equivalent to prove that $\text{Ext}^1(\mathcal{U}_x, \mathcal{B}) = 0$. Otherwise we obtain some $B$ in $\mathcal{B}$ and exact sequences

\[
0 \to B \to B_n \to U_n \to 0 \quad (n = 1, 2, \ldots),
\]

having the properties of Lemma 4. We show that each $B_n$ actually belongs to $\mathcal{B}$. Consider a morphism $f: A \to B_n$ with $A \in \mathcal{A}$, and from the pull-back

\[
\begin{array}{c}
0 \to A' \to A \to U_n \to 0 \\
\downarrow f' \quad \quad \quad \downarrow f \\
0 \to B \to B_n \to U_n \to 0.
\end{array}
\]

Using $x \notin \mathcal{Y}$ we see that $A' \in \mathcal{A}$, hence $f' = 0$, so $f$ induces a morphism $U_n \to B_n$ which is zero, implying $\text{Hom}(A, B_n) = 0$. Since $[B_n] = [B] + nw_x$, we obtain $\langle [F], B_n \rangle > 0$ for large $n$, thus contradicting $\text{Hom}(A, B) = 0$.

Step 2. Each $X$ in $\mathcal{H}$ has the form $X \cong A \oplus B$ with $A \in \mathcal{A}, B \in \mathcal{B}$.

By noetherianness $X$ has a maximal subobject $A$ from $\mathcal{A}$, accordingly $X/A$ has no non-zero subobject from $\mathcal{A}$, hence belongs to $\mathcal{B}$. Obviously, $\mathcal{A}$ is closed under $\tau^{-}$, therefore $D \text{Ext}^1(X/A, A) = \text{Hom}(\tau^{-} X, X/A) = 0$, and $X \cong A \oplus X/A$.

Step 3. $\text{Hom}(\mathcal{B}, \mathcal{A}) = 0$.

Otherwise we find $A \in A \cap \mathcal{H}_+ + B \in B \cap \mathcal{H}_+$ and a non-zero morphism $f: B \to A$. Let $A' = \text{im}(f)$. We choose $A' \subseteq A'' \subseteq A$ such that $A''/A' = t(A/A')$. Since $A \in \mathcal{A}$ and $A/A'' \in \mathcal{H}_+$ we see that $A'' \in \mathcal{A}$. Replacing $A$ by $A''$ we may thus assume that $A/A' \in \mathcal{H}_0$, in fact $A/A' \in \bigoplus_{x \in X \setminus \mathcal{Y}} \mathcal{U}_x$. By Step 1 the sequence $0 \to A' \to A \to A/A' \to 0$ splits, so $A'$ belongs to $\mathcal{A}$. We thus arrive at an epimorphism $B \to A' \to S_x$ where $S_x$ is simple in $\mathcal{U}_x$ and $x \in X \setminus \mathcal{Y}$, contradiction, which proves the assertion of Step 3.

Combining Steps 2 and 3 with $\text{Hom}(\mathcal{A}, \mathcal{B}) = 0$ we see that $\mathcal{H} = \mathcal{A} \bigoplus \mathcal{B}$. Since $\mathcal{H}$ is connected and $0 \neq F \in \mathcal{A}$, it follows that $\mathcal{B} = 0$, hence $\mathcal{Y} = \emptyset$ as claimed.

By $K'_0(\mathcal{H})$ we denote the image of the natural map $K_0(\mathcal{H}_0) \to K_0(\mathcal{H})$, i.e. the subgroup of $K_0(\mathcal{H})$ generated by the classes of simple objects.

Lemma 6. There exists a torsionfree object $L$ of rank one. $K_0(\mathcal{H}) = \mathbb{Z}[L] \oplus K'_0(\mathcal{H})$. Moreover for any $x, y \in X, w_x$ and $w_y$ are proportional.

