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HEINZ'S INEQUALITY AND BERNSTEIN'S INEQUALITY

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Dedicated to Professor Tien-Hoh Lin on his seventieth birthday and his retirement

ABSTRACT. The purpose of the present account is to sharpen Heinz's inequality, and to investigate the equality and the bound of the inequality. As a consequence of this we present a Bernstein type inequality for nonselfadjoint operators. The Heinz inequality can be naturally extended to a more general case, and from which we obtain in particular Bessel's equality and inequality. Finally, Bernstein's inequality is extended to n eigenvectors, and shows that the bound of the inequality is preserved.

The well-known Heinz inequality is as follows: The relation

$$|(Tx,y)|^2 \le (|T|^{2\alpha}x,x)(|T^*|^{2(1-\alpha)}y,y) \tag{$*$}$$

holds for any bounded linear operator T on a complex Hilbert space $H, x, y \in H$, and any real number α with $0 \le \alpha \le 1$, where $|T|$ is the positive square root of the operator T^*T. It is possible to sharpen the inequality $(*)$ if T^*y is orthogonal to a vector z with $Tz \neq 0$. The new inequality is naturally extended to a more general case when T^*y is orthogonal to a set of vectors in which the bound of the inequality is retained. In particular we obtain Bessel's equality. By a similar method we present a Bernstein type inequality for nonselfadjoint operators. Finally, Bernstein's inequality is generalized to n eigenvectors for a selfadjoint operator and shows that the bound is preserved.

Theorem 1. *Let T be a bounded linear operator on a complex Hilbert space H and $0 \neq y \in H$. If T^*y is orthogonal to a vector $z \in H$ with $Tz \neq 0$, then*

$$|(Tx,y)|^2 + \frac{(|T^*|^{2(1-\alpha)}y,y)|(|T|^{2\alpha}x,z)|^2}{(|T|^{2\alpha}z,z)} \le (|T|^{2\alpha}x,x)(|T^*|^{2(1-\alpha)}y,y)$$

for every $x \in H$ and $\alpha \in [0,1]$. The equality holds if and only if the two vectors T^*y and $|T|^{2\alpha}x - \frac{(|T|^{2\alpha}x,z)|T|^{2\alpha}z}{(|T|^{2\alpha}z,z)}$ are proportional, equivalently, the two vectors $Tx - \frac{(|T|^{2\alpha}x,z)Tz}{(|T|^{2\alpha}z,z)}$ and $|T^*|^{2(1-\alpha)}y$ are proportional for $0 < \alpha < 1$.

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Proof. Let us define a vector $u = x - \frac{(|T|^{2\alpha}x,z)z}{(|T|^{2\alpha}z,z)}$, and write $a = (|T|^{2\alpha}z, z)$. Then $(u, |T|^{2\alpha}z) = 0$, so that

$$\begin{aligned}(|T|^{2\alpha}x, x) &= (|T|^{2\alpha}u + \tfrac{1}{a}(|T|^{2\alpha}x, z)|T|^{2\alpha}z, u + \tfrac{1}{a}(|T|^{2\alpha}x, z)z) \\ &= (|T|^{2\alpha}u, u) + \tfrac{1}{a}|(|T|^{2\alpha}x, z)|^2.\end{aligned}$$

Also,

$$(Tx, y) = (Tu + \tfrac{1}{a}(|T|^{2\alpha}x, z)Tz, y) = (Tu, y)$$

since T^*y and z are orthogonal by assumption. Hence,

$$\begin{aligned}&(|T|^{2\alpha}x, x)(|T^*|^{2(1-\alpha)}y, y) - |(Tx, y)|^2 \\ &\quad= [(|T|^{2\alpha}u, u) + \tfrac{1}{a}|(|T|^{2\alpha}x, z)|^2](|T^*|^{2(1-\alpha)}y, y) - |(Tu, y)|^2 \\ &\quad= \tfrac{1}{a}(|T^*|^{2(1-\alpha)}y, y)|(|T|^{2\alpha}x, z)|^2 \\ &\qquad+ [(|T|^{2\alpha}u, u)(|T^*|^{2(1-\alpha)}y, y) - |(Tu, y)|^2] \\ &\quad\geq \tfrac{1}{a}(|T^*|^{2(1-\alpha)}y, y)|(|T|^{2\alpha}x, z)|^2\end{aligned}$$

