PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 7, July 1997, Pages 1903–1912
S 0002-9939(97)04166-X

THE BOUNDARY OF A BUSEMANN SPACE

PHILIP K. HOTCHKISS

(Communicated by James E. West)

Abstract. Let X be a proper Busemann space. Then there is a well defined boundary, ∂X, for X. Moreover, if X is (Gromov) hyperbolic (resp. non-positively curved), then this boundary is homeomorphic to the hyperbolic (resp. non-positively curved) boundary.

§0. Introduction

The boundary of a (Gromov) hyperbolic space (and hence of a (Gromov) hyperbolic group) was introduced in Gromov's now famous article on hyperbolic groups [G1]. Since then, this notion has received much attention and provided many interesting results (see [F], [G1], [GH], [Sw]). This notion of boundary has been generalized to non-positively curved spaces and automatic groups (see [G2] and [NS] respectively) although it appears that the proof of this for non-positively curved spaces has not been published anywhere. However, the notion of boundary can also be extended to more general class of spaces called *Busemann spaces* which were defined in [Bo]. In this paper, we provide an elementary proof that the boundary of a Busemann space is well defined.

Definition. Let X be a proper Busemann space, and let $x_0 \in X$. We define the *boundary of X relative to x_0* as

$$\partial_{x_0} X = \{f : [0, \infty) \to X \mid \text{where } f(0) = x_0 \text{ and } f \text{ is an isometry}\}$$

and we give $\partial_{x_0} X$ the compact-open topology.

Our main theorem is the following:

Main Theorem. *Let X be a proper Busemann space and let x_0 and x_1 be two distinct points in X. Then $\partial_{x_0} X$ is homeomorphic to $\partial_{x_1} X$.*

Corollary. *If a Busemann space X is (Gromov) hyperbolic (resp. non-positively curved), then $\partial_{x_0} X$ is homeomorphic to the hyperbolic (resp. non-positively curved) boundary.*

In §1 we give the basic definitions and some background material, and in §2 we define the boundary of a Busemann space. The author would like to thank his advisor, Edward C. Turner, for all his helpful suggestions and support.

Received by the editors November 19, 1995.
1991 *Mathematics Subject Classification.* Primary 20F32.
Key words and phrases. Busemann space, geodesic, proper.

§1. Background

Busemann spaces.

Before we begin, we set some terminology and notation. Most, if not all, of these terms are standard and can be found in most of the sources cited in this paper and in many others. Throughout this paper, all our metric spaces will be *geodesic* and *proper*. Recall that a metric space is *proper* if all closed and bounded sets are compact. A metric space, X, is said to be *geodesic* if, for any two points $x,\ y \in X$, there is a path $p : [a, b] \to X$ such that $p(a) = x,\ p(b) = y$ and p is an isometry. Such a path is called a *geodesic segment* or a *geodesic*. A *(geodesic) ray* will denote an isometry $p : \mathbb{R}^+ = [0, \infty) \to X$. If X is geodesic and $x,\ y \in X$, then $[x, y]$ will denote a geodesic from x to y (note that there may be more than one). We will also denote the metric on X by $|\cdot|$.

Definition. A path $p : [0, 1] \to X$ is a *geodesic parameterized proportionally to arc length* if there exists a geodesic segment $\tilde{p} : [a, b] \to X$ such that $p(t) = \tilde{p}(a+(b-a)t)$.

Definition ([Bo])**.** A geodesic space X is a *Busemann space* if for all geodesic segments, α_0 and α_1, that are parameterized proportionally to arc length with $\alpha_0(0) = \alpha_1(0)$, we have that

$$|\alpha_0(t) - \alpha_1(t)| \le t|\alpha_0(1) - \alpha_1(1)|$$

for all $t \in [0, 1]$.

The following proposition follows easily from the definition.

Proposition 1.1. *If X is a Busemann space, then*

(1) *X is contractible.*
(2) *X has unique geodesic segments. Moreover, for any $x_0 \in X$ geodesic rays from x_0 diverge; i.e. if f and g are distinct geodesic rays beginning at x_0, then*

$$\lim_{t\to\infty} |f(t) - g(t)| = \infty.$$

In fact, if $s < t$, then $|f(s) - g(s)| < |f(t) - g(t)|$.

Remark. It should be noted that any non-positively curved space is a Busemann space so the proof that follows will work for non-positively curved spaces as well.

Hausdorff distance.