Proof. Choose a torsionfree object $L \neq 0$ of minimal rank. Each $a \in K'_0(\mathcal{H})$ satisfies $\text{rk} a = 0$, on the other hand $\text{rk} L > 0$, which implies $\mathbb{Z}[L] \cap K'_0(\mathcal{H}) = 0$. We show by induction on $r = \text{rk} A$ that the class $[A]$ of an indecomposable object $A$ belongs
to $\mathbb{Z}[L] + K_0'(\mathcal{H})$. For $r = 0$ this follows from Lemma 3. For $r > 0$, $A$ is torsionfree. We choose $x \in \mathcal{X}$ and a simple object $S_x$ from $\mathcal{U}_x$ with $\text{Ext}^1(S_x, A) \neq 0$. Invoking Lemma 4 we get an exact sequence $0 \to A \to A \to U \to 0$, where $A$ is torsionfree, $U$ is in $\mathcal{U}_x$, and a non-zero homomorphism $f : L \to A$. Because $L$ has minimal rank and $A$ is torsionfree, $f$ is a monomorphism hence $C = \text{coker}(f)$ has strictly smaller rank than $A$. By induction $[C]$, hence $[A]$, belongs to $\mathbb{Z}[L] + K_0'(\mathcal{H})$, proving the splitting of $K_0(\mathcal{H})$. Surjectivity of the rank function now implies that $L$ has rank one.

Since the linear forms $\langle -, w_x \rangle$ and $\langle -, w_y \rangle$ vanish on $K_0' \mathcal{H}$ they are determined by their non-zero value on $[L]$, hence are proportional. Because the Euler form on $K_0(\mathcal{H})$ is non-degenerate, the corresponding assertion holds for $w_x$ and $w_y$. 

Through the formation of perpendicular categories [4] we are going to reduce the weight function $u : \mathcal{X} \to \mathbb{N}$. An object $E$ from $\mathcal{H}$ is called exceptional if $E$ has a trivial endomorphism ring $k$ and no self-extensions. Invoking Lemma 5 it follows from [7] that each indecomposable object $E \in \mathcal{H}$ without self-extensions is exceptional. Notice that every simple object $S$ in $\mathcal{U}_x$ with $u(x) > 1$ is exceptional.

**Proposition 1.** Let $x \in \mathcal{X}$ be exceptional, i.e. $u(x) > 1$, and let $S$ be simple in $\mathcal{U}_x$.

(a) The right perpendicular category $\mathcal{H}' = S^\perp$ of $S$ in $\mathcal{H}$, consisting of all $X \in \mathcal{H}$ with $\text{Hom}(S, X) = 0 = \text{Ext}^1(S, X)$, satisfies conditions (iii) (a), (b), (c) again.

(b) $\mathcal{X}$ serves naturally also as a parametrizing set for the decomposition of $\mathcal{H}_0' = S^\perp \cap \mathcal{H}_0$ into connected uniserial categories $\mathcal{U}_y' = S^\perp \cap U_y$. The number of isomorphism classes of simples from $\mathcal{U}_y'$ is given by $u'(x) = u(x) - 1$ and $u'(y) = u(y)$ for $y \neq x$. Moreover, inclusion $\mathcal{H}' \hookrightarrow \mathcal{H}$ preserves the rank.

(c) $K_0(\mathcal{H}) = \mathbb{Z}[S] \oplus K_0(\mathcal{H}')$.

**Proof.** Since $\mathcal{H}$ is hereditary and $S$ is exceptional, inclusion $\mathcal{H}' \hookrightarrow \mathcal{H}$ admits a left adjoint $\ell$ and a right adjoint $r$ [4]. Since $S$ is simple, the category $\mathcal{H}_0'$ of torsion objects in $\mathcal{H}'$ agrees with $\mathcal{H}' \cap \mathcal{H}_0$, and therefore $\mathcal{H}_0' = \prod_{y \in \mathcal{X}} \mathcal{U}_y'$ with $\mathcal{U}_y' = U_y$ if $y \neq x$; further $\mathcal{U}_y' = S^\perp \cap U_x$ is connected as well with $u(x) - 1$ isomorphic classes of simple objects. Moreover, $w_y' = w_y$ for each $y \in \mathcal{X}$, so inclusion $\mathcal{H}' \hookrightarrow \mathcal{H}$ preserves the rank.

If $E$ is torsionfree in $\mathcal{H}$ without self-extensions, then $\text{Hom}(S, E) = 0$ and $\ell E$ is given by the middle term of the $S$-universal extension

$$0 \to E \to \ell E \to \text{Ext}^1(S, E) \otimes_k S \to 0,$$

in particular $\ell E$ is non-zero torsionfree. From the exactness of

$$0 = \text{Ext}^1(E, E) \to \text{Ext}^1(E, \ell E) \to \text{Ext}^1(S, E) \otimes_k \text{Ext}^1(S, S) = 0$$

we get $\text{Ext}^1(E, \ell E) = 0$. By definition $\text{Ext}^1(S, \ell E) = 0$ holds, hence $\text{Ext}^1(\ell E, \ell E) = 0$, which proves the existence of a torsionfree object in $\mathcal{H}'$ without self-extensions.

