FACTORIZATION OF HOLOMORPHIC MAPPINGS ON $C(K)$–SPACES

JARI TASKINEN

(Communicated by Theodore W. Gamelin)

Abstract. We prove a universal mapping theorem for a large class of holomorphic mappings $F$ on a $C(K)$–space, stating that $F$ can be locally written in the form $F(f) = B(1/(1 - Af))$, where $A$ and $B$ are bounded linear operators on certain Banach spaces consisting of functions on $K$, and the division is taken pointwise.

Introduction

We prove a linearization theorem for a class of holomorphic mappings $F$ on $C(K)$–spaces. We show in Theorem 3.4 that such an $F$ can be presented as a compose of bounded linear operators $A$, $B$ and the holomorphic mapping $H(f)(t) := 1/(1 - f(t))$, where $f \in U$ (the open unit ball of $C(K)$ ) and $t \in K$:

$$F(f) = BH(Af) = B\left(\frac{1}{1 - Af}\right).$$

Here only the operator $B$ depends on $F$ so that this result can be considered as a universal mapping theorem where both the universal map ($= H \circ A$) and also the universal space are of a very special form. The point is that the non–linearity of the universal map comes only from the simple scalar holomorphic map $z \mapsto 1/(1 - z)$.

Our result is only local: it deals only with mappings $F$ defined on open discs. Moreover, there are some unsolved problems concerning the operators $A$ and $B$. We refer to Theorem 3.4 and Remark 3.5.

Universal mapping theorems for holomorphic mappings on Banach or locally convex spaces have previously been studied for example in [Ma, Mu1, Mu2, Mu–N, G–G–M], see also [R].

1. Notation. Integral holomorphic mappings

We denote by $\mathbb{N}$ the set $\{1, 2, 3, \ldots\}$ and by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$. The closed unit interval $[0, 1]$ is denoted by $I$. All Banach spaces are over the complex scalar field. The space of bounded linear operators between the Banach spaces $X$ and $Y$ is denoted by $L(X, Y)$, or by $L(X)$, if $X = Y$; the dual of $X$ is denoted by $X^*$. The absolutely convex hull of a subset $A$ of a Banach space is denoted by $\Gamma(A)$.

For general topology we refer to [Ku]. If $K$ is a compact metric space, we denote by $C(K)$ (resp. $\ell_\infty(K)$ ) the Banach space of continuous (resp. bounded), complex
valued mappings $K \to \mathbb{C}$, endowed with the sup–norm. If $K_1$ and $K_2$ are compact metric spaces and $\varphi : K_1 \to K_2$ is a continuous surjection, we denote by $\varphi^o$ the linear isometry from $C(K_2)$ into $C(K_1)$ given by $\varphi^of = f \circ \varphi$. If $\varphi^o(C(K_2))$ is 1–complemented in $C(K_1)$, i.e., if there exists a contractive projection from $C(K_1)$ onto $\varphi^o(C(K_2))$, we say that $\varphi$ admits a regular averaging operator. (Note that in this case the map $\varphi^o$ also has a contractive left inverse.) For more details we recommend the reference [LT], Sections II.4.h,i, and [P].

For complex analysis in infinite dimensional spaces we refer to [D2] and [C]. If $X$ and $Y$ are Banach spaces and $n \in \mathbb{N}$, we denote by $P(nX,Y)$ the space of continuous $n$–homogeneous polynomials $X \to Y$.

Recall that a continuous $n$–linear form $F$ on $C(K)^n$ is called integral, if there exists a $\mu(F) \in C(K^n)^*$ such that

$$F(f_1, \ldots , f_n) = \prod_{k=1}^{n} f_k \circ \pi_n^{(k)}(\mu(F)),$$

where $f_k \in C(K)$, $\pi_n^{(k)}$ is the canonical projection from $K^n$ onto the $k$:th coordinate space and the product on the right-hand side is taken pointwise.