Definition. If $A \subseteq (X, |\cdot|)$ and $K \in \mathbb{R}^+$, then the K *neighborhood of* A, $N_K(A)$, is the set $\{x \in X \mid |x - A| \le K\}$.

Definition ([GH])**.** Let A and B be subsets of X. The *Hausdorff distance between A and B* is given by

$$\mathcal{H}(A, B) = \inf\{K > 0 \bigm| A \subset N_K(B) \text{ and } B \subset N_K(A)\}.$$

Definition. If $f,\ g : \mathbb{R}^+ \to X$, then their Hausdorff distance is defined by

$$\mathcal{H}(f, g) = \mathcal{H}(f(\mathbb{R}^+), g(\mathbb{R}^+))$$

and we will write $g \subseteq N_K(f)$ for $g(\mathbb{R}^+) \subseteq N_K(f(\mathbb{R}^+))$.

The following lemma, which is part of Proposition 7.2 in [GH] provides a useful criteria for determining if two rays are a finite Hausdorff distance apart. (In [Bo], such rays are called *parallel*.)

Lemma 1.2. *Let f, g be geodesic rays from a base point x_0 in a geodesic metric space, X. Then $\mathcal{H}(f, g) < \infty$ if and only if there exists a constant M, such that $|f(t) - g(t)| \leq M$ for all $t \in \mathbb{R}^+$.*

§2. The boundary of a Busemann space

Definition. Let X be a proper Busemann space. Let $x_0 \in X$. We define the *boundary of X relative to x_0* as

$$\partial_{x_0} X = \{f : \mathbb{R}^+ \to X \mid f(0) = x_0 \text{ and } f \text{ is an isometry}\}$$

endowed with the compact-open topology.

Our goal is to prove the following theorem:

Theorem 2.1. *Let X be a proper Busemann space and let x_0 and x_1 be two distinct points in X. Then $\partial_{x_0} X$ is homeomorphic to $\partial_{x_1} X$.*

We establish this theorem in several stages. First we show that for distinct points x_0 and x_1 in X, the set $\partial_{x_1} X$ is bijectively equivalent to $\partial_{x_0} X$.

Proposition 2.2. *Let X be a proper Busemann space and let x_0 and x_1 be distinct points in X. Then, for each $f \in \partial_{x_0} X$, there exists a unique $g \in \partial_{x_1} X$, such that $\mathcal{H}(f, g) < \infty$.*

Proof. Let $f \in \partial_{x_0} X$.

Step 1. The construction of g. For each $n \in \mathbb{N}$, let g_n be the geodesic segment from x_1 to $f(n)$. If $L_n = |x_1 - f(n)|$, then $g_n : [0, L_n] \to X$, and we extend g_n to $\mathbb{R}^+$ via $g_n(t) = g_n(L_n) = f(n)$ for $t \geq L_n$. These extensions make the set $\{g_n\}$ equicontinuous on $\mathbb{R}^+$. Since X is proper, the set $\mathcal{F}_t = \{g_n(t)\}$ has compact closure for each $t \in \mathbb{R}^+$. Therefore, by Ascoli's Theorem (Theorem 7.6.1 in [M]), $\{g_n\}$ has a convergent subsequence, $\{g_{n_k}\}$, that converges uniformly on compact subsets of X. Let $g = \lim_{k \to \infty} g_{n_k}$.

Claim. $g \in \partial_{x_1} X$.

Proof. Since $g_{n_k}(0) = x_1$ for each k, we have that $g(0) = x_1$. Hence, it remains to show that g is a geodesic ray. Let $\epsilon > 0$ be arbitrary and let $s, t \in \mathbb{R}^+$. Since $\{s, t\}$ is compact, there exists a number N, so that if $n_k \geq N$, then $|g_{n_k}(s) - g(s)| < \frac{\epsilon}{2}$ and $|g_{n_k}(t) - g(t)| < \frac{\epsilon}{2}$. Pick $n_K \geq N$, and hence

$$\begin{aligned} |g(s) - g(t)| &\leq |g(s) - g_{n_K}(s)| + |g_{n_K}(s) - g_{n_K}(t)| + |g_{n_K}(t) - g(t)| \\ &< \frac{\epsilon}{2} + |s - t| + \frac{\epsilon}{2} \\ &= |s - t| + \epsilon. \end{aligned}$$

Similarly,

$$|s - t| = |g_{n_K}(s) - g_{n_K}(t)| < |g(s) - g(t)| + \epsilon.$$

Since ϵ was arbitrary, $|g(s) - g(t)| = |s - t|$ and so g is a ray. □

We now want to show that $\mathcal{H}(f, g) < \infty$.

