PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 8, August 1997, Pages 2471–2479
S 0002-9939(97)03786-6

SELF-CONTRAGREDIENT SUPERCUSPIDAL REPRESENTATIONS OF GL_n

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(Communicated by Roe W. Goodman)

Abstract. Let F be a non-archimedean local field of residual characteristic p. Then $\mathrm{GL}_n(F)$ has tamely ramified self-contragredient supercuspidal representations if and only if n or p is even. When such representations exist, they do so in abundance.

Suppose G is a symplectic or split special orthogonal group over a p-adic field F (of either zero or positive characteristic), and P is a maximal parabolic subgroup of G with Levi factor M isomorphic to $\mathrm{GL}_n(F)$. If ρ is a unitary supercuspidal representation of M, then we can form the induced representation $\pi_\rho = \mathrm{Ind}_P^G \rho$ of G. In order for π_ρ to be reducible, it is a necessary condition that ρ be self-contragredient. (See [12], which also investigates sufficiency conditions when F has characteristic zero.) For if π_ρ is reducible, this implies that the nontrivial element w of the Weyl group of M in G fixes the isomorphism class of ρ. However, w acts on M by $m \longmapsto {}^t m^{-1}$, and so by a result of Gelfand and Kazhdan [3, Theorem 2], ρ is self-contragredient.

Thus, it is of interest to know which p-adic general linear groups have self-contragredient supercuspidal representations, and to have examples.

We exploit a construction of supercuspidal representations due to Howe [4], and call the resulting representations *tamely ramified*. (Moy [7] has shown that if p and n are relatively prime and F has characteristic zero, then all supercuspidals of $\mathrm{GL}_n(F)$ are tamely ramified.) Since these representations are constructed from "admissible" characters of extension fields, it is necessary to examine the structure of these fields and their characters in some detail.

In §2, we write down the p-adic analogue of polar coordinates. In the two-dimensional, ramified case, this is due to Gelfand-Graev [2] and Sally [10]. The p-adic analogue of the unit circle comes equipped with a natural filtration, which we study in §3. Since an extension field E of F can have many intermediate subfields, E can have many polar decompositions, all of which we need to consider. We compare them in §4.

In §5, we give necessary and sufficient conditions for a character to parametrize a self-contragedient supercuspidal representation. This allows us to prove our main result, Theorem 6.1, which implies that $\mathrm{GL}_n(F)$ has tamely ramified self-contragredient supercuspidal representations if and only if n or p is even, in which

Received by the editors December 1, 1995 and, in revised form, February 12, 1996.
1991 *Mathematics Subject Classification.* Primary 22E50; Secondary 20G05, 11F70.

case one can attach such representations to most tamely ramified extensions E/F of degree n.

For the sake of explicitness, we count the examples of depth zero in §7.

This leaves open the question of whether self-contragredient supercuspidal representations (not tamely ramified) exist when p and n are odd, and $p|n$. In order to find the answer, one has to deal with wildly ramified extension fields of F. But in this situation, the supercuspidal representations of $\mathrm{GL}_n(F)$ are no longer parametrized by admissible characters. Therefore, although much of our study of admissible characters carries over (in a more complicated form) to the wild case, we have restricted ourselves to the tame case wherever convenient.

I wish to thank Paul Sally and Mark Reeder for very helpful discussions, and David Goldberg for clarifying for me the connection between self-contragredience and reducibility.

1. Notation and conventions

For any p-adic field F, let ν_F denote its normalized valuation, k_F its residue field, and q_F the order of k_F. We denote the prime ideal in F by $\wp_F$, and a uniformizing element by ϖ_F. We let U_F be the group of units, and let $U_{F,i} = 1 + \wp_F^i$ for all $i \in \mathbf{N}$. We can (and will) identify the multiplicative group $k_F^\times$ with the group of roots of unity in F of order prime to p.

By a *character* of $F^\times$, we mean a continuous homomorphism $F^\times \longrightarrow \mathbf{C}^\times$.

