GROTHENDIECK OPERATORS ON TENSOR PRODUCTS

P. DOMAŃSKI, M. LINDSTRÖM, AND G. SCHLÜCHTERMANN

(Communicated by Palle E. T. Jorgensen)

Abstract. We prove that for Banach spaces $E, F, G, H$ and operators $T \in \mathcal{L}(E, G), S \in \mathcal{L}(F, H)$ the tensor product $T \otimes S : E \otimes \varepsilon F \to G \otimes \varepsilon H$ is a Grothendieck operator, provided $T$ is a Grothendieck operator and $S$ is compact.

1. Introduction

J. Diestel and B. Faires proved in '76 that for Banach spaces $E, F, G, H$, for $T \in \mathcal{A}(E, G)$ and compact $S \in \mathcal{L}(F, H)$ the tensor product of $T$ and $S$ defined as $T \otimes S : E \otimes \varepsilon F \to G \otimes \varepsilon H$ belongs again to the operator ideal $\mathcal{A}$, provided $\mathcal{A}$ is closed and injective [DF]. For the ideal of weakly compact operators E. Saksman and H. O. Tylli [ST] have obtained similar results for both the projective and injective tensor product.

The mentioned results open a natural, and interesting in itself, question on stability of non-injective operator ideals with respect to injective tensor products. We solve this problem for the non-injective, closed ideal of Grothendieck operators. We are interested in exactly that ideal because the corresponding problem of tensor stability turns out to be closely related to the question of existence of complemented copies of $c_0$ in injective tensor products even for Fréchet spaces. In fact, it was shown [R, p. 98] that for a large class of Banach spaces $E$ (containing all $E = C(K)$) we have that $E$ is a Grothendieck space (that is, weak* and weak sequential convergence coincide on equicontinuous subsets) if and only if $E$ contains no complemented copy of $c_0$.

On the other hand, by a surprising result of Freniche [Fr1] (compare [C]), each completed injective tensor product $E \otimes \varepsilon F$ of a Fréchet space $E$ containing a copy of $c_0$ and a Fréchet space $F$ satisfying the Josefson-Nissenzweig type theorem (that is, weak* and strong convergence do not coincide for sequences in the dual) contains always a complemented copy of $c_0$. A fortiori, such a tensor product cannot be a Grothendieck space. All infinite dimensional Banach spaces satisfy the Josefson-Nissenzweig theorem and for Fréchet spaces it was proved by Bonet, Lindström and Valdivia [BLV] that this property exactly characterizes the non-Montel spaces.

This development leads to two natural questions: Let $E$ be a Fréchet space and $F$ a Fréchet-Montel space. When exactly does $E \otimes \varepsilon F$ contain a complemented copy of $c_0$ and when exactly is it a Grothendieck space? Both problems can also be interpreted in terms of tensor stability.

Received by the editors August 29, 1995 and, in revised form, January 9, 1996.

1991 Mathematics Subject Classification. Primary 47A80.
In case of $E = C(K)$ it follows immediately from results of Freniche [Fr2] (compare [DL, Cor. 3.7]) that $C(K, F)$ is a Grothendieck space if and only if $C(K)$ is a Grothendieck space which automatically implies that $C(K, F)$ contains a complemented copy of $c_0$ if and only if $C(K)$ contains a complemented copy of $c_0$.

For general injective tensor products, the known results are contained in [DL, Th. 2.3, Th. 3.6]:

(i) if $F$ has the approximation property, then $E \otimes F$ contains a complemented copy of $c_0$ if and only if $E$ contains a complemented copy of $c_0$.

(ii) if either $F$ or $E''$ has the approximation property, then $E \otimes F$ is a Grothendieck space if and only if $E$ is a Grothendieck space.

Using our injective tensor stability result for the ideal of Grothendieck operators we are able to remove the approximation property assumption from the second result when $E$ is a Banach space and $F$ is a Schwartz space.

Let us now fix some notations and definitions. $E, F, G, H$ are Banach spaces. $B(E)$ stands for the unit ball, while $E^*$ denotes the topological dual. By an operator $T$ from $E$ into $F$ we mean a bounded linear map. Let us call an operator $T \in \mathcal{L}(E, F)$ approximable if there exists a sequence of finite rank operators $(v_n) \subset \mathcal{L}(E, F)$ such that $\|T - v_n\| \to 0$ (cf. [Jh]).