Step 2. There exists a number R, such that $g_{n_k} \subseteq N_R(f)$ for all n_k. For each k, let f_k denote the geodesic segment from $f(n_k)$ to x_0 and $g_{n_k}^-$ be the geodesic segment from $f(n_k)$ to x_1, with both segments parameterized proportionally to arc length. Since X is a Busemann space we have that for each $t \in [0,1]$,

$$|f_k(t) - g^-_{n_k}(t)| \le t|f_k(1) - g^-_{n_k}(1)| = t|x_0 - x_1| \le |x_0 - x_1|. \tag{2-1}$$

Since $g^-_{n_k}$ is just the reverse of g_{n_k}, parameterized proportionally to arc length, we have that for each n_k, $g_{n_k} \subseteq N_R(f)$ where $R = |x_0 - x_1|$.

Step 3. There exists a number M, so that $g \subseteq N_M(f)$. Let $t \in \mathbb{R}^+$ and choose n_K so that $|g_{n_K}(t) - g(t)| \le 1$. Then, since $g_{n_K} \subseteq N_R(f)$, there exists $s_t \in \mathbb{R}^+$ such that $|g_{n_K}(t) - f(s_t)| \le R$. Thus,

$$|g(t) - f(s_t)| \le R + 1 = |x_0 - x_1| + 1.$$

Hence, $g \subseteq N_M(f)$ where $M = |x_0 - x_1| + 1$.

Step 4. $f \in N_M(g)$, where $M = |x_0 - x_1| + 1$. Fix $t \in \mathbb{R}^+$; then for each $n_k \ge t$, there exists $s_k \in \mathbb{R}^+$ such that $|f(t) - g_{n_k}(s_k)| \le R$ by (2-1).

Claim. $\{s_k\}$ is bounded.

Proof. Now $g_{n_k}(0) = x_1$ for each n_k so

$$\begin{aligned} s_k = |s_k - 0| = |g_{n_k}(s_k) - x_1| &\le |g_{n_k}(s_k) - f(t)| + |f(t) - x_0| + |x_0 - x_1| \\ &\le R + t + R = 2R + t. \end{aligned}$$

Since t and R are fixed $\{s_k\}$ is bounded. □

Since $\{s_k\}$ is bounded, it has a convergent subsequence, but, without loss of generality, we will assume $\{s_k\}$ converges. Let $s_0 = \lim_{k\to\infty} s_k$. Then since $g_{n_k} \to g$, it follows that $g_{n_k}(s_k) \to g(s_0)$. Choose K so that $|g_{n_K}(s_K) - g(s_0)| \le 1$. Then

$$|f(t) - g(s_0)| \le |f(t) - g_{n_K}(s_K)| + |g_{n_K}(s_K) - g(s_0)| \le R + 1 = |x_0 - x_1| + 1.$$

Thus $f \subseteq N_M(g)$ as desired, and so $\mathcal{H}(f,g) < \infty$.

The uniqueness of g follows from Lemma 1.2 and Proposition 1.1. □

Corollary 2.3. *Let X be a proper Busemann space and let x_0 and x_1 be distinct points in X. Then there exists a bijection $\Phi : \partial_{x_0}X \to \partial_{x_1}X$, defined by letting $\Phi(f)$ be the unique ray in $\partial_{x_1}X$, such that $\mathcal{H}(f, \Phi(f)) < \infty$.*

Proof. This follows from Propositions 1.1 and 2.2 since the steps in the proof of Proposition 2.2 can be reversed. □

Remark. Note that the constant $M = |x_0 - x_1| + 1$ from Proposition 2.2 is universal, i.e. it does not depend on f or g. This observation and Lemma 1.2 show that there is a universal constant $K = 2M$, such that for all $f \in \partial_{x_0}X$ and for all $t \in \mathbb{R}^+$,

$$|f(t) - \Phi(f)(t)| \le K. \tag{2-2}$$

The goal now is to prove that Φ (as defined above) is a homeomorphism. Now for every $w \in X$, the topology on $\partial_w X$ is the compact-open topology which has a subbasis of the form $\{S(C,U)\}$ where C is compact in $\mathbb{R}^+$, U is open in X and

$$S(C,U) = \{f \in \partial_w X \mid f(C) \subset U\}.$$

However, if X is a Busemann space, then there is a subbasis that will be easier for us to use.