Let $U \subset C(K)$ be open and $F : U \to Y$ holomorphic. We write the Taylor series of $F$ at the point $y \in U$ as

$$F(x) = F_0 + \sum_{n=1}^{\infty} F_n^{(y)}(x-y),$$

where $F_0 \in Y$ and $F_n^{(y)} \in P(nC(K),Y)$; we denote by $F_n^{(y)}$ the corresponding symmetric $n$–linear mapping.

The following definition was given in [T2].

1.1. Definition. Let $Y$ be a Banach space, let $U \subset C(K)$ be open, let $F : U \to Y$ be a holomorphic mapping, let $B \subset U$ be an open ball with center $y$ and radius $r$, and let $S \subset Y^*$ be a bounded subset. We say that $F$ is uniformly $(S,B)$–integral, if

1°. for every $t \in S$, $n \in \mathbb{N}$, the $n$–linear form

$$(f_1, \ldots , f_n) \mapsto \langle F_n^{(y)}(f_1, \ldots , f_n), t \rangle$$

is integral (write $\mu(F,n,t)$ for the corresponding element of $C(K^n)^*$ as in (1.1)),

2°. the mapping

$$||F||_{S,B} := \sup_{t \in S} \{ ||\langle F_0 , t \rangle || + \sum_{n=1}^{\infty} \sup_{h \in C(K^n)} ||\langle h , \mu(F,n,t) \rangle || r^n \} < \infty,$$

and

3°. the mapping

$$t \to \sum_{n=1}^{\infty} \langle h_n , \mu(F,n,t) \rangle r^n$$

is, for arbitrary $h_n \in C(K^n)$ with $||h_n|| \leq 1$, continuous $S \to \mathbb{C}$, when $S$ is endowed with the weak* topology.
We remark that this concept of integral holomorphic mappings does not coincide with the definition of mappings of integral holomorphy type in [D1] and [A]. Nevertheless, the definition is quite natural and gives quite a large class of holomorphic mappings.

1.2. Examples. 1° The operator \( f \mapsto f^n \) (pointwise multiplication; \( n \in \mathbb{N} \)) is uniformly integral \( C(K) \to C(K) \) for every \( S \) and \( B \) as in Definition 1.1. We especially see that the identity operator on \( C(K) \) is uniformly integral. (We refer to [T2] for the details of this and the following examples.)

2° Let \( U \subset C(K) \) be the open unit ball, let \( Y = C(K) \) and let \( h \) be a scalar valued holomorphic mapping on the open unit disc of \( C \) such that its Taylor coefficients at 0 form an absolutely summable sequence. Then the map \( (Hf)(t) := h(f(t)) \), \( f \in U, \ t \in K \), is uniformly \((K, U)\)-integral on \( U \); here the set \( K \) is identified in the canonical way with a subset of \( C(K)^* \).

3° Denote by \( U \subset C(I) \) the open unit ball. If \( F_n : I \times I^n \to C \) is for all \( n \in \mathbb{N}_0 \) a continuous function satisfying \( \sum_{n=0}^{\infty} ||F_n||_{C(I^{n+1})} < \infty \), then the holomorphic integral operator

\[
f \mapsto \sum_{n=0}^{\infty} \int_{I^n} F_n(\cdot, s_1, \ldots, s_n) f(s_1) \ldots f(s_n) ds,
\]

where \( s = (s_1, \ldots, s_n) \), is uniformly \((I, U)\)-integral \( U \to C(I) \).

2. Preliminary results

In this section we present some results necessary for the proof of the main result.

In the following universal mapping theorem we denote by \( K \) a compact metric uncountable space and by \( U \) the open unit ball of \( C(K) \). The set \( K \) is also considered as a subset of \( C(K)^* \): for every \( t \in K \) there corresponds the point evaluation \( \delta_t : f \mapsto f(t), \ f \in C(K) \). This identification is a homeomorphism, when \( C(K)^* \) is endowed with the weak*–topology.