Let us define the class $\mathcal{G}R(E, F)$ of all operators $T : E \to F$ such that for any pair of Banach spaces $E_1, F_1$ and any operator $T \in \mathcal{A}(E_1, F_1)$ the map $T \otimes S : E_1 \otimes c_0 \to F_1 \otimes c_0$ belongs to $\mathcal{A}$ as well. J. Diestel and B. Faires (see [DF, Th. 1 and Th. 2]) proved that $\mathcal{A}_0$ is always a closed operator ideal which is injective whenever $\mathcal{A}$ and $\alpha$ are injective. Analogously, it is easily seen that if $\mathcal{A}$ is surjective and $\alpha$ is projective, then $\mathcal{A}_0$ is surjective. Thus we obtain immediately:
Proposition 2.1. Let $E,F,G,H$ be Banach spaces, $\mathcal{A}$ be a closed operator ideal, $\alpha$ be a tensor norm and $\pi \in \mathcal{A}(E,G)$.

(a) If $S \in \mathcal{L}(F,H)$ is approximable, then $T \otimes S : E \otimes_{\alpha} F \to G \otimes_{\alpha} H$ is again in $\mathcal{A}$.
(b) If $\alpha$ and $\mathcal{A}$ are injective, then (a) holds even for compact $S$.
(c) If $\alpha$ is projective and $\mathcal{A}$ is surjective, then (a) holds even for compact $S$.

Proof. The ideal of approximable operators is the smallest closed operator ideal. Similarly, the ideal of compact operators is the smallest surjective (injective) closed operator ideal.

Since $\mathcal{GR}$ is surjective, we can state the following result.

Corollary 2.2. Let $E,F,G,H$ be Banach spaces. Then $T \otimes S : E \otimes_{\pi} F \to G \otimes_{\pi} H$ belongs to $\mathcal{GR}$ whenever $T$ is Grothendieck and $S$ is compact.

Remark 2.3. If we apply Theorem 2.2 and Remark 2.9 of [S] the following criterion of weak compactness in the dual space $(E \otimes_{\pi} F)^* = \mathcal{L}(E,F^*)$ can be obtained: Let $(T_n) \subset \mathcal{L}(E,F^*)$ be a bounded sequence. Then $T_n \overset{w}{\rightharpoonup} 0$ weakly if and only if $\{(T_n(x),y^{**})\}_{n \in \mathbb{N}} : x \in B(E), y^{**} \in B(F^{**}) \}$ is relatively weakly compact.

This result or Theorem 1 in [K] can be used to give a direct proof of the above corollary. It also follows from the proof that $E \otimes_{\pi} F$ is a Grothendieck space if $E$ is a Grothendieck space, $F$ is reflexive and every operator from $E$ into $F^*$ is compact.

At this stage we mention that from P. Enflo’s famous example [E] it is an easy consequence that there is a Banach space $E$ for which there is a non-approximable but compact operator from $E$ into itself. In [A] F. A. Alexander obtained a similar result for a closed subspace $E$ of $l^p$ when $2 < p < \infty$.

The ideal of Grothendieck operators is not injective. Thus our main aim is to improve 2.1 in that case and to obtain injective tensor stability with compact operators. First we reduce the problem to reflexive $F$ and $H$.

Lemma 2.4. Let $E,F,G,H$ be Banach spaces, $T \in \mathcal{L}(E,G)$ and $S \in \mathcal{L}(F,H)$ is compact. Then there exist reflexive Banach spaces $G_1, H_1$ and operators $S_1 \in \mathcal{L}(E,G_1), S_2 \in \mathcal{L}(G_1,H_1), S_3 \in \mathcal{L}(H_1,H)$, such that

$$T \otimes S = (id_G \otimes S_3) \circ (T \otimes S_2) \circ (id_E \otimes S_1).$$

Proof. Every compact $S \in \mathcal{L}(F,H)$ admits a compact factorization through a reflexive Banach space according to a result of T. Figel and W. Johnson [Fi, Jo] (see also [DU, p. 260]). Then the proof is straightforward.

We write $Bo(B(E^*))$ for the Borel sets on $B(E^*)$ w.r.t. the $w^*$-topology. If $m : Bo(B(X^*)) \to F$ is a vector measure of bounded variation, then $\|m\|$ is the variation norm. Let us recall the representation of the dual of $E \otimes_{\pi} F$, provided $F$ is reflexive.