by $(*)$, and the inequality is proved. For $0 < \alpha < 1$ the equality holds if and only if $|(Tu, y)|^2 = (|T|^{2\alpha}u, u)(|T^*|^{2(1-\alpha)}y, y)$, equivalently, $|T|^{2\alpha}u$ and T^*y are proportional, or, Tu and $|T^*|^{2(1-\alpha)}y$ are proportional by [2, p. 91], where $u = x - \frac{(|T|^{2\alpha}x,z)z}{(|T|^{2\alpha}z,z)}$. □

Let us rewrite the inequality in Theorem 1 in a different form when T is positive and $Ty \neq 0$:

$$\frac{|(T^{2\alpha}x, z)|^2}{(T^{2\alpha}z, z)} \leq \frac{(T^{2\alpha}x, x)(T^{2(1-\alpha)}y, y) - |(Tx, y)|^2}{(T^{2(1-\alpha)}y, y)} = \frac{\|T^\alpha x\|^2\|T^{1-\alpha}y\|^2 - |(Tx, y)|^2}{\|T^{1-\alpha}y\|^2}.$$

We will show next that the bound in the above inequality is indeed the best of the bounds that can be obtained from a class of squares of ratios of shifted norm of vectors to the number shifted by the same amount. More precisely, we have

Theorem 2. *Under the hypothesis of Theorem 1, if T is positive and $Ty \neq 0$, then*

$$\frac{\|T^\alpha x\|^2\|T^{1-\alpha}y\|^2 - |(Tx, y)|^2}{\|T^{1-\alpha}y\|^2} \leq \frac{\|T^{1-\alpha}y - \gamma T^\alpha x\|^2}{\gamma^2}$$

for any real number $\gamma \neq 0$.

Proof. Let f be a function of γ defined by

$$\begin{aligned}f(\gamma) &= \|T^{1-\alpha}y\|^2\|T^{1-\alpha}y - \gamma T^\alpha x\|^2 - \gamma^2[\|T^\alpha x\|^2\|T^{1-\alpha}y\|^2 - |(Tx, y)|^2] \\ &= \|T^{1-\alpha}y\|^2[\|T^{1-\alpha}y\|^2 - 2\gamma\,\mathrm{Re}(Tx, y) + \gamma^2\|T^\alpha x\|^2] \\ &\qquad - \gamma^2[\|T^\alpha x\|^2\|T^{1-\alpha}y\|^2 - |(Tx, y)|^2] \\ &= \gamma^2|(Tx, y)|^2 - 2\gamma\,\mathrm{Re}(Tx, y)\|T^{1-\alpha}y\|^2 + \|T^{1-\alpha}y\|^4 \geq 0,\end{aligned}$$

since $\mathrm{Re}(Tx, y) \leq |(Tx, y)|$ and $f(0) > 0$. □

Remark that in Theorem 2 if (Tx, y) is real, then equality holds if and only if $\gamma = \frac{(T^{2(1-\alpha)}y,y)}{(Tx,y)}$.

Bernstein's inequality [1, p. 319] which is used in testing convergence of eigenvector calculations is as follows: if e is a unit eigenvector corresponding to an eigenvalue λ of a selfadjoint operator S, then

$$|(x,e)|^2 \leq \frac{\|x\|^2\|Sx\|^2-(x,Sx)^2}{\|(S-\lambda I)x\|^2}$$

for every $x \in H$ for which $Sx \neq \lambda x$. The bound of the inequality is the best in the sense that $\frac{\|x\|^2\|Sx\|^2-(x,Sx)^2}{\|(S-\lambda I)x\|^2} \leq \frac{\|(S-\gamma I)x\|^2}{(\lambda-\gamma)^2}$ for any real number $\gamma \neq \lambda$.