2.1. Theorem. There exists a universal holomorphic mapping \( \psi : U \to C(K) \) such that for every uniformly \((K, U)\)-integral holomorphic \( F : U \to C(K) \) there exists \( B_F \in L(C(K)) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
U & \xrightarrow{F} & C(K) \\
\downarrow{\psi} & & \downarrow{B_F} \\
C(K) & & 
\end{array}
\]

This result was proved in [T2], Theorem 2.1.

For the proof of the main theorem of this paper we shall need the following

Remark. Having a look at the proof of Theorem 2.1 of [T2] (especially (2.2) there) one easily verifies that

\[
(2.1) \quad \psi(f) = f_0 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} \psi^{(k)}_n f
\]

where \( f_0 \) is an element of \( C(K) \), \( \psi^{(k)}_n \in L(C(K)) \) and \( ||\psi^{(k)}_n|| \leq 1 \) for all \( n \) and \( k \).
2.2. Lemma. Let \( n \in \mathbb{N} \), let \( J \) be a finite set and let \( (x_j)_{j \in J} \) be a sequence of complex numbers satisfying \( |x_j| < 1 \). For all sequences of complex numbers \( (\lambda_j)_{j \in J} \)

\[
\sup_{t \in [0,1]} \left| \sum_{j \in J} \lambda_j \frac{1}{1 - e^{2i\pi x_j}} \right| \geq \left| \sum_{j \in J} \lambda_j x_j^n \right|.
\]

(2.2)

Proof. Using the Taylor series of the analytic function \( z \mapsto 1/(1-z) \), \( |z| < 1 \), we easily get

\[
\int_0^1 e^{-it2n\pi} \sum_{j \in J} \lambda_j \frac{1}{1 - e^{2i\pi x_j}} dt = \sum_{j \in J} \lambda_j x_j^n
\]

This implies the inequality (2.2).

2.3. Proposition. There exist a strictly increasing sequence \( (\tau(t))_{t=0}^\infty \), \( \tau(0) = 1 \), and, for every \( m \in \mathbb{N} \), \( n \in \mathbb{N}_0 \), complex numbers \( a_{m,n} \) and \( b_{m,n} \) such that the following holds (convention: \( 0^0 = 1 \)).

For every \( m \in \mathbb{N} \), \( n \in \mathbb{N}_0 \), we have \( |a_{m,n}| < 1 \), \( |b_{m,n}| \leq e^7 \).

For all \( z \in \mathbb{C} \), \( |z| < 1 \), for all \( n \in \mathbb{N}_0 \),

\[
\sum_{t \in \mathbb{N}_0} \sum_{m=\tau(t)}^{\tau(t+1)-1} \sum_{k \in \mathbb{N}_0} b_{m,n} a_{m,n}^k z^k = z^n,
\]

(2.3)

and

\[
\sum_{t \in \mathbb{N}_0} \sum_{m=\tau(t)}^{\tau(t+1)-1} b_{m,0} = 1.
\]

(2.4)

This result is contained in Theorem 9 of [T4].

3. Holomorphic mapping as a compose of linear operators and a scalar holomorphic function

Let \( U \subset C(K) \) be the open unit ball and let \( r > e \). In this section we show that a uniformly \((K, rU)\)-integral holomorphic \( F : rU \to C(K) \) can be presented as a product of linear operators \( A, B \) and the mapping \( H : f(t) \mapsto 1/(1 - f(t)) \), \( f \in U \), \( t \in K \). More precisely, we show that the equality

\[
F(f) = BH(Af)
\]

(3.1)

holds for \( f \in U \).