For any finite-dimensional field extension E/F, let $C_{E/F}$ denote the kernel of the norm map $\mathrm{N}_{E/F}$ from E to F. For $i > 0$, let $C_{E/F,i} = C_{E/F} \cap U_{E,i}$. Let $C_{E/F,0} = C_{E/F}$. We will sometimes write $C_{E/F}^{(0)}$ for $C_{E/F,1}$.

As usual, we let $e(E/F)$ denote the ramification degree of E/F. The extension E/F is *tamely ramified* (or *tame*) if $e(E/F)$ is relatively prime to p.

There is a canonical decomposition

$$C_{E/F} = \overline{C}_{E/F} \times C_{E/F}^{(0)}, \tag{1.1}$$

where

$$\overline{C}_{E/F} = \left\{ x \in k_E \;\middle|\; \mathrm{N}_{k_E/k_F}(x^{e(E/F)}) = 1 \right\}. \tag{1.2}$$

If A and B are elements or subsets of a group, then $\langle A, B\rangle$ denotes the subgroup generated by A and B.

2. Polar decomposition

Let E/F be a finite extension of p-adic fields. Then $E^\times$ is almost a direct product of $F^\times$ and $C_{E/F}$. The purpose of this section is to make this statement more precise in two special cases.

Lemma 2.1. *Let E/F be an extension of degree n. Then the norm map $\mathrm{N}_{E/F}$ induces an isomorphism*

$$U_{E,1}/C_{E/F}^{(0)} \cdot U_{F,1} \xrightarrow{\sim} \mathrm{N}_{E/F}(U_{E,1})/(U_{F,1})^n.$$

Proof. Let $x \in U_{E,1}$, and suppose $\mathrm{N}_{E/F}(x) \in (U_{F,1})^n$. Pick $y \in U_{F,1}$ such that $y^n = \mathrm{N}_{E/F}(x)$. Then $xy^{-1} \in C_{E/F}^{(0)}$, so $x \in C_{E/F}^{(0)} \cdot U_{F,1}$.

The surjectivity of the map is clear. □

Note that if E/F is tame, then $\mathrm{N}_{E/F}(U_{E,1}) = U_{F,1}$, as we will see from Lemma 3.1. If, furthermore, $(n,p)=1$, then $(U_{F,1})^n = U_{F,1}$ and $C^{(0)}_{E/F} \cap U_{F,1} = \{1\}$, so we have a direct product decomposition

$$U_{E,1} = C^{(0)}_{E/F} \times U_{F,1}.$$

Proposition 2.2. *Let E/F be an unramified extension of degree n. Then $\mathrm{N}_{E/F}$ induces an isomorphism*

$$E^\times / C^{(0)}_{E/F} \cdot k_E^\times F^\times \xrightarrow{\sim} \mathrm{N}_{E/F}(E^\times)/k_F^\times (F^\times)^n \cong U_{F,1}/(U_{F,1})^n.$$

Proof. Similar to that of the lemma. □

In particular, if $(n,p)=1$, we get a direct product decomposition

$$E^\times = C^{(0)}_{E/F} \times k_E^\times F^\times.$$

Proposition 2.3. *Let E/F be a totally and tamely ramified extension of degree n. Write $E = F(\varpi_E)$, where ϖ_E^n is a prime in F. Then*

$$E^\times = C^{(0)}_{E/F} \times \langle F^\times, \varpi_E \rangle.$$

Proof. Similar to that of the lemma. □

3. The filtration $\{C_{E/F,i}\}_{i=0}^\infty$

We want to find the successive quotients of this filtration. We start by recalling how $\mathrm{N}_{E/F}$ behaves with respect to the filtration $\{U_{E,i}\}$.