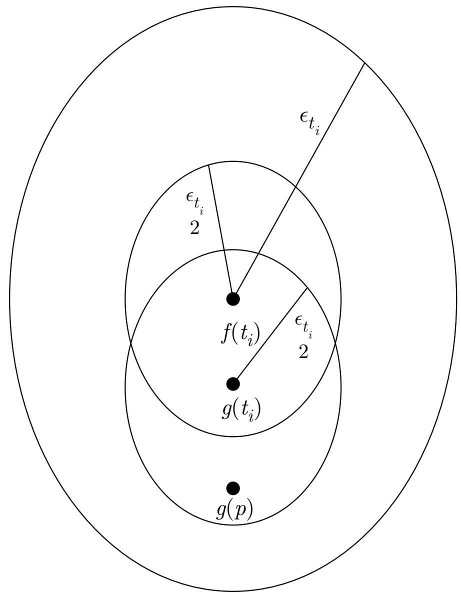

FIGURE 1

Proposition 2.4. *If X is a proper Busemann space, then, for each w in X, the set*

$$\{S(\{p\}, B_\epsilon(y)) \bigm| p \in \mathbb{R}^+ \text{ and } y \in X\}$$

is a subbasis for the compact-open topology on $\partial_w X$.

Proof. Let $S(C, U)$ be a subbasis element, and, without loss of generality, assume that $U = B_\epsilon(y)$ for some $y \in X$. Let $f \in S(C, B_\epsilon(y))$. For each $t \in C$, pick ϵ_t such that $B_t = B_{\epsilon_t}(f(t)) \subset B_\epsilon(y)$. Let $B'_t = B_{\frac{\epsilon_t}{2}}(f(t))$ and let $A_t = f^{-1}(B'_t) \subset B_{\frac{\epsilon_t}{2}}(t)$. The collection $\{A_t\}$ forms an open cover of C and, hence, there exists a finite subcover $\{A_{t_1}, \ldots, A_{t_n}\}$ of C. Let

$$S_f = \bigcap_{i=1}^{n} S(\{t_i\}, B'_{t_i}).$$

Note that $S_f \neq \emptyset$, since $f \in S_f$ by construction.

Claim. $S_f \subset S(C, B_\epsilon(y))$.

Proof. Let $g \in S_f$ and let $p \in C$; then there exists t_i such that $p \in A_{t_i}$ for some $1 \leq i \leq n$. Since $g \in S_f$, we have that $g(t_i) \in B'_{t_i} = B_{\frac{\epsilon_{t_i}}{2}}(f(t_i))$ and $|g(t_i) - g(p)| < \dfrac{\epsilon_{t_i}}{2}$. Thus,

$$\begin{aligned} |g(p) - f(t_i)| &\leq |g(p) - g(t_i)| + |g(t_i) - f(t_i)| \\ &< \frac{\epsilon_{t_i}}{2} + \frac{\epsilon_{t_i}}{2} = \epsilon_{t_i}, \end{aligned}$$

since g is a ray (see Figure 1).

Hence, $g(p) \in B_{t_i}$, and since p was arbitrary and $\bigcup_{i=1}^{n} B_{t_i} \subset B_\epsilon(y)$, we have that $g(C) \subset B_\epsilon(y)$ and so $S_f \subset S(C, B_\epsilon(y))$ as desired. □

Now clearly

$$S(C, B_\epsilon(y)) = \bigcup_{f \in S(C, B_\epsilon(y))} S_f$$

and since each S_f is a finite intersection of sets of the form $S(\{t\}, B_\epsilon(y))$, we have that the collection $\{S(\{t\}, B_\epsilon(y))\}$ forms a subbasis for the compact-open topology on $\partial_w X$. □

Remark. Note that this proof did not use the fact that X was a Busemann space, it only required that f and g be geodesic rays.

The following theorem will be very helpful in proving that Φ is a homeomorphism.

Theorem 2.5. *Let X be a proper Busemann space and let $w \in X$. Then $\partial_w X$ with the compact-open topology is compact and metrizable.*

Proof. Compactness follows from Ascoli's Theorem (Theorem 7.6.1 in [M]). To show metrizability, we will show that $\partial_w X$ is regular and second countable and so by the Urysohn Metrization Theorem (Theorem 4.1 in [M]) $\partial_w X$ will be metrizable.