Finally, assume that $A$ and $B$ are in $\mathcal{H}'$. Since

$$\text{DExt}^1(A, B) = \text{Hom}(\ell r_{\mathcal{H}} B, A) = \text{Hom}(B, r_{\mathcal{H}} A),$$

projectivity of $A$ (resp. injectivity of $B$) in $\mathcal{H}'$ implies $r_{\mathcal{H}} A = 0$ (resp. $r_{\mathcal{H}} B = 0$) which, by the construction of $r$ and $\ell$ [4], implies $A \in \mathcal{H}_0'$ (resp. $B \in \mathcal{H}_0'$). But any projective (resp. injective) in $\mathcal{U}_y'$, $y \in \mathcal{X}$, is zero, hence $A = 0$ (resp. $B = 0$). Invoking Auslander-Reiten theory this implies that $\tau_{\mathcal{H}} = r_{\mathcal{H}} A$ and $\tau_{\mathcal{H}} = \ell_{\mathcal{H}} A$ are inverse equivalences. For assertion (c) we refer to [4]. Further $\langle [S], K_0(\mathcal{H}') \rangle = 0$ implies that the Euler forms on $K_0(\mathcal{H})$ and $K_0(\mathcal{H}')$ do have the same determinant. \qed
The existence of an exceptional object $S \in \mathcal{H}$ of rank one, note that $\text{End}(L) = k$ hence $\langle [L], [L] \rangle = 1 - g$, where $g = \dim_k \text{Ext}^1(L, L)$, and define the degree function $\deg: K_0(\mathcal{H}) \to \mathbb{Z}$ by
\[
\deg z = \langle [L], z \rangle - (1 - g) \text{rk } z.
\]

We thus obtain a Riemann-Roch formula
\[
(a, b) = (1 - g) \text{rk } a \text{rk } b + \begin{vmatrix}
\text{rk } a & \text{rk } b \\
\deg a & \deg b
\end{vmatrix}
\quad \text{for all } a, b \in K_0(\mathcal{H}).
\]

The existence of an exceptional object $E$ of $\mathcal{H}$ implies $0 < \langle [E], [E] \rangle = (1 - g)(\text{rk } E)^2$ hence $g = 0$, and so $L$ is exceptional.

Next we choose $x \in \mathcal{X}$ such that $d = \deg w_x'$ is minimal. Note that $d > 0$. For $y \in \mathcal{X}$ write $\deg w_y' = d \cdot q + r$ with $q, r \in \mathbb{N}$ and $0 \leq r < d$. Extending $L$ by $S_y$ and $q$ times by $S_x$ we obtain torsionfree rank one objects $L', L''$ with
\[
[L'] = [L] + w_y', \quad [L''] = [L] + q \cdot w_x',
\]
hence
\[
\langle [L''], [L'] \rangle = \langle [L], [L] \rangle + \langle [L], w_y' \rangle - q \cdot \langle [L], w_x \rangle \geq 1.
\]

We thus arrive at an exact sequence $0 \to L'' \to L' \to U \to 0$, where $U \in \mathcal{H}$ has degree $< d$. This shows $U = 0$, hence $w_y' = q \cdot w_x'$. In view of Lemma 6 the classes $[L], [S_x]$ thus form a basis of $K_0(\mathcal{H})$, accordingly the determinant of the Euler form is $\pm 1 = \langle [L], [S_x] \rangle^2$, proving $\dim \text{Hom}(L, S_x) = 1$, therefore $\deg S_x = 1$.

Put $L_1 = L$, and let $L_2$ be the middle term of a non-split exact sequence $\mu: 0 \to L_1 \xrightarrow{\xi} L_2 \to S_x \to 0$. Application of $\text{Hom}(L_1, -)$ to $\mu$ shows $\text{Hom}(L_1, L_2) \cong k^2$ and $\text{Ext}^1(L_1, L_2) = 0$, while application of $\text{Hom}(\cdot, L_1)$ leads to $\text{Hom}(L_2, L_1) = 0$ and $\text{Ext}^1(L_2, L_1) = 0$.