Recall that a complex number $\lambda \neq 0$ is a normal eigenvalue for an operator T if $Te = \lambda e$ and $T^*e = \overline{\lambda} e$ associated with the same eigenvector $e \neq 0$. For example, if λ is an eigenvalue for a hyponormal operator, then λ is a normal eigenvalue. By a similar method as in Theorem 1 we have the following Bernstein type inequality for nonselfadjoint operators.

Theorem 3. *If T is a bounded linear operator on a complex Hilbert space H which has a normal eigenvalue λ associated with a unit eigenvector e, and if $0 \neq y \in H$, e and y are orthogonal, and $T^*y \neq 0$, then*

$$|\lambda|^2|(x,e)|^2 = |(Tx,e)|^2 \leq \frac{\|Tx\|^2\|T^*y\|^2-|(Tx,T^*y)|^2}{\|T^*y\|^2}$$

for every $x \in H$. Equality holds if and only if the two vectors $Tx-\lambda(x,e)e$ and T^*y are proportional. The bound of the inequality is as follows:

$$\frac{\|Tx\|^2\|T^*y\|^2-|(Tx,T^*y)|^2}{\|T^*y\|^2} \leq \frac{\|T^*y-\beta Tx\|^2}{\beta^2}$$

for any real number $\beta \neq 0$.

Proof. Set $u = x-(x,e)e$; then $(u,e) = 0$. It follows that $\|Tx\|^2 = \|Tu\|^2 + |\lambda|^2|(x,e)|^2$, and $(Tx,T^*y) = (Tu,T^*y)$. Thus,

$$\begin{aligned}&\|Tx\|^2\|T^*y\|^2-|(Tx,T^*y)|^2\\&\quad = |\lambda|^2|(x,e)|^2\|T^*y\|^2 + [\|Tu\|^2\|T^*y\|^2-|(Tu,T^*y)|^2].\end{aligned}$$

Hence, the inequality follows. The bound is clear by Theorem 2. □

The next result is an extension of Theorem 1. We see that the bound of the inequality is retained as in Theorem 1.

Theorem 4. *Under the hypotheses of Theorem 1, if T^*y is orthogonal to a set of vectors $\{z_1,\ldots,z_n\} \subseteq H$ with $Tz_i \neq 0$, $i = 1,\ldots,n$, then*

$$\begin{aligned}&|(Tx,y)|^2 + (|T^*|^{2(1-\alpha)}y,y)\sum_{i=1}^{n}\frac{|(|T|^{2\alpha}u_{i-1},z_i)|^2}{(|T|^{2\alpha}z_i,z_i)}\\&\quad \leq (|T|^{2\alpha}x,x)(|T^*|^{2(1-\alpha)}y,y)\end{aligned}$$

for every $x \in H$, where $u_i = u_{i-1} - \frac{(|T|^{2\alpha}u_{i-1},z_i)z_i}{(|T|^{2\alpha}z_i,z_i)}$, $i = 1,\ldots,n$, with $u_0 = x$. In case $0 < \alpha < 1$, equality holds if and only if $|T|^{2\alpha}u_n$ and T^*y are proportional, or Tu_n and $|T^*|^{2(1-\alpha)}y$ are proportional.

Proof. Proceeding as in Theorem 1 we obtain

$$(|T|^{2\alpha}x,x)(|T^*|^{2(1-\alpha)}y,y)-|(Tx,y)|^2$$
$$=\frac{(|T^*|^{2(1-\alpha)}y,y)|(|T|^{2\alpha}x,z_1)|^2}{(|T|^{2\alpha}z_1,z_1)}+[(|T|^{2\alpha}u_1,u_1)(|T^*|^{2(1-\alpha)}y,y)-|(Tu_1,y)|^2]$$