First we note that $\partial_w X$ is Hausdorff because if $f \neq g$, there exists $p \in \mathbb{R}^+$ so that $f(p) \neq g(p)$. Pick $\epsilon < \frac{1}{4}|f(p) - g(p)|$; then it is easy to show that the subbasic open sets $S(\{p\}, B_\epsilon(f(p)))$ and $S(\{p\}, B_\epsilon(g(p)))$ are disjoint.

Since $\partial_w X$ is compact and Hausdorff we have that it is normal and hence regular. It remains to show that $\partial_w X$ is second countable. Since $\partial_w X$ is compact it suffices to show that $\partial_w X$ is first countable.

Let $f \in \partial_w X$. Order the rationals $r_1, r_2, r_3, \ldots$, and consider the subbasic sets:

$$\begin{array}{cccc} S(\{r_1\}, B_1(f(r_1))), & S(\{r_1\}, B_{\frac{1}{2}}(f(r_1))), & S(\{r_1\}, B_{\frac{1}{3}}(f(r_1))), & \ldots \\ S(\{r_2\}, B_1(f(r_2))), & S(\{r_2\}, B_{\frac{1}{2}}(f(r_2))), & S(\{r_2\}, B_{\frac{1}{3}}(f(r_1))), & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

This is countable by a Cantor diagonalization argument, so denote the resulting order by $S_1, S_2, \ldots$. Let $\mathcal{B}_f$ denote the set $\{U_1, U_2, \ldots\}$ where $U_n = \bigcap_{i=1}^n S_i$. Clearly $\mathcal{B}_f$, is countable; to see that it is a basis at f, let V be an open set containing f. Without loss of generality, we can assume that V is a subbasic open set as given by Proposition 2.4, i.e., $V = S(\{p\}, B_\epsilon(y))$. Since $f \in V$ we have that $|f(p) - y| < \epsilon$. Choose $n \in \mathbb{N}$ and $r \in \mathbb{Q}$ so that $\frac{1}{n} < \frac{\epsilon - |f(p) - y|}{2}$ and $|r - p| < \frac{1}{2n}$. Consider the subbasic open set $S_k = S(\{r\}, B_{\frac{1}{2n}}(f(r)))$; we will show $S_k \subset V$. Now $f \in S_k$ by definition, so let $g \in S_k$. Then

$$\begin{aligned} |g(p) - y| &< |g(p) - g(r)| + |g(r) - f(r)| + |f(r) - f(p)| + |f(p) - y| \\ &< \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n} + |f(p) - y| \\ &< \frac{3}{2}(\frac{\epsilon - |f(p) - y|}{2}) + |f(p) - y| \\ &= \frac{3}{4}\epsilon - \frac{3}{4}|f(p) - y| + |f(p) - y| \\ &= \frac{3}{4}\epsilon + \frac{1}{4}|f(p) - y| \\ &< \epsilon. \end{aligned}$$

Thus $S_k \subset V$; and since $U_k \subset S_k$, we have that $f \in U_k \subset V$, which shows that $\mathcal{B}_f$ is a countable basis at f and completes the proof that $\partial_w X$ is metrizable. □

Since both $\partial_{x_0} X$ and $\partial_{x_1} X$ are compact and metrizable, to prove that Φ is a homeomorphism we need only show that Φ is continuous.

The following lemma is key to the proof that Φ is a homeomorphism.

Lemma 2.6. *Let X be a proper Busemann space and let $\Phi : \partial_{x_0} X \to \partial_{x_1} X$ be the bijection given by Corollary 2.3. Suppose that $\{f_n\}$ is a sequence of rays in $\partial_{x_0} X$ that converges uniformly on compact sets to a ray $f \in \partial_{x_0} X$. Then, there is a subsequence $\{f_{n_k}\}$, such that the sequence $\{\Phi(f_{n_k})\}$ converges uniformly on compact sets to the ray $\Phi(f)$.*

Proof. Consider the sequence of rays $\{\Phi(f_n)\}$ in $\partial_{x_1} X$. Since $\partial_{x_1} X$ is equicontinuous and compact (by Theorem 2.5), there exists a subsequence $\{\Phi(f_{n_k})\}$ that converges uniformly on compact sets to a ray r, but since Φ is surjective, there exists a ray $g \in \partial_{x_0} X$ such that $\Phi(g) = r$. To show that $g = f$, it suffices to show that $\mathcal{H}(\Phi(g), \Phi(f)) < \infty$ since Φ is a bijection.