Extend $\xi$ to a $k$-basis $\xi, \eta$ of $\text{Hom}(L_1, L_2)$. Invoking the properties of rank and degree, for each $a \in k$ the cokernel $T_a$ of $\eta - a\xi: L_1 \to L_2$ is seen to be simple. In fact, each simple object $S_y$ ($y \neq x$) is isomorphic to some $T_a$: Setting $\xi^* = \text{Hom}(\xi, S_y), \eta^* = \text{Hom}(\eta, S_y)$ we see that $\xi^*$ is an isomorphism because $x \neq y$; moreover, since $k$ is algebraically closed there exists an eigenvalue $a \in k$ such that the kernel $\text{Hom}(T_a, S_y) \in \text{Hom}(L_2, S_y) \to \text{Hom}(L_a, S_y)$ is non-zero. Hence $S_y \cong T_a$, in particular $[S_y] = [L_1] - [L_2] = [S_x]$.

\[ \square \]
Proposition 3. $\mathcal{H}$ is a tilting object $\Delta$

\[
\begin{array}{cccc}
S_1^{[p_1-1]} & \cdots & S_1^{[l]} \\
\eta_1 & & \\
\hline
L_1 \xrightarrow{\xi_1} L_2 & \eta_2 & S_2^{[p_2-1]} & \cdots & S_2^{[l]} \\
\vdots & & \vdots & & \\
\eta_t & & S_t^{[p_t-1]} & \cdots & S_t^{[l]} \\
\end{array}
\]

where $L_1, L_2$ are torsionfree rank one objects satisfying (1), and $S_i^{[l]}$ is an exceptional torsion object of length $l$.

$\Delta$ is a squid algebra in the sense of [10], i.e. equivalent to the path category of the above quiver subject to the relations

\[
\eta_1 \circ \xi_1 = 0, \quad \eta_2 \circ \xi_2 = 0, \quad \eta_i \circ (\xi_2 - \lambda_i \xi_1) = 0, \quad i = 3, \ldots, t,
\]

where $\lambda_3, \ldots, \lambda_t$ are pairwise distinct non-zero elements from $k$.

$\Delta$ also has a realization as a tilting object on the weighted projective line $\mathbb{X}(p, \lambda)$ given by the data $p = (p_1, \ldots, p_t)$, $\lambda = (\lambda_3, \ldots, \lambda_t)$.

Proof. Let $x_1, \ldots, x_t$ denote the sequence of pairwise distinct exceptional elements from $\mathbb{X}$. We put $p_i = u(x_i)$, $U_i = U_{x_i}$. Let $S_i$ be simple in $U_i$ and $\mathcal{H}'$ denote the right perpendicular category with respect to the system

\[
\tau^l S_i, \quad l = 0, \ldots, p_i - 2, \quad i = 1, \ldots, t,
\]

and take a system $L_1, L_2$ of torsionfree rank one objects from $\mathcal{H}'$ as established in Proposition 2.

By induction on the rank it is easily verified (compare proof of Lemma 6) that each object from $\mathcal{H}$ is contained in the smallest subcategory of $\mathcal{H}$ which is closed under extensions, kernels of epimorphisms, cokernels of monomorphisms and contains

\[
L_1, L_2, S_i, \tau S_i, \ldots, \tau^{p_i-2} S_i, \quad i = 1, \ldots, t;
\]

in particular (4) generates $D^b(\mathcal{H})$.

Let $S_i^{[n]}$ denote the indecomposable object of $U_i$ with top $S_i$ and length $n$. Then the system

\[
L_1, L_2, S_i = S_i^{[1]}, S_i^{[2]}, \ldots, S_i^{[p_i-1]}, \quad i = 1, \ldots, t,
\]

generates $D^b(\mathcal{H})$ as well, moreover satisfies

\[
\text{Ext}^1(S_i^{[a]}, S_i'^{[a']}) = 0, \quad \text{Ext}^1(S_i^{[a]}, L_j) = 0, \quad \text{Ext}^1(L_j, S_i^{[a]}) = 0
\]

for all $i, i' = 1, \ldots, t, \quad a = 1, \ldots, p_i - 1, \quad a' = 1, \ldots, p_{i'} - 1, \quad j = 1, 2$. 