if $u_1=x-\frac{(|T|^{2\alpha}x,z_1)z_1}{(|T|^{2\alpha}z_1,z_1)}$, and

$$(|T|^{2\alpha}u_i,u_i)(|T^*|^{2(1-\alpha)}y,y)-|(Tu_i,y)|^2$$
$$=\frac{(|T^*|^{2(1-\alpha)}y,y)|(|T|^{2\alpha}u_i,z_{i+1})|^2}{(|T|^{2\alpha}z_{i+1},z_{i+1})}$$
$$+[(|T|^{2\alpha}u_{i+1},u_{i+1})(|T^*|^{2(1-\alpha)}y,y)-|(Tu_{i+1},y)|^2]$$

if $u_i=u_{i-1}-\frac{(|T|^{2\alpha}u_{i-1},z_i)z_i}{(|T|^{2\alpha}z_i,z_i)}$, $i=2,\dots,n$. Therefore,

$$(|T|^{2\alpha}x,x)(|T^*|^{2(1-\alpha)}y,y)-|(Tx,y)|^2$$
$$=(|T^*|^{2(1-\alpha)}y,y)\left[\frac{|(|T|^{2\alpha}x,z_1)|^2}{(|T|^{2\alpha}z_1,z_1)}+\sum_{i=1}^{n-1}\frac{|(|T|^{2\alpha}u_i,z_{i+1})|^2}{(|T|^{2\alpha}z_{i+1},z_{i+1})}\right]$$
$$+(|T|^{2\alpha}u_n,u_n)(|T^*|^{2(1-\alpha)}y,y)-|(Tu_n,y)|^2$$
$$\geq(|T^*|^{2(1-\alpha)}y,y)\sum_{i=1}^{n}\frac{|(|T|^{2\alpha}u_{i-1},z_i)|^2}{(|T|^{2\alpha}z_i,z_i)}.$$

For $0<\alpha<1$, equality holds if and only if

$$(|T|^{2\alpha}u_n,u_n)(|T^*|^{2(1-\alpha)}y,y)=|(Tu_n,y)|^2$$

and the desired result follows. □

As an application of Theorem 4 we shall show that Bessel's equality can be derived directly from it, but let us state the next results which may be of some interest in themselves.

Corollary 1. *If y is a unit vector which is orthogonal to a set $\{z_1,z_2,\dots,z_n\}$ of unit vectors, then*

(1) $|(x,y)|^2+\sum_{i=1}^n|(u_{i-1},z_i)|^2+\|u_n\|^2-|(u_n,y)|^2=\|x\|^2$,
(2) $|(x,y)|^2+\sum_{i=1}^n|(u_{i-1},z_i)|^2\leq\|x\|^2$,

for every $x\in H$, where $u_i=u_{i-1}-(u_{i-1},z_i)z_i$, $i=1,\dots,n$, with $u_0=x$. Equality in (2) *holds if and only if u_n and y are proportional.*

Proof. In the proof of Theorem 4 let T be the identity operator. □

Corollary 2 (Bessel's equality and inequality)**.** *If $\{z_1,\dots,z_n\}\subseteq H$ is a set of orthonormal vectors, then, for every $x\in H$,*

(1) $\|x-\sum_{i=1}^n(x,z_i)z_i\|^2=\|x\|^2-\sum_{i=1}^n|(x,z_i)|^2$,

hence

(2) $\sum_{i=1}^n|(x,z_i)|^2\leq\|x\|^2$.

Proof. In Corollary 1 if $\{y,z_1,\dots,z_n\}$ is a set of orthonormal vectors, then

$$(u_i,z_{i+1})=(x,z_{i+1}),\qquad i=1,\dots,n-1,$$

and so

$$u_n = x - \sum_{i=1}^{n} (x, z_i) z_i.$$

Thus, (1) follows by (1) in Corollary 1, since $(u_n, y) = (x, y)$. ☐

Similarly to Theorem 4, Theorem 3 can be generalized as follows.