Let $t \in \mathbb{R}^+$ and choose $N \in \mathbb{N}$ such that for all $n_k \geq N$,

$$|f_{n_k}(t) - f(t)| \leq 1 \text{ and } |\Phi(f_{n_k})(t) - \Phi(g)(t)| \leq 1.$$

Then, for any $n_k \geq N$,

$$\begin{aligned} |\Phi(g)(t) - \Phi(f)(t)| &\leq |\Phi(g)(t) - \Phi(f_{n_k})(t)| + |\Phi(f_{n_k})(t) - f_{n_k}(t)| \\ &\quad + |f_{n_k}(t) - f(t)| + |f(t) - \Phi(f)(t)| \\ &\leq 1 + K + 1 + K = 2K + 2, \end{aligned}$$

where K is the constant in (2-2). Then since t was arbitrary,

$$|\Phi(g)(t) - \Phi(f)(t)| \leq 2K + 2$$

for all $t \in \mathbb{R}^+$ and so $\mathcal{H}(\Phi(f), \Phi(g)) < \infty$ as desired. □

We can now prove Theorem 2.1.

Proof of 2.1. As we noted above, it suffices to show that Φ is continuous. Let $f \in \partial_{x_0} X$ and let $S(\{p\}, B_\epsilon(w))$ be a subbasis element containing $\Phi(f)$ as given by Proposition 2.4. Without loss of generality, we may assume that $w = \Phi(f)(p)$. It suffices to show that there exist $q \geq 0$ and a $\delta > 0$ such that if $|g(q) - f(q)| < \delta$, then $|\Phi(g)(p) - \Phi(f)(p)| < \epsilon$, because then it follows that

$$\Phi(S(\{q\}, B_\delta(f(q)))) \subset S(\{p\}, B_\epsilon(\Phi(f)(p))).$$

Suppose that this is not the case. Then for each n, there exists a ray $g_n \in \partial_{x_0} X$ with $|f(n) - g_n(n)| \leq \dfrac{1}{n}$ but $|\Phi(f)(p) - \Phi(g)(p)| \geq \epsilon$. Since $\partial_{x_0} X$ is compact and equicontinuous, we have by Lemma 2.6 that there exist a subsequence $\{g_{n_k}\}$ and a ray $g \in \partial_{x_0} X$, such that $g_{n_k} \to g$ uniformly on compact sets and $\Phi(g_{n_k}) \to \Phi(g)$ uniformly on compact sets.

Claim. $f = g$.

Proof. Let $t \in \mathbb{R}^+$ and pick $n_r \geq t$. Since $g_{n_k} \to g$ pointwise, there exists $K \in \mathbb{N}$, such that for all $k \geq K$, $|g_{n_k}(n_r) - g(n_r)| \leq 1$. Now pick $n_m \geq \max\{K, n_r\}$. Then by Proposition 1.1,

$$|f(t) - g(t)| \leq |f(n_r) - g(n_r)|$$

and

$$|f(n_r) - g_{n_m}(n_r)| < |f(n_m) - g_{n_m}(n_m)|$$

since $t \le n_r \le n_m$. Thus,

$$\begin{aligned} |f(t) - g(t)| &\le |f(n_r) - g(n_r)| \\ &\le |f(n_r) - g_{n_m}(n_r)| + |g_{n_m}(n_r) - g(n_r)| \\ &< |f(n_m) - g_{n_m}(n_m)| + |g_{n_m}(n_r) - g(n_r)| \\ &< \frac{1}{n_m} + 1 < 2. \end{aligned}$$

As a result, $\mathcal{H}(f, g) < \infty$ which implies that $f = g$. □

Hence $\Phi(g_{n_k}) \to \Phi(g) = \Phi(f)$ uniformly on compact sets; but

$$|\Phi(f)(p) - \Phi(g_{n_k})(p)| \ge \epsilon$$

for all n_k which is a contradiction. Therefore, there exist $q \in \mathbb{R}^+$ and a $\delta > 0$ such that if $|g(q) - f(q)| < \delta$, then $|\Phi(f)(p) - \Phi(g)(p)| < \epsilon$, showing that Φ is continuous. □

We conclude by proving that if a proper Busemann space X is (Gromov) hyperbolic, then $\partial_{x_0} X$ is homeomorphic to the standard hyperbolic boundary (Corollary 2.9 below).

To this end, we first define the hyperbolic boundary. For details on (Gromov) hyperbolic spaces we refer the reader to [GH].