Lemma 3.1. *Let E/F be a tamely ramified extension of ramification degree e. Then*

$$\mathrm{N}_{E/F}(U_{E,i}) = U_{F,[(i-1)/e]+1}.$$

Proof. If E/F is unramified, then this is just [11, V, Prop. 3]. If E/F is Galois and totally ramified, then this is a special case of [11, V, §6, Cor. 3]. Therefore, the proposition is true if the ramified part of E/F is Galois.

By adjoining roots of unity to E, we can obtain a field E' such that E'/E is unramified and E'/F (and thus its ramified part) is Galois. Then, from the previous paragraph,

$$\mathrm{N}_{E'/E}(U_{E',i}) = U_{E,i}$$

and

$$\mathrm{N}_{E'/F}(U_{E',i}) = U_{F,[(i-1)/e]+1},$$

so

$$\begin{aligned}\mathrm{N}_{E/F}(U_{E,i}) &= \mathrm{N}_{E/F}\,\mathrm{N}_{E'/E}(U_{E',i}) \\ &= \mathrm{N}_{E'/F}(U_{E',i}) \\ &= U_{F,[(i-1)/e]+1}. \quad \square\end{aligned}$$

Proposition 3.2. *For any extension E/F and any $i>0$,*

$$\left|\frac{C_{E/F,i}}{C_{E/F,i+1}}\right| = q_E \cdot \left|\frac{\mathrm{N}_{E/F}(U_{E,i})}{\mathrm{N}_{E/F}(U_{E,i+1})}\right|^{-1}.$$

For $i=0$, replace q_E by $q_E - 1$.

Proof. For any $i \geq 0$, we have a commutative diagram

$$\begin{array}{ccc} U_{E,i+1}/C_{E/F,i+1} & \hookrightarrow & U_{E,i}/C_{E/F,i} \\ \downarrow & & \downarrow \\ \mathrm{N}_{E/F}(U_{E,i+1}) & \hookrightarrow & \mathrm{N}_{E/F}(U_{E,i}) \end{array}$$

where the horizontal arrows are induced by inclusion, and the vertical arrows are isomorphisms induced by the norm map. Thus,

$$\left|\frac{\mathrm{N}_{E/F}(U_{E,i})}{\mathrm{N}_{E/F}(U_{E,i+1})}\right| = \left|\frac{U_{E,i}}{U_{E,i+1}}\right| \cdot \left|\frac{C_{E/F,i}}{C_{E/F,i+1}}\right|^{-1},$$

which implies our conclusion. ☐

Corollary 3.3. *If E/F is tame and $i > 0$, then*

$$\left|\frac{C_{E/F,i}}{C_{E/F,i+1}}\right| = \begin{cases} q_E/q_F & \text{if } e(E/F)|i, \\ q_E & \text{otherwise.} \end{cases}$$

Also,

$$\left|\frac{C_{E/F,0}}{C_{E/F,1}}\right| = \frac{q_E - 1}{q_F - 1} \cdot (e, q_F - 1).$$

Proof. The first statement is immediate from Lemma 3.1 and Proposition 3.2, and the second follows from the latter and (1.2). ☐

4. The family of subgroups $\{C_{E/L,i}\}_{E\supset L\supseteq F}$

Lemma 4.1. *If $E/L\cap L'$ is separable, then* $\ker \mathrm{Tr}_{E/L} + \ker \mathrm{Tr}_{E/L'} = \ker \mathrm{Tr}_{E/L\cap L'}$.

This actually makes sense for *any* field E (p-adic or not) and any subfields L and L' such that $E/(L \cap L')$ is finite-dimensional and separable.