Corollary 3. *If T is a bounded linear operator on a complex Hilbert space H which has a set $\{\lambda_1, \ldots, \lambda_n\}$ of normal eigenvalues associated with a set $\{e_1, \ldots, e_n\}$ of unit eigenvectors, and if $0 \neq y \in H, e_i$ and y are orthogonal for $i = 1, \ldots, n$, and $T^*y \neq 0$, then*

$$\sum_{i=1}^{n} |\lambda_i|^2 |(u_{i-1}, e_i)|^2 = \sum_{i=1}^{n} |(Tu_{i-1}, e_i)|^2 \leq \frac{\|Tx\|^2 \|T^*y\|^2 - |(Tx, T^*y)|^2}{\|T^*y\|^2}$$

for every $x \in H$, where $u_i = u_{i-1} - (u_{i-1}, e_i)e_i$, $i = 1, \ldots, n$, with $u_0 = x$. Equality holds if and only if Tu_n and T^*y are proportional.

Proof. If $i = 1$, from the proof in Theorem 3 we have

$$\begin{aligned} &\|Tx\|^2 \|T^*y\|^2 - |(Tx, T^*y)|^2 \\ &\quad = \|T^*y\|^2 |(Tx, e_1)|^2 + [\|Tu_1\|^2 \|T^*y\|^2 - |(Tu_1, T^*y)|^2]. \end{aligned}$$

For $i = 2, \ldots, n$ we have

$$\begin{aligned} &\|Tu_{n-1}\|^2 \|T^*y\|^2 - |(Tu_{n-1}, T^*y)|^2 \\ &\quad = \|T^*y\|^2 |(Tu_{n-1}, e_n)|^2 + [\|Tu_n\|^2 \|T^*y\|^2 - |(Tu_n, T^*y)|^2]. \end{aligned}$$

It follows that

$$\begin{aligned} &\|Tx\|^2 \|T^*y\|^2 - |(Tx, T^*y)|^2 \\ &\quad = \|T^*y\|^2 \sum_{i=1}^{n} |(Tu_{i-1}, e_i)|^2 + [\|Tu_n\|^2 \|T^*y\|^2 - |(Tu_n, T^*y)|^2], \end{aligned}$$

and we have the desired conclusion. ☐

Corollary 4. *Besides the hypotheses of Corollary 3, if $\{e_1, \ldots, e_n\}$ is a set of orthonormal vectors, then*

$$\sum_{i=1}^{n} |\lambda_i|^2 |(x, e_i)|^2 \leq \frac{\|Tx\|^2 \|T^*y\|^2 - |(Tx, T^*y)|^2}{\|T^*y\|^2}$$

for every $x \in H$. Equality holds if and only if the two vectors $Tx - \sum_{i=1}^{n} \lambda_i e_i (x, e_i)$ and T^*y are proportional.

Proof. If $\{e_1, \ldots, e_n\}$ is a set of orthonormal vectors, then

$$(u_{i-1}, e_i) = (x, e_i), \qquad i = 1, \ldots, n,$$

since $u_i = u_{i-1} - (u_{i-1}, e_i)e_i$, $i = 1, \ldots, n$, with $u_0 = x$. Hence, inequality holds by Corollary 3. As in the proof of Corollary 2 we have $u_n = x - \sum_{i=1}^{n} (x, e_i)e_i$, and so $Tu_n = Tx - \sum_{i=1}^{n} \lambda_i (x, e_i) e_i$. Thus, by Corollary 3 again we have the case of equality. ☐

Finally, we give a straightforward generalization of Bernstein's inequality to n eigenvectors. Firstly, we require the next lemma which can be easily proved.