Definition. Let X be a (Gromov) hyperbolic space (not necessarily Busemann) and let $\mathcal{R}_{x_0} = \{f : \mathbb{R}^+ \to X \mid f(0) = x_0 \text{ and } f \text{ is an isometry }\}$. Define an equivalence relation on $\mathcal{R}_{x_0}$ via $f \sim g \Leftrightarrow \mathcal{H}(f, g) < \infty$. Then the *hyperbolic boundary* is given by $\partial_{x_0} X = \mathcal{R}_{x_0}/\sim$. We give $\mathcal{R}_{x_0}$ the compact-open topology and give $\partial_{x_0} X$ the quotient topology.

It seems to be well-known that if X is hyperbolic, then $\partial_{x_0} X$ with the quotient topology is homeomorphic to the standard hyperbolic boundary (see [GH] for the standard definition), but we have not found a proof of this in the literature. In the interest of completeness, we include a proof of this fact (Theorem 2.8 below).

In [Sw] Swenson describes a topology on $\mathcal{R}_{x_0}/\sim$ that makes it homeomorphic to the hyperbolic boundary. We provide his definitions and the result (Proposition 2.7 below) that we will need but we refer the reader to [Sw] for details.

Definition ([Sw])**.** Let $f \in \mathcal{R}_{x_0}$ and $t \in \mathbb{R}^+$. The *half-space*, $H(f, t)$, determined by f and t is the set

$$H(f, t) = \{x \in X \bigm| |x - f[t, \infty)| \le |x - f[0, t)|\}.$$

Set $H^-(f, t) = X - H(f, t)$ and call it the complementary half-space (see Figure 2).

Definition ([Sw])**.** Given a half-space $H(f, t)$, define an (open) disk in $\mathcal{R}_{x_0}/\sim$ by

$$D([f], t) = \left\{ [g] \bigm| \liminf_{s \to \infty} |g(s) - H^-(f, t)| = +\infty \right\}.$$

Proposition 2.7 ([Sw])**.** *These disks form a basis for a topology on $\mathcal{R}_{x_0}/\sim$ that is homeomorphic to the hyperbolic boundary.*

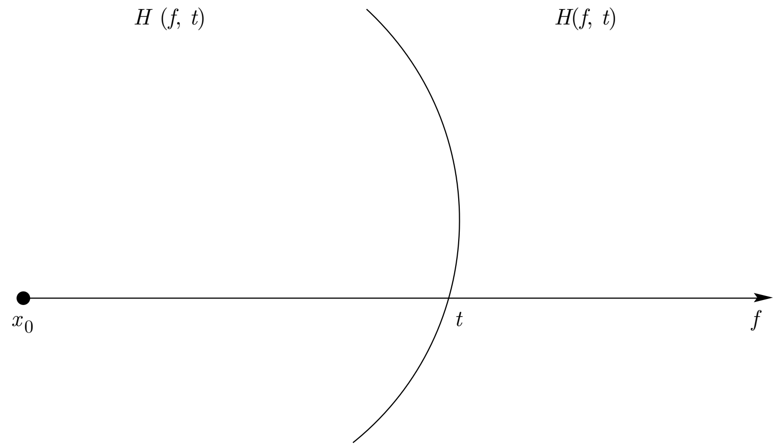

FIGURE 2

Theorem 2.8. *Let X be a δ-hyperbolic metric space (in the sense of Gromov). If $\partial_{x_0}X$ denotes $\mathcal{R}_{x_0}/\sim$ with the quotient topology defined above, and $\partial_{r_h}X$ denotes $\mathcal{R}_{x_0}/\sim$ with Swenson's topology, then $\partial_{x_0}X$ and $\partial_{r_h}X$ are homeomorphic.*

Proof. By Theorem 2.5, $\partial_{x_0}X$ is compact (and metrizable) and since $\partial_{r_h}X$ is metrizable, it is Hausdorff. Thus, to prove that $\partial_{x_0}X$ is homeomorphic to $\partial_{r_h}X$, it suffices to find a continuous surjection, φ, from $\mathcal{R}_{x_0}$ to $\partial_{r_h}X$, that is constant on each equivalence class $[f]$. Then, φ will be a quotient map, and by Theorem 3.11.2 in [M], φ will induce the desired homeomorphism from $\partial_{x_0}X$ to $\partial_{r_h}X$. Define $\varphi : \mathcal{R}_{x_0} \to \partial_{r_h}X$ by $\varphi(f) = [f]$. Clearly, φ is surjective since the elements of $\partial_{r_h}X$ are, by definition, the equivalence classes of elements in $\mathcal{R}_{x_0}$. Thus, it remains to show that φ is continuous. Let $[f] \in \partial_{r_h}X$ and let $D([f], t)$ be a neighborhood of $[f]$. Let $p = t + 24\delta$; then, clearly, $f(p) \in H(f, t + 24\delta)$. Therefore, by Lemmas 2.4 and 2.5 in [F], we have that $|f(p) - H^-(f, t + 8\delta)| > 4\delta$. Now pick $\epsilon < \delta$ and let $S = S(\{p\}, B_\epsilon(f(p)))$. Then, $f \in S$ and we claim that $\varphi(S) \subseteq D([f], t)$. To do this, we must show that for $g \in S$,