Proof. $(x, y) \longmapsto \mathrm{Tr}_{E/L\cap L'}(xy)$ is a nondegenerate, symmetric, $L\cap L'$-bilinear form on E. For any subset $S \subset E$, let

$$S^{\perp} = \left\{ x \in E \,\middle|\, \mathrm{Tr}_{E/L\cap L'}(xs) = 0 \text{ for all } s \in S \right\}.$$

Then for any intermediate field $E \supset K \supset L \cap L'$,

$$\begin{aligned} \ker \mathrm{Tr}_{E/K} &= \left\{ x \,\middle|\, \mathrm{Tr}_{E/K}(x) = 0 \right\} \\ &= \left\{ x \,\middle|\, K \cdot \mathrm{Tr}_{E/K}(x) = 0 \right\} \\ &= \left\{ x \,\middle|\, K \cdot \mathrm{Tr}_{E/K}(x) \subset \ker \mathrm{Tr}_{K/L\cap L'} \right\} \\ &\qquad \text{(since } K \cdot \mathrm{Tr}_{E/K}(x) \text{ is a } K\text{-subspace of } K) \\ &= \left\{ x \,\middle|\, \mathrm{Tr}_{E/L\cap L'}(xK) = 0 \right\} \\ &= K^{\perp}. \end{aligned}$$

Therefore, we need to show $L^{\perp} + L'^{\perp} = (L \cap L')^{\perp}$, which is elementary. ☐

Proposition 4.2. *Suppose that $E/L \cap L'$ is tame, and $i > 0$. Then*

$$C_{E/L,i} \cdot C_{E/L',i} = C_{E/L\cap L',i}.$$

Proof. It is clear that the left-hand side is contained in the right. Let $c = c_0 \in C_{E/L\cap L',i}$. Then $c = 1+x_1$, with $x_1 \in \wp_E^i$, and $\mathrm{N}_{E/L\cap L'}(c) = 1$. But $\mathrm{N}_{E/L\cap L'}(c) \equiv 1 + \mathrm{Tr}_{E/L\cap L'}(x_1) \bmod U_{E,2i}$, so $\mathrm{Tr}_{E/L\cap L'}(x_1) \equiv 0 \bmod \wp_E^{2i}$. From the lemma and (3.1), we can find $t_1 = 1 + y_1$ and $t_1' = 1 + y_1'$ in $C_{E/L}$ and $C_{E/L'}$, respectively, so that $y_1 + y_1' \equiv x_1 \bmod \wp_E^{2i}$. Therefore, $t_1 t_1' \equiv c_0 \bmod C_{E/L\cap L',2i}$. Let $c_1 = c_0 t_1^{-1} t_1'^{-1} \in C_{E/L\cap L',2i}$. Repeating this process, we may write

$$c = \prod_{j=1}^{\infty} t_j \cdot \prod_{j=1}^{\infty} t_j',$$

where the infinite products converge, and they lie in $C_{E/L,i}$ and $C_{E/L',i}$ respectively. □

5. Definition of admissible and self-contragredient characters

Let E/F be a finite extension of p-adic fields. For any character θ of $E^\times$, define the *level* of θ to be the smallest nonnegative integer i such that $\theta|_{U_{E,i}}$ is trivial.

Definition 5.1. A character θ of $E^\times$ is *admissible over* F if

(1) $\theta|_{C_{E/L}}$ is nontrivial for all $E \supset L \supseteq F$, and
(2) $\theta|_{C_{E/L}^{(0)}}$ is nontrivial for all $E \supset L \supseteq F$ such that E/L is ramified.

Note that in the definition we may restrict ourselves to *maximal* subfields L.

If E/F is a tamely ramified extension of degree n and θ is an admissible character of $E^\times$, then let π_θ denote the supercuspidal representation of $\mathrm{GL}_n(F)$ that arises from θ via the Howe construction. (For details, see [4] or [7]. While the latter uses a blanket assumption that $(n,p) = 1$, the section devoted to the construction of supercuspidal representations works for any tamely ramified E.) From [7], we know that all supercuspidal representations of $\mathrm{GL}_n(F)$ arise in this way if $(n,p) = 1$ and F has characteristic zero.

Two characters θ and θ' of $E^\times$ and $E'^\times$ (respectively) are *conjugate* if there is an F-isomorphism $\sigma\colon E \longrightarrow E'$ such that $\theta = \theta' \circ \sigma$ (sometimes denoted θ'^σ).