Definition and Lemma 2.5. Let $E,F$ be Banach spaces with $F$ reflexive. $\mathcal{PI}(E,F) \subset \mathcal{L}(E,F)$ are the Pietsch-integral operators, defined as:

$$T \in \mathcal{PI}(E,F) \iff \exists m : Bo(B(E^*)) \to F \text{ vector measure of bounded variation}$$

$$\forall x \in E : T(x) = \int_{B(E^*)} x(x^*) \, dm(x^*).$$
We equip $\mathcal{P}(E,F)$ with the integral norm, i.e. $\|T\|_{\mathcal{P}F} := \inf \{ \|m\| ; \forall x \in E : T(x) = \int_{B(E^*)} x(x^*) \, dm(x^*) \}$ (cf. [DFI, p. 522]). Then $\mathcal{P}(E,F^*)$ is isometric isomorphic to $(E \otimes F)^*$ by the identity $T(x \otimes y) = \langle y, \int_{B(E^*)} x(x^*) \, dm(x^*) \rangle$ (cf. [DFI, p. 522]).

**Notation.** Let $E, F$ be Banach spaces with $F$ reflexive, and let $(z_n^*) \subset B((E \otimes F)^*)$. According to 2.5 for all $n \in \mathbb{N}$ we choose a vector measure $m_n := m_n(z_n^*) : Bo(B(X^*)) \rightarrow F$ of bounded variation, satisfying:

i) $\lim_{n \rightarrow \infty} \|m_n\| - \|z_n^*\| = 0$,

ii) $\forall e \in E, f \in F, n \in \mathbb{N} : z_n^*(e \otimes f) = (f, \int_{B(E^*)} e(x^*) \, dm_n(x^*))$.

Furthermore we define a finite scalar-valued measure $\mu(\cdot) := \mu((z_n^*))(\cdot) := \sum_{n \in \mathbb{N}} 2^{-n} \text{var}(m_n(z_n^*), \cdot)$, where $\text{var}$ denotes the variation of the corresponding measure. Then $m_n$ is absolutely continuous w.r.t. $\mu$ for all $n \in \mathbb{N}$.

We write

$$L_{H_1} := \left\{ f \in L_1(\mu, H_1^*) : \forall e \in E : \int_{B(E^*)} e(x^*) f(e^*) \, dm(e^*) = 0 \right\}$$

for a subspace $H_1 \subset H$. For a Banach space $H$ we denote by $q_H : L_1(\mu, H^*) \rightarrow L_1(\mu, H^*)/L_H$ the canonical quotient map. If $\mu = \mu(z_n^*)$ and $H_1 \subset H$, then let $r_{H_1} : L_1(\mu, H_1^*)/L_{H_1} \rightarrow (E \otimes \varrho H_1)^*$ be the canonical injection.

**Theorem 2.6.** Let $E, F, G, H$ be Banach spaces. If $T \in \mathcal{G}(E,G)$ and $S \in \mathcal{L}(F,H)$ is compact, then $T \otimes S : E \otimes x F \rightarrow G \otimes x H$ is Grothendieck.

**Proof.** By Lemma 2.4 we assume that $F,H$ are reflexive. W.l.o.g. let $\|T\|, \|S\| \leq 1$.

Let $(x_n^*) \subset B((G \otimes H)^*)$ be $w^*$-converging to 0. First we consider the map $T \otimes id : E \otimes \varrho H \rightarrow G \otimes \varrho H$. For a finite dimensional subspace $H_1 \subset H$, according to 2.1, we have that for $(z_n^*) := (T \otimes id)^*(x_n^*)$ the restriction

$$\langle (z_n^*)_{E \otimes H_1} \rangle \overset{n \rightarrow \infty}{\longrightarrow} 0 \text{ weakly.}$$

Consider now $id \otimes S : E \otimes x F \rightarrow E \otimes x H$. For $n \in \mathbb{N}$ let $h_n \in L_1(\mu, H^*)$ be the density of $m_n$ with respect to $\mu := \mu(z_n^*)$. To show that $(id \otimes S)^*((z_n^*))$ is weakly null (then we are done), we have to show that for all $\overline{g} \in B((E \otimes x F)^*)$:

$$\langle \overline{g}(id \otimes S)^*((z_n^*)) \rangle = \int_{B(E^*)} \langle q_{F^*} \circ r_{F^*}(\overline{g}), S^* \circ h_n \rangle \, d\mu$$

$$= \int_{B(E^*)} \langle S \circ q_{F^*} \circ r_{F^*}(\overline{g}), h_n \rangle \, d\mu \rightarrow 0. \quad (2)$$

