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A PRIMITIVE RING WHICH IS A SUM OF TWO WEDDERBURN RADICAL SUBRINGS

A. V. KELAREV

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Abstract. We give an example of a primitive ring which is a sum of two Wedderburn radical subrings. This answers an open question and simplifies the proof of the known theorem that there exists a ring which is not nil but is a sum of two locally nilpotent subrings.

Kegel [3] proved that a ring is nilpotent if it is a sum of two nilpotent subrings. The question of whether every ring must be nil if it is the sum of two nil or locally nilpotent subrings was asked in [4] and was considered by several authors. Herstein and Small proved that a PI-ring is locally nilpotent if it is a sum of two nil subrings ([2, Theorem 2]). There was a conjecture that every ring is locally nilpotent if it is a sum of two locally nilpotent subrings (it is mentioned on p. 775 of [2]). Ferrero and Puczyłowski [1] proved, in particular, that every ring must be locally nilpotent if it is a sum of a right or left T-nilpotent subring and a locally nilpotent subring.

The interest in the question increased after Ferrero and Puczyłowski [1] had shown that the famous Koethe problem is equivalent to the fact that every ring is nil if it is a sum of a nil subring and a nilpotent subring. In the survey [6] Puczyłowski suggested that an example be sought of a ring which is not nil but is a sum of two nil subrings.

Indeed, in [5] the author constructed an example of a ring which is not nil but is a sum of two locally nilpotent subrings. A few years later a family of such examples using the semigroup of all partial translations was given by Salwa [7].

However, there is another related question which still remains open. Namely, Puczyłowski [6] asked whether a ring, which is a sum of two Wedderburn radical subrings, must be Baer radical. A ring is said to be *Wedderburn radical* if it is equal to the sum of its nilpotent ideals. Is is written in [6, p. 224] that the answer seems to be 'yes'.

The aim of the present note is twofold. First, we show that there exists a primitive (and so prime) ring which is a sum of two Wedderburn radical subrings. Thus, surprisingly, the answer to the open question above is in fact 'no'. Secondly, we seriously simplify the proof of the main theorem of [5] which answered the first long-standing question above.

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Theorem 1. *There exists a primitive ring which is a sum of two Wedderburn radical subrings.*

Proof. Following [5], let S be the free semigroup with two generators a and b. For $s \in S$, let $n_a(s)$ (respectively, $n_b(s)$) denote the number of letters a (respectively, b) in s. Put $d(s) = n_a(s) - n_b(s)\sqrt{2}$, $A = \{s \in S \mid d(s) > 0\}$, $B = S \backslash A$. Consider the ideal I generated in S by all s with $|d(s)| > 3$. Factoring out the ideal I, we put $\overline{S} = S/I$, $\overline{A} = A \cup I/I$, $\overline{B} = B \cup I/I$. Let $\mathbb{R}$ be the ring of real numbers. Consider the contracted semigroup ring $\mathbb{R}\overline{S}$. Clearly, $\mathbb{R}\overline{S} = \mathbb{R}\overline{A} + \mathbb{R}\overline{B}$, as in [5].

Take any element $0 \neq x \in \mathbb{R}\overline{A}$, say $x = \sum_{i=1}^{m} r_i s_i$, where $r_i \in \mathbb{R}$, $0 \neq s_i \in \overline{A}$. Denote by N the ideal generated in $\mathbb{R}\overline{A}$ by x, and put $q = 3/\min_{i=1}^{m} |d(s_i)|$. Then it is routine to verify that $N^q = 0$. Thus $\mathbb{R}\overline{A}$ is the sum of its nilpotent ideals. Similarly, $\mathbb{R}\overline{B}$ is a Wedderburn radical ring, too.

Let us inductively define a sequence of elements $t_1, t_2, \ldots \in \{a, b\}$. Put $t_1 = a$. Suppose that $t_1, \ldots, t_k$ have been defined. Let $t_{k+1} = a$ if $d(t_1 \cdots t_k) < 0$, and let $t_{k+1} = b$ otherwise. An easy induction on k shows that $-\sqrt{2} < d(t_1 \cdots t_k) < 1$, for all $k \geq 1$. Hence $t_1 \cdots t_k \neq 0$ in $\mathbb{R}\overline{S}$. As in the second paragraph of the proof of the main theorem of [5], it follows that $a + b$ is not nilpotent in $\mathbb{R}\overline{S}$, because $t_1 \cdots t_k$ is a summand of $(a+b)^k$. Thus $\mathbb{R}\overline{S}$ is not nil.

By the classical theorem of Amitsur the Jacobson radical of every finitely generated algebra over a nondenumerable field is nil. Hence $\mathcal{J}(\mathbb{R}\overline{S}) \neq \mathbb{R}\overline{S}$. Therefore there exists an ideal P of $\mathbb{R}\overline{S}$ such that $\mathbb{R}\overline{S}/P$ is a primitive ring. This completes the proof. □

Theorem 1 answers negatively all questions asked in [6, §2.4].

Note that the main result of [5] follows from our Theorem 1.

For any function $f : (A \cup B) \to \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, let I_f be the ideal generated in S by all products $s_1 \cdots s_k$ such that $k > \max\{1, f(s_1), \ldots, f(s_k)\}$ and either $\{s_1, \ldots, s_k\} \subseteq A$ or $\{s_1, \ldots, s_k\} \subseteq B$. It is proved in [5] that, for any function f, the contracted semigroup ring $\mathbb{R}(S/I_f)$ is the sum of two locally nilpotent rings $\mathbb{R}(A/I_f)$ and $\mathbb{R}(B/I_f)$, and that there exists a function g such that the ring $\mathbb{R}(S/I_f)$ is not nil for every function f satisfying $f(s) \geq g(s)$ for all $s \in A \cup B$.

If we take $f(s) = \max\{g(s), 3/|d(s)|\}$, for all $s \in A \cup B$, then clearly $I_f \subseteq I$, where I is the ideal used in the proof of Theorem 1. Thus the ring $\mathbb{R}(S/I)$ from the proof of our Theorem 1 is a homomorphic image of the ring $\mathbb{R}(S/I_f)$ introduced in [5].

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Department of Mathematics, University of Tasmania, G.P.O. Box 252 C, Hobart, Tasmania 7001, Australia
E-mail address: kelarev@hilbert.maths.utas.edu.au