Proposition 5.2 (Howe)**.** *If θ and θ' are admissible characters of $E^\times$ and $E'^\times$, respectively, then $\pi_\theta \cong \pi_{\theta'}$ if and only if θ and θ' are conjugate.*

Proposition 5.3. *Let E/F be tame, and let θ be an admissible character of $E^\times$. Then π_θ is self-contragredient if and only if one of the following conditions holds:*

(a) *there is some $E \supset L \supseteq F$ such that $[E:L] = 2$ and $\theta|_{\mathrm{N}_{E/L}(E^\times)}$ is trivial;*
(b) *$p = 2$ and θ has order two.*

Proof. Note that $\widetilde{\pi}_\theta = \pi_{\theta^{-1}}$. In the case where θ has level one, this follows from the analogous fact for Deligne-Lusztig virtual representations [5]. Otherwise, it follows from the fact that π_θ is (unitarily) induced from an extension of θ to a subgroup of $\mathrm{GL}_{[E:F]}(F)$ containing an embedded image of $E^\times$. (See [4] for details.)

From Proposition 5.2, $\widetilde{\pi}_\theta \cong \pi_\theta$ if and only if $\theta^{-1} = \theta^\sigma$ for some $\sigma \in \mathrm{Aut}_F(E)$. In particular, $\theta^{\sigma^2} = \theta$.

Suppose σ is nontrivial and has odd order. Then $\theta^{\sigma^2} = \theta$ implies that $\theta = \theta^\sigma$, so θ is trivial on $\{\sigma(x)/x \mid x \in E^\times\}$, which equals C_{E/E^σ} by Hilbert's Theorem 90. But this contradicts the admissibility of θ.

Suppose that σ is nontrivial and has even order. Then $\theta = \theta^{\sigma^2}$ so, by reasoning similar to that used above, θ is trivial on $C_{E/E^{\sigma^2}}$. The admissibility of θ then implies that $\sigma^2 = 1$. Let $L = E^\sigma$. Then $[E : L] = 2$, and for all $x \in E^\times$,

$$\theta(\mathrm{N}_{E/L}(x)) = \theta(\sigma(x) \cdot x) = \theta^{-1}(x) \cdot \theta(x) = 1.$$

This reasoning also works in reverse.

Now suppose $\sigma = 1$. Then θ has order two. The fact that p must equal 2 in this case follows from the next result. □

Proposition 5.4. *Let E/F be any extension of p-adic fields, with p odd. Then $E^\times$ has no real-valued admissible characters over F.*

Proof. Suppose that E/F is ramified and θ is a real admissible character of $E^\times$. Then θ is nontrivial on $C^{(0)}_{E/F}$. But this is a pro-p-group, so the order of any of its characters must be a power of p, a contradiction.

Now suppose that E/F is unramified. Then it will be enough to show that $C_{E/F} \subset (E^\times)^2$, since this will imply that all real characters of $E^\times$ are trivial on $C_{E/F}$. Write $E = F(\tau)$, where τ is a root of unity. Choose a minimal positive r such that τ^r is congruent mod $\wp$ to an element of $C_{E/F}$, i.e., such that the corresponding element $\bar{\tau}^r$ in k_E lies in C_{k_E/k_F}. It is enough to show that r is even. In fact,

$$r = |k_E^\times|/|C_{k_E/k_F}| = |k_F^\times|,$$

which is even. □

Definition 5.5. By an abuse of language, let us call a character θ *self-contragredient* if it satisfies either of the conditions of (5.3).

Thus, we are interested in the existence of self-contragredient admissible characters. These characters are necessarily unitary, as are the corresponding supercuspidal representations.