The following arguments are devoted to proving this. We define $g := S \circ q_{F^*} \circ r_{F^*}(\overline{g})$. Then $g \in L_\infty(\mu, H)$, since $H$ is reflexive. Further, $g$ has relatively compact range, since $S$ is compact. We assume that (2) is not true. Then

$$\exists(h_{n_k}) \text{ subsequence } \exists \varepsilon > 0 : \inf_{k \in \mathbb{N}} \left| \int_{B(E^*)} \langle g, h_{n_k} \rangle \, d\mu \right| > \varepsilon. \quad (3)$$

For the sake of simplicity assume that $(h_n)$ satisfies (3). Since $g$ has relatively compact range, there is an increasing sequence of finite $Bo(B(E^*))$-partitions $(\pi_k)$, such that

$$\|E_{\pi_k}(g) - g\|_{\infty} \rightarrow 0 \text{ and } \forall n \in \mathbb{N} : \|E_{\pi_k}(h_n) - h_n\|_1 \rightarrow 0. \quad (4)$$
We define $\Sigma_0 := \sigma(\bigcup_{k \in \mathbb{N}} \pi_k)$. Since $H$ is reflexive, for all $k \in \mathbb{N}$ the sequence $(E_{\pi_k}(h_n))$ is relatively weakly compact in $L_1(\mu, H^*)$. Hence, for all $k \in \mathbb{N}$ there is an $m_k \in L_1(\mu, H^*)$, so that $E_{\pi_k}(h_n) \to m_k$ weakly (for at least going to a subsequence by a diagonalization argument). $(\pi_k)$ is increasing, thus, $(m_k, \pi_k)$ is a bounded martingale, which converges in the $L_1(\mu, H^*)$-norm to an $M \in L_1(\mu, H^*)$ (note that the $(h_n)$ are bounded and $H^*$ has the RNP as a reflexive space). We show now that for all $G \in L_\infty(\mu|_{\Sigma_0}, H)$ with relatively compact range:

\[ \exists \text{ subsequence } (h_{n_j}) \text{ such that } \forall \delta > 0 \exists N \in \mathbb{N} \forall j \geq N : |\langle G, M \rangle - \langle G, h_{n_j} \rangle| < \delta. \]

**Proof of (5).** $G$ has relatively compact range, thus there exists an increasing sequence of finite $Bo(B(E^\ast), w^\ast)$-partitions $(\pi_k(G))$ such that $\pi_k \subset \sigma(\pi_k(G))$ for $k \in \mathbb{N}$ and $\|G - E_{\pi_k(G)}(G)\| \to 0$ (cf. [DU, p. 67, Lemma 1]). Let $(h_{n_j})$ be a subsequence with $E_{\pi_k(G)}(h_{n_j}) \to m_k(G) \in L_1(\mu, H^*)$ weakly for all $k \in \mathbb{N}$ (subsequence argument like above). Thus, since $\pi_k \subset \sigma(\pi_k(G))$, it follows $E_{\pi_k(G)}(E_{\pi_k(G)}(h_{n_j})) \to m_k(G)$ (as above). Again $(m_k(G))$ is a martingale. Hence, as above, there exists an $M(G) \in L_1(\mu, H^*)$, such that $m_k(G) \to M(G)$. We have

\[ M = M(G). \]

Let $A \in \bigcup_{k \in \mathbb{N}} \pi_k$. Then

\[ \int_A M(G) \, d\mu - \int_A M \, d\mu = \lim_{k \to \infty} \int_A m_k(G) - m_k \, d\mu = 0, \]

since $(m_k(G))$ and $(m_k)$ are martingales and there is a $k_0 \in \mathbb{N}$, such that $A \in \pi_{k_0} \subset \sigma(\pi_k(G))$. Hence, for all $B \in \Sigma_0 : \int_B M(G) \, d\mu = \int_B M \, d\mu$. Thus to prove (5) we first note that it suffices to demonstrate (5) for all $G = E_{\pi_k(G)}(G)$ ($k \in \mathbb{N}$), since $G$ has relatively compact range and $M, h_n, n \in \mathbb{N}$, are measurable w.r.t. $\Sigma_0$. But then (5) follows by:

\[ \| \langle E_{\pi_k(G)}(G), M \rangle - \langle E_{\pi_k(G)}(G), h_{n_j} \rangle \| = \| \langle E_{\pi_k(G)}(G), m_k(G) \rangle - \langle E_{\pi_k(G)}(G), E_{\pi_k(G)}(h_{n_j}) \rangle \| = \| \langle E_{\pi_k(G)}(G), m_k(G) - E_{\pi_k(G)}(h_{n_j}) \rangle \|. \]