Lemma ([1, p. 319]). *Let S be a selfadjoint operator on a complex Hilbert space H, $x \in H$, and let γ be real. Then*

$$\|x\|^2\|Sx\|^2 - (x, Sx)^2 = \|x\|^2\|(S-\gamma I)x\|^2 - (x, (S-\gamma I)x)^2.$$

Theorem 5. *Let S be a selfadjoint operator on a complex Hilbert space H. If $\{e_1, \ldots, e_n\}$ is a set of unit eigenvectors corresponding to a set $\{\lambda_1, \ldots, \lambda_n\}$ of eigenvalues of S, then*

(1) $\|(S-\lambda_i I)u_{i-1}\|^2|(u_{i-1}, e_i)|^2 \leq \|u_{i-1}\|^2\|Su_{i-1}\|^2 - (u_{i-1}, Su_{i-1})^2$,

(2) $\sum_{i=1}^n |(u_{i-1}, e_i)|^2 \leq \frac{\|x\|^2\|Sx\|^2 - (x,Sx)^2}{\|(S-\lambda_j I)x\|^2}$

for every $x \in H$ for which $Sx \neq \lambda_j x$, where $u_i = u_{i-1} - (u_{i-1}, e_i)e_i$, $i = 1, \ldots, n$, with $u_0 = x$, and $j \in \{1, \ldots, n\}$. Equality in (1) holds if and only if u_i is an eigenvector of S, and equality in (2) holds if and only if u_n is an eigenvector of S.

Proof. This can be proved by the analogous methods as in [1, Theorem 1] and our Theorem 3.

Clearly, $(u_i, e_i) = 0$, $i = 1, \ldots, n$, and so $\|u_{i-1}\|^2 = \|u_i\|^2 + |(u_{i-1}, e_i)|^2$.

(1) By the Lemma we may replace $S - \lambda_i I$ by S, which allows us to assume without loss of generality $\lambda_i = 0$, so that $Su_i = Sx$, and hence $(u_{i-1}, Sx) = (u_i, Sx)$, $i = 1, \ldots, n$. Therefore,

$$\begin{aligned}
&\|u_{i-1}\|^2\|Su_{i-1}\|^2 - (u_{i-1}, Su_{i-1})^2 \\
&\quad = [\|u_i\|^2 + |(u_{i-1}, e_i)|^2]\|Su_{i-1}\|^2 - (u_i, Su_{i-1})^2 \\
&\quad = \|Su_{i-1}\|^2|(u_{i-1}, e_i)|^2 + [\|u_i\|^2\|Su_i\|^2 - (u_i, Su_i)^2] \\
&\quad \geq \|Su_{i-1}\|^2|(u_{i-1}, e_i)|^2
\end{aligned}$$

for $n = 1, \ldots, n$.

(2) We proceed as follows: In the above proof if $i = 1$ (recall that $Su_i = Sx$, $i = 1, \ldots, n$), then

$$\|x\|^2\|Sx\|^2 - (x, Sx)^2 = \|Sx\|^2|(x, e_1)|^2 + [\|u_1\|^2\|Sx\|^2 - (u_1, Sx)^2],$$

otherwise

$$\begin{aligned}
&\|u_{i-1}\|^2\|Sx\|^2 - (u_{i-1}, Sx)^2 \\
&\quad = \|Sx\|^2|(u_{i-1}, e_i)|^2 + [\|u_i\|^2\|Sx\|^2 - (u_i, Sx)^2]
\end{aligned}$$

for $i = 2, \ldots, n$. It follows as in Theorem 3 that

$$\begin{aligned}
&\|x\|^2\|Sx\|^2 - (x, Sx)^2 \\
&\quad = \|Sx\|^2 \sum_{i=1}^n |(u_{i-1}, e_i)|^2 + [\|u_n\|^2\|Su_n\|^2 - (u_n, Su_n)^2].
\end{aligned}$$

By applying the Lemma once more, we have the required result. □

Corollary 5. *Besides the hypotheses of Theorem 5, if $\{e_1, \ldots, e_n\}$ is a set of orthonormal vectors, then*

$$\sum_{i=1}^n |(x, e_i)|^2 \leq \frac{\|x\|^2\|Sx\|^2 - (x, Sx)^2}{\|(S-\lambda_j I)x\|^2}$$

for every $x \in H$ for which $Sx \neq \lambda_j x$, $j \in \{1, \ldots, n\}$. Equality holds if and only if $x - \sum_{i=1}^n (x, e_i)e_i$ is an eigenvector of S.

Proof. See the proofs of Corollary 4 and Theorem 5. □

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