Corollary 5.6. *Let E/F be a tamely ramified extension, and let $L_1, L_2, \dots, L_r$ be the maximal intermediate fields. Then a character θ of $E^\times$ is self-contragredient and admissible if and only if both of the following conditions hold:*

(1) *θ is nontrivial on C_{E/L_i} for all i, and*
(2) *θ is nontrivial on $C^{(0)}_{E/L_i}$ for all i such that E/L_i is ramified,*

and one of the following conditions holds:

(3) *there is some $1 \le i \le r$ such that E/L_i is quadratic and θ is trivial on $\mathrm{N}_{E/L_i}(E^\times)$,*
(3′) *$p = 2$ and θ has order 2.*

6. Existence

Here is a our main theorem.

Theorem 6.1. *Suppose that E/F is any tame extension of degree n. Then $E^\times$ has self-contragredient admissible characters if and only if either $p = 2$ or E is quadratic over some intermediate field.*

First we need a lemma.

Lemma 6.2. *Let A be a topological abelian group, let $C_1, \dots, C_r$ be closed subgroups, and let N be a closed subgroup that does not contain any C_i. Then there exists a character θ of A such that θ is trivial on N and θ is nontrivial on every C_i.*

Proof. Replacing A by A/N and each C_i by its image in A/N, we may reduce to the case where N is trivial.

It is now enough to find a partition of $\{1, \dots, r\}$ into subsets S_j such that the groups $H_j = \cap_{i \in S_j} C_i$ are all nontrivial, but have trivial pairwise intersections. For then we may choose any nontrivial characters θ_j of each H_j, let θ be the corresponding character of $H_1 H_2 \cdots \subset A$, and extend θ to A.

To construct such a partition, let S_1 be any maximal subset of $\{1, \dots, r\}$ such that $\cap_{i \in S_1} C_i$ is nontrivial, and then proceed by induction. $\square$

Proof of Theorem 6.1. If n and p are both odd, then it is clear from Corollary 5.6 that for no tame extension E/F of degree n does $E^\times$ have a self-contragredient admissible character.

Suppose that L_0 is an intermediate field such that E/L_0 is quadratic. From (6.2), it is enough to show that $\mathrm{N}_{E/L_0}(E^\times)$ does not contain any $C^{(0)}_{E/L_i}$, where L_i is an intermediate field. Suppose on the contrary that $C^{(0)}_{E/L_i} \subset \mathrm{N}_{E/L_0}(E^\times)$ for some L_i. Then $C^{(0)}_{E/L_i} \subset U_{L_0,1}$. Since $C^{(0)}_{E/L_0} \cap U_{L_0,1}$ is finite, so is $C^{(0)}_{E/L_0} \cap C^{(0)}_{E/L_i}$. This implies that, as a manifold over F, the product $C^{(0)}_{E/L_i} \cdot C^{(0)}_{E/L_0}$ has dimension

$$(n - \dim L_i) + (n - \dim L_0) \geq n.$$

But from (4.2), this product is $C^{(0)}_{E/L_0 \cap L_i}$, which has dimension at most $n-1$, a contradiction.

Now suppose that p is even. From (6.2), it is enough to show that $(E^\times)^2$ does not contain any $C^{(0)}_{E/L_i}$. Suppose that $C^{(0)}_{E/L_i} \subset (E^\times)^2$ for some L_i. It is elementary that $U_{E,1} \cap (E^\times)^2 \subset U_{E,2}$, so $C_{E/L_i,1} = C_{E/L_i,2}$. But this contradicts (3.3). $\square$

7. Examples at depth zero

The *depth* of an irreducible representation is defined in [8]. For our purposes, it will be enough to say that a representation has *depth zero* if it has nontrivial P^+-fixed vectors, where P^+ is the maximal normal pro-p-subgroup of some parahoric subgroup P of $\mathrm{GL}_n(F)$.

From work of Bushnell-Kutzko [1], Morris [6], or Moy-Prasad [9], all supercuspidal representations of $\mathrm{GL}_n(F)$ of depth zero are tamely ramified in the sense we are using here. That is, they all arise via the Howe construction from admissible characters of level one. (Note that there is no restriction on n or p.)