$$\liminf_{s\to\infty} |g(s) - H^-(f, t)| = +\infty.$$

Let $g \in S$; then $|g(p) - f(p)| < \epsilon < \delta$, and so $|g(p) - H^-(f, t + 8\delta)| > 3\delta$, which implies that $g(p) \in H(f, t+8\delta)$. It follows from Lemma 2.4 in [F] that if $s > p$, then $g(s) \in H(f, t+8\delta)$. Thus, by Lemma I.12 in [Sw], we have that $|g(s) - H^-(f, t)| > |g(s) - f(\mathbb{R}^+)| - 4\delta$. Now, if $g \not\sim f$, i.e $\varphi(g) \neq \varphi(f)$, then $\mathcal{H}(f, g) = \infty$ and so,

$$\liminf_{s\to\infty} |g(s) - H^-(f, t)| = +\infty.$$

Thus, φ is continuous as desired. □

Remark. If X is a simply connected, geodesic, hyperbolic metric space, then the metric is convex by Proposition 2.16 in [P]. Hence it is Busemann; thus geodesics diverge by Proposition 1.1 and so the equivalence relation on $\mathcal{R}_{x_0}$ is empty and we get the following corollary.

Corollary 2.9. *If a Busemann space X is also a (Gromov) hyperbolic (resp. non-positively curved) space, then $\partial_{x_0} X$ is homeomorphic to the hyperbolic (resp. non-positively curved) boundary.*

References

[Bo] B. H. Bowditch, *Minkowskian Subspaces of Non-positively Curved Subspaces*, preprint.

[Br] M. Bridson, *Geodesics and Curvature in Metric Simplicial Complexes*, Group Theory from a Geometrical Viewpoint (E. Ghys, A. Haefliger, and A. Verjovsky, eds.), Proceedings of ICTP, Trieste, Italy, World Scientific, 1991, pp. 373–463. MR **94c:**57040

[F] E. Freden, *Negatively Curved Groups have the Convergence Property.* I, II,Ann. Acad. Sci. Fenn. Math. **20** (1995), 333–348; **21** (1996), 133–150. MR **96g:**20054; CMP 96:08

[GH] E. Ghys and P. de la Harpe (eds.), *Sur le Groupes Hyperboliques d'après Mikhael Gromov*, Progress in Mathematics **83**, Birkhäuser, Boston, 1990. MR **92f:**53050

[G1] M. Gromov, *Hyperbolic Groups*, Essays on Group Theory (S.M. Gersten, ed.), MSRI Publications **8**, Springer-Verlag, 1987, pp. 75–263. MR **89e:**20070

[G2] ———, *Asymptotic Invariants of Infinite Groups*, Geometric Group Theory, Volume 2 (Graham A. Niblo and Martin Roller, eds.), London Mathematical Society Lecture Note Series **182**, Cambridge University Press, 1993. MR **95m:**20041

[M] J. Munkres, *Topology: A First Course*, Prentice Hall, Englewood Cliffs, 1975. MR **57:**4063

[NS] W. Neumann and M. Shapiro, *Equivalent Automatic Structures and Their Boundaries*, Inter. Jour. of Algebra and Comp. **2** (1992), 443–469. MR **93j:**20081

[P] F. Paulin, *Constructions of Hyperbolic Groups via Hyperbolization of Polyhedra*, Group Theory from a Geometrical Viewpoint (E. Ghys, A. Haefliger, and A. Verjovsky, eds.), Proceedings of ICTP, Trieste, Italy, World Scientific, 1991, pp. 313–372. MR **93d:**57005

[Sw] E. Swenson, *Negatively Curved Groups and Related Topics*, PhD Thesis, Brigham Young University (1993).

Department of Mathematics, The University of St. Thomas, St. Paul, Minnesota 55015

E-mail address: pkhotchkiss@stthomas.edu