There are two major difficulties. First, one needs to solve at least approximately an infinite system of polynomially nonlinear equations. The solution is presented in detail in the paper [T4] and only the result is mentioned here; see Proposition 2.3. The second difficulty is to make the operators \( A \) and \( B \) well defined and continuous. We are not able to solve this problem in the optimal way. Accordingly, \( A \) becomes a bounded operator in the sup-norm, but the functions \( Af \), where \( f \in C(K) \), need not be continuous everywhere in \( K \). (However, the discontinuity is in some sense only “mild”. ) We are in general able to define the operator \( B \) only in the closed linear span of \( H \circ A \), not in the whole space \( C(K) \) (or \( \ell_\infty(K) \)). Finally, we need to assume that the given map \( F \) is holomorphic in \( rU \), not only in \( U \) (see above). We refer to Remark 3.5 for some explanations.
3.1. Definitions. We choose for every \( n, j \in \mathbb{N}_0, 1 \leq j \leq 2^n \), a closed subinterval \( I_{n,j} \subset I = [0, 1] \) such that \( I_{n,j} \cap I_{n',j'} = \emptyset \), if \( n \neq n' \) or \( j \neq j' \), and such that for every \( \varepsilon > 0 \) the interval \([0, 1 - \varepsilon]\) contains only finitely many intervals \( I_{n,j} \).

We choose for every \( n, j \) a continuous surjection \( \varphi_{n,j} : I_{n,j} \to I^3 \) with a regular averaging operator. (See \([H]\), Theorem 2.2., also \([T3]\), Theorem 2.) We denote by \( \hat{\varphi}_{n,j} \) a contractive left inverse of \( \varphi_{n,j} \).

We fix for every \( n, j \) a Borsuk–Kakutani extension operator \( E_{n,j} : C(I_{n,j}) \to C(I) \) such that \( \text{supp} E_{n,j} f \cap \text{supp} E_{n',j'} g = \emptyset \) for all \( f, g \), if \( n \neq n' \) or \( j \neq j' \). (See [LT], Theorem II.4.14.; since the sets \( I_{n,j} \) have mutually disjoint open neighbourhoods for different indices \( n, j \), a simple trick shows that our requirement for the disjointness of the sets \( \text{supp} E_{n,j} f \) can be satisfied.) We fix some disjoint closed intervals \( J_{s,j} \supset I_{n,j}, J_{n,j} \subset I \), such that \( \text{supp} E_{n,j} f \subset J_{n,j} \) for all \( f \in C(I_{n,j}) \).

For every \( m \in \mathbb{N} \) and \( n, j \) we denote by \( K_{m,n,j} \) a subspace of \( I^3 \) of the form \( I_{n,j} \times \{s_2\} \times \{s_3\} \), where the numbers \( s_2, s_3 \in I \) are chosen such that

\[
s_2 e^{i2\pi s_3} = a_{m,n}.
\]

Here \( a_{m,n} \) is as in Proposition 2.3. We denote by \( \eta_{m,n,j} : I_{n,j} \to K_{m,n,j} \) the homeomorphism \( t \mapsto t \times \{s_2\} \times \{s_3\} \).

3.2. Lemma. Use the notation of Proposition 2.3 and Definition 3.1, and fix some indices \( n \in \mathbb{N}_0, j \in \mathbb{N}, 1 \leq j \leq 2^n \). Let \( \alpha \in L(C(I^3)) \) be a contraction. Define \( \alpha^{(j)}_n \in L(C(I^3)) \) by

\[
(\alpha^{(j)}_n f)(t_1, t_2, t_3) = t_2 e^{i2\pi t_3} (E_{n,j} \hat{\varphi}_{n,j}^{(3)} \alpha f)(t_1),
\]

where \((t_1, t_2, t_3) \in I^3 \).

1°. We have \( ||\alpha^{(j)}_n f|| \leq ||\alpha f|| \) for all \( f \).

2°. For every \( f \in H(\alpha^{(j)}_n(U)), \) where \( U \subset C(I^3) \) is the open unit ball, the sum

\[
\beta^{(j)}_n f := \sum_{\tau=0}^{\infty} \sum_{m=\tau(t)} b_{m,n} \hat{\varphi}_{m,n,j} \alpha^{(j,j)} f
\]

converges pointwise in \( I^3 \) and defines a continuous mapping from \( H(\alpha^{(j)}_n(U)) \) into \( C(I^3) \) which can be extended as a bounded linear operator to the subspace \( \text{sp} H(\alpha^{(j)}_n(U)) \). Denoting the extension again by \( \beta^{(j)}_n \) we have \( ||\beta^{(j)}_n|| \leq 1 \).