Theorem 7.1. *The number of self-contragredient supercuspidal representations of $\mathrm{GL}_n(F)$ of depth zero is*

$$\begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{2}{n}(q^{n/2} - 1) & \text{if } n \text{ is a power of } 2 \text{ and } p \text{ is odd,} \\ \frac{2}{n}(q^{n/2}) & \text{if } n \text{ is a power of } 2 \text{ and } p = 2, \\ \frac{2}{n} \displaystyle\sum_{S \subseteq \{1, \dots, t\}} (-1)^{|S|} q_F^{n/\left(2 \prod_{i \in S} p_i\right)} & \text{otherwise,} \end{cases}$$

where $\left\{ p_i \mid 1 \leq i \leq t \right\}$ is the set of odd prime divisors of n.

Proof. We start by counting the self-contragredient admissible characters of level one of tame extensions E/F of degree n. From Definition 5.1, such characters can only exist when E/F is unramified. No such characters can have order 2, so they can only exist when n is even, and they must satisfy condition (3) of Corollary 5.6.

Let $p_0 = 2$. For each $0 \leq i \leq t$, let k_i be the intermediate field in k_E/k_F such that $[k_E : k_i] = p_i$. Then the k_i are the maximal intermediate fields. Let C_i be the image in $k_E^\times/k_2^\times$ of C_{k_E/k_i}. Let $q = q_F$.

Recall that for any positive integers b, c, and d, $q^b - 1$ divides $q^d(q^{bc} - 1) = q^{bc+d} - q^d$. Therefore,

$$(q^b - 1, q^{bc+d} + 1) = (q^b - 1, q^d + 1). \tag{7.2}$$

We have

$$\begin{aligned} |C_i| &= |C_{k_E/k_i}|/|C_{k_E/k_i} \cap k_2^\times| \\ &= \frac{(q^n - 1)/(q^{n/p_i} - 1)}{\left(\frac{q^n-1}{q^{n/p_i}-1}, q^{n/2} - 1\right)} \\ &= \frac{(q^n - 1)/(q^{n/p_i} - 1)}{\left(\frac{q^n-1}{q^{n/p_i}-1}, \frac{q^n-1}{q^{n/2}+1}\right)} \\ &= \frac{\operatorname{lcm}(q^{n/p_i} - 1, q^{n/2} + 1)}{q^{n/p_i} - 1} \\ &= \frac{q^{n/2} + 1}{(q^{n/p_i} - 1, q^{n/2} + 1)}. \end{aligned}$$

Using (7.2), we can simplify this to

$$\frac{q^{n/2} + 1}{r_i},$$

where

$$r_i = \begin{cases} q^{n/2p_i} + 1 & \text{if } i > 0, \\ 2 & \text{if } i = 0 \text{ and } p \text{ is odd}, \\ 1 & \text{if } i = 0 \text{ and } p = 2. \end{cases}$$

For any subset $S \subseteq \{0, \ldots, t\}$, the product of all C_i with $i \in S$ has order

$$\frac{q^{n/2} + 1}{\gcd\left\{ r_i \,\middle|\, i \in S \right\}}.$$

Therefore, the number of characters of $k_E^\times/k_2^\times$ that are nontrivial on every C_i is

$$(q^{n/2} + 1) + \sum_{\emptyset \neq S \subseteq \{0,\ldots,t\}} (-1)^{|S|} \gcd\left\{ r_i \,\middle|\, i \in S \right\}.$$

If n is a power of two, then this simplifies to $q^{n/2} - 1$ if p is odd, and $q^{n/2}$ if $p = 2$.