For a finite dimensional subspace $H_1 \subset H$ we consider the canonical restriction operator $\text{rest}_{H_1} : L_1(\mu, H^*)/L_{H^*} \to L_1(\mu, H_1^*)/L_{H_1^*}$. Then according to (1) we have:

\[ \forall z^{**} \in (E \otimes_c H_1)^{**} : \int_{B(E^\ast)} \langle q_H^* \circ \text{rest}_{H_1} \circ q_{H_1}^*(z^{**}), h_n \rangle \, d\mu \to 0. \]

Since $q_H^* \circ \text{rest}_{H_1} \circ q_{H_1}^*(z^{**})$ has relatively compact range for all $z^{**} \in (E \otimes_c H_1)^{**}$ ($H_1$ is finite dimensional), (5) and (6) imply:

\[ \forall z^{**} \in (E \otimes_c H_1)^{**} : \langle z^{**}, r_{H_1} \circ \text{rest}_{H_1} \circ q_{H}(M) \rangle = 0. \]

Note that since $M \in L_1(\mu|_{\Sigma_0}, H^*)$ we may assume that $q_H^* \circ \text{rest}_{H_1} \circ q_{H_1}^*(z^{**})$ is measurable w.r.t. $\Sigma_0$. Thus

\[ \forall H_1 \subset H \text{ finite dimensional } r_{H_1} \circ \text{rest}_{H_1} \circ q_{H}(M) = 0. \]

But (7) implies

\[ r_H \circ q_{H}(M) = 0. \]
Hence we compute
\[ 0 \langle (id \otimes S)^* (\mathfrak{g}), r_H \circ q_H (M) \rangle = \langle \mathfrak{g}, (id \otimes S)^* (r_H \circ q_H (M)) \rangle \]
\[ = \int_{B(E^*)} \langle q^*_F \circ r^*_F (\mathfrak{g}), S^* \circ M \rangle \, d\mu = \int_{B(E^*)} \langle S \circ q^*_F \circ r^*_F (\mathfrak{g}), M \rangle \, d\mu \]
\[ = \int_{B(E^*)} \langle g, M \rangle \, d\mu. \]
Thus, this contradicts (3) and (5), and we are done.

We shall now apply Theorem 2.6 and an operator ideal approach to obtain the announced result avoiding the assumption of the approximation property.

**Corollary 2.7.** Let \( E \) be a Schwartz space and \( F \) a Banach space with the Grothendieck property. Then \( E \otimes_\varepsilon F \) is a Grothendieck space.

**Proof.** By a well-known representation of \( \varepsilon \)-tensor products as projective limits \( E \otimes_\varepsilon F = \text{proj}_{\varepsilon} \{ E_U \otimes_\varepsilon F \} \), where \( U_E \) is a \( 0 \)-basis in \( E \). A locally convex space \( X \) is Grothendieck if and only if every continuous linear map from \( X \) into \( c_0 \) maps bounded sets into relatively weakly compact ones. Now, each continuous linear map \( T : E \otimes_\varepsilon F \to c_0 \) factorizes through \( E_U \otimes_\varepsilon F \) for some \( U \in U_E \). Since \( E \) is a Schwartz space we can apply our main theorem so that for every \( U \in U_E \) there exists a \( V \in U_E \) contained in \( U \) such that the canonical map \( E_V \otimes_\varepsilon F \to E_U \otimes_\varepsilon F \) is a Grothendieck operator. The result follows immediately.

**References**


[C] P. Cembranos, \( C(K, E) \) contains a complemented copy of \( c_0 \), Proc. Amer. Math. Soc. 91 (1984), 556–558. MR 85g:46025


Department of Mathematics, A. Mickiewicz University, 60-769 Poznań, Poland
E-mail address: domanski@math.amu.edu.pl

Department of Mathematics, Åbo Akademi University, FIN-20500 Åbo, Finland
E-mail address: mikael.lindstrom@abo.fi

Mathematisches Institut der Universität München, Theresienstrasse 39, D-80333 München, Germany
E-mail address: schluech@rz.mathematik.uni-muenchen.de