3°. We can define, without increasing the norm of \( \beta^{(j)}_n \),

\[
\beta^{(j)}_n \sum_{k} \lambda_k H(f_k + g_k) = \beta^{(j)}_n \sum_{k} \lambda_k H(f_k)
\]

for all finite sequences \( (\lambda_k) \subset \mathbb{C} \), \((f_k) \subset \alpha^{(j)}_n(U) \) and \((g_k) \subset \ell_{\infty}(I^3) \) such that \( ||g_k|| < 1, \text{supp}(g_k) \cap J_{n,j} \times I^2 = \emptyset \).

4°. We have for \( n \geq 1 \)

\[
\beta^{(j)}_n H(\alpha^{(j)}_n f) = (\alpha f)^n
\]

for all \( f \in U \), and \( \beta^{(j)}_0 H(0) = 1 \).

Proof. 1°. This is clear.

2°. Assume that \( f \in H(\alpha^{(j)}_n(U)), f = H(\alpha^{(j)}_n g) \) for some \( g \in U \). Because of the definition of \( K_{m,n,j} \) we have for every \( m \in \mathbb{N}_0 \)

\[
f \circ \eta_{m,n,j} = H(a_{m,n} \hat{\varphi}_{n,j}^{(3)} \alpha g) = \hat{\varphi}_{n,j}^{(3)} H(a_{m,n} \alpha g).
\]
Hence,
\[
\beta_n^{(j)} f = \sum_{t=0}^{\infty} \sum_{m=\tau(t)}^{\tau(t+1)-1} b_{m,n} H(a_{m,n} \alpha g) = \sum_{t=0}^{\infty} \sum_{m=\tau(t)}^{\tau(t+1)-1} b_{m,n} \sum_{k=0}^{\infty} (a_{m,n} \alpha g)^k,
\]
and Proposition 2.3 implies the desired pointwise convergence of (3.4). Moreover, by Proposition 2.3 and (3.6),
\[
\beta_n^{(j)} f = (\alpha g)^n,
\]
so that \( \beta_n^{(j)} f \in C(I^3) \).

We extend \( \beta_n^{(j)} \) linearly to \( \text{sp}H(\alpha_n^{(j)}(U)) \) and prove that the extension is a bounded operator. To this end let \( J \subset \mathbb{N} \) be a finite sequence, let for every \( k \in J \) the functions \( f_k \in U \) and the complex numbers \( \lambda_k \) be arbitrary. We apply Lemma 2.2 to get the estimate
\[
\|\beta_n^{(j)} \sum_{k \in J} \lambda_k H(\alpha_n^{(j)} f_k)\| \leq \|\sum_{k \in J} \lambda_k (\alpha f_k)^n\|
\]
(3.8)
\[
\leq \sup_{t_2 \in I_2} \sup_{t_3 \in I_3} \left| \sum_{k \in J} \frac{\lambda_k}{1 - t_2 e^{2 \pi t_3}} \left( \alpha f_k(t) \right) \right|.
\]
Recall that each \( \varphi_{n,j} \) is a surjection and each \( E_{n,j} \) is an extension operator. Hence, (3.8) is not greater than
\[
\sup_{t_2 \in I_2} \sup_{t_3 \in I_3} \left| \sum_{k \in J} \frac{\lambda_k}{1 - t_2 e^{2 \pi t_3}} \left( E_{n,j} \varphi_{n,j}^{\circ} \alpha f_k \right)(t) \right| = \left| \sum_{k \in J} \lambda_k H(\alpha_n^{(j)} f_k) \right|.
\]
(3.9)
This proves the boundedness of \( \beta_n^{(j)} \) in \( E := \text{sp}H(\alpha_n^{(j)}(U)) \) and the desired norm estimate.