Suppose n is even, but not a power of two. The terms involving subsets S containing 0 are all $(-1)^{|S|} r_0$. These terms cancel each other out, and we are left with

$$(q^{n/2} + 1) + \sum_{\emptyset \neq S \subseteq \{1,\ldots,t\}} (-1)^{|S|} \gcd\left\{ q^{n/2p_i} + 1 \,\middle|\, i \in S \right\},$$

which simplifies to

$$\sum_{S\subseteq\{1,\dots,t\}} (-1)^{|S|} q^{n/\left(2\prod_{i\in S} p_i\right)}.$$

All self-contragredient admissible characters of level one of $E^\times$ over F arise by inflating such characters of $k_E^\times$ to U_E and extending to $E^\times$. There are always two such extensions, as ϖ_E can be sent to ± 1.

Two such characters give the same representation of $\mathrm{GL}_n(F)$ if and only if they are in the same orbit of the action of $\mathrm{Gal}(E/F)$ on the characters of $E^\times$. Our result follows from the fact that admissible characters lie in orbits of size n, from reasoning used in (5.3). □

To get concrete examples, take E/F unramified of degree n, take any m that divides $q^{n/2}+1$ but that does not divide any $q^{n/p_i}-1$ (for example, $m = q^{n/2}+1$), take any character of $k_E^\times/(k_E^\times)^m$ with trivial kernel, inflate to a character of U_E, and extend to $E^\times$ by sending ϖ to either 1 or -1.

Given a self-contragredient admissible character θ of level one of $E^\times$, it is easy to obtain examples at any higher level. As before, let L_0 be the intermediate field such that E/L_0 is quadratic. Let χ be any character of $C^{(0)}_{E/L_0}$ of level $\ell > 1$. (If $p = 2$, we require that $\chi(-1) = 1$.) Then χ extends to a character of $U_{E,1}$ trivial on $U_{L_0,1}$. Extend this character trivially to U_E, and then one can further extend to get a character $\overline{\chi}$ on $E^\times$ that is trivial on $\mathrm{N}_{E/L_0}(E^\times)$. Then $\theta\cdot\overline{\chi}$ is a self-contragredient admissible character of level ℓ.

References

[1] C. J. Bushnell and P. Kutzko, *The admissible dual of GL_N via restriction to compact open subgroups*, Ann. of Math. Studies, vol. 129, Princeton University Press, 1993. MR **94h:**22007

[2] I. M. Gelfand and M. I. Graev, *The group of matrices of second order with coefficients in a locally compact field and special functions on locally compact fields*, Uspekhi Mat. Nauk **18** (1963), 29–99. MR **27:**5864

[3] I. M. Gelfand and D. A. Kazhdan, *Representations of* $\mathrm{GL}(n,K)$ *where K is a local field*, Lie groups and their representations: Proc. Summer School on representation theory, Hungary, 1971 (I. M. Gelfand, ed.), 1975, pp. 95–118. MR **53:**8334

[4] R. Howe, *Tamely ramified supercuspidal representations of GL_n*, Pac. J. Math. **73** (1977), no. 2, 437–460. MR **58:**11241

[5] G. Lusztig, *Characters of reductive groups over a finite field*, Ann. of Math. Studies, vol. 107, Princeton University Press, 1984. MR **86j:**20038

[6] L. Morris, *Level zero G-types*, preprint.

[7] A. Moy, *Local constants and the tame Langlands correspondence*, Amer. J. Math. **108** (1986), 863–930. MR **88b:**11081

[8] A. Moy and G. Prasad, *Unrefined minimal K-types for p-adic groups*, Inv. Math. **116** (1994), 393–408. MR **95f:**22023

[9] ———, *Jacquet functors and unrefined minimal K-types*, Comm. Math. Helv., to appear. CMP 96:07

[10] P. J. Sally, Jr., *Invariant subspaces and Fourier-Bessel transforms on the $\mathfrak{p}$-adic plane*, Math. Ann. **174** (1967), 247–264. MR **58:**28317

[11] J.-P. Serre, *Corps locaux*, Hermann, 1962. MR **27:**133

[12] F. Shahidi, *Twisted endoscopy and reducibility of induced representations for p-adic groups*, Duke Math. J., **66** (1992), 1–41. MR **93b:**22034

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