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Invoking $w_x = w$ for each $x \in X$, we obtain exact sequences

$$0 \to L_1 \xrightarrow{\xi_i} L_2 \xrightarrow{\eta_i} S_i^{[p_i]} \to 0 \quad (1 \leq i \leq t),$$

defining $t$ mutually disjoint 1-dimensional subspaces $k \xi_i$ of $\text{Hom}(L_1, L_2)$, thus $\xi_i = \xi_2 - \lambda_i \xi_1$ for $2 \leq i \leq t$, where $0 = \lambda_0, \lambda_1, \ldots, \lambda_t$ are pairwise distinct. Setting $\eta_i = [L_2 \xrightarrow{\tau_i} S_i^{[p_i]} \to S_i^{[p_i - 1]}]$ and calculating dimensions of Hom-spaces by means of the Euler form further establishes that (2) is a complete set of relations for the subcategory $\Delta$.

It is known (cf. [8]) and not difficult to prove that $\Delta$ can be realized as a tilting object $\mathbb{X}(p, \lambda)$ arising from the above data $p = (p_1, \ldots, p_t)$ and $\lambda = (\lambda_3, \ldots, \lambda_t)$ [3]. Since moreover $\Delta$ has finite global dimension (actually $\text{gl. dim} \; \Delta \leq 2$) we may invoke Beilinson’s lemma [1] to conclude that $\Delta$ is also a tilting object in $\mathcal{H}$.

\textbf{Proof of (iii) }$\Rightarrow$ \textbf{(i).} Abbreviating $\mathbb{X}(p, \lambda)$ to $X$ from now on, the right derived functor of $\text{Hom}_X(\Delta, -) : \text{coh}(X) \to \text{mod}(\Delta)$ induces an equivalence of triangulated categories $\alpha : D^b(\text{coh}(X)) \to D^b(\text{mod}(\Delta))$, similarly the right derived functor of $\text{Hom}_H(\Delta, -) \to \text{mod}(\Delta))$. Since $\alpha$ and $\beta$ preserve $\Delta$ the composition $\phi = \alpha^{-1} \circ \beta : D^b(\mathcal{H}) \to D^b(\text{coh}(X))$ preserves the rank and maps $\mathcal{H}$ onto the category $\text{coh}_0(X)$ of finite length sheaves.

An indecomposable object $X$ from $D^b(\mathcal{H})$ lies in $\mathcal{H}$ if and only if $X$ is either in $\mathcal{H}_0$ or has rank $> 0$ and satisfies $\text{Hom}(X, \mathcal{H}_0) \neq 0$. Similarly an indecomposable object $Y$ from $D^b(\text{coh}(X))$ belongs to $\text{coh}(X)$ if and only if $Y$ is in $\text{coh}_0(X)$ or $Y$ has rank $> 0$ and satisfies $\text{Hom}(Y, \text{coh}_0(X)) \neq 0$. Because $\phi(\mathcal{H}_0) = \text{coh}_0(X)$, and $\phi$ preserves the rank, it hence follows that $\phi$ induces an equivalence $\mathcal{H} \to \text{coh}(X)$, thus concluding the proof of the theorem.

Given $\mathcal{H}$ satisfying (ii) or (iii), the theorem implies the existence of a “natural” bijection between the set $X$, parametrizing the decomposition of $\mathcal{H}_0$ into connected components, and the projective line $\mathbb{P}_1(k)$, the point set underlying $X$. To determine the weighted projective line corresponding to $\mathcal{H}$, one fixes a simple object $S$ with $\tau S \cong S$ further a rank one object $L$ of $\mathcal{H}$, and forms a non-split extension $0 \to L \to \mathbb{L} \to S \to 0$. The $k$-space $\text{Hom}(L, \mathbb{L})$ has dimension two; each non-zero $v : L \to \mathbb{L}$ is a monomorphism whose cokernel is indecomposable of finite length, therefore belongs to $\mathcal{U}_{\psi(v)}$ for a unique $\psi(v) \in X$. The induced mapping $\mathbb{P}_1(k) = \mathbb{P}(\text{Hom}(L, \mathbb{L})) \to X$, $kv \mapsto \psi(v)$, is a bijection turning the function $u$ from Lemma 1 into a weight function on $\mathbb{P}_1(k)$, determining thus completely the data for the weighted projective line.

\textbf{References}


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