3°. The operator \( \beta_n^{(j)} \) is extended above to \( E \). We have \( \text{supp} f \subset J_{n,j} \times I^2 \) for all \( f \in \alpha_n^{(j)}(C(I^3)) \). Hence, for all \( \lambda_k \in \mathbb{C}, f_k \in \alpha_n^{(j)}(U) \) and \( g_k \in \ell_\infty(I^3) \) such that \( \|g_k\| < 1 \) and \( \text{supp}g_k \cap J_{n,j} \times I^2 = \emptyset \),
\[
\|\beta_n^{(j)} \sum_k \lambda_k H(f_k + g_k)\| = \|\beta_n^{(j)} \sum_k \lambda_k H(f_k)\|
\]
\[
\leq \|\sum_k \lambda_k H(f_k)\| \leq \|\sum_k \lambda_k H(f_k + g_k)\|.
\]
(The assumption on the supports is used to get the last inequality.)

4°. Follows from (3.7) and (2.4).

3.3. Lemma. Let \((A_n^{(j)})_{j=1}^{2^n} \) be a sequence of linear contractions \( C(I^3) \to C(I^3) \) and let \( \varepsilon > 0 \). Let \( \psi : U \to C(I^3) \) be the holomorphic mapping
\[
\psi(f)(t) := f_0(t) + \sum_{n=1}^{\infty} \sum_{j=1}^{2^n} \varepsilon_n^{(j)} (A_n^{(j)} f)(t)^n
\]
(3.10)
where \( \varepsilon_n^{(j)} \in \mathbb{C}, |\varepsilon_n^{(j)}| \leq (2 + \varepsilon)^{-n} \) and \( f_0 \in C(I^3) \) is fixed.
There exist linear operators \( A \in L(C(I^3), \ell_\infty(I^3)) \) and \( B_1 \in L(E, C(I^3)) \), where \( E \subset \ell_\infty(I^3) \) is the closed linear span of \( H(A(U)) \), such that

\[
\psi(f) = B_1 H(Af)
\]

for all \( f \in U \).

Proof. We use the notations of Proposition 2.3, Definition 3.1 and Lemma 3.2. For every \( n \in \mathbb{N}_0 \) and \( j = 1, \ldots, 2^n \) we choose the operators \( \alpha_n^{(j)} \) and \( \beta_n^{(j)} \) as in Lemma 3.2, taking \( \alpha = A_n^{(j)} \) and \( \alpha = 0 \) in the case \( n = 0, j = 1 \). We define

\[
A = \sum_{n,j} \alpha_n^{(j)},
\]

\[
B_1 = \beta_0^{(1)} + \sum_{n,j=1} 2^n \beta_n^{(j)}.
\]

That \( A \in L(C(I^3), \ell_\infty(I^3)) \) follows from 1° of Lemma 3.2 and from the assumption on the supports of the functions \( E_{n,j} f \) (Definition 3.1). The boundedness of \( B_1 \) follows from the facts that \( ||\beta_n^{(j)}|| \leq 1 \) and \( ||\beta_n^{(j)}|| \leq (2 + \varepsilon)^{-n} \) for every \( n, j \), and from 3° of Lemma 3.2. The statements 3° and 4° of Lemma 3.2 yield, for \( f \in U \),

\[
B_1 H(Af) = f_0\beta_0^{(1)} H(0) + \sum_{n,j=1} 2^n \sum_{j=1}^n \varepsilon_n^{(j)} \beta_n^{(j)} H(\alpha_n^{(j)} f)
\]

\[
= f_0 + \sum_{n,j=1} 2^n \sum_{j=1}^n \varepsilon_n^{(j)} (A_n^{(j)} f)^n.
\]

\( \square \)

In the following theorem we denote by \( K \) a compact metrizable uncountable space which has a closed subspace homeomorphic to \( I \) and which is a Peano space (i.e. a continuous image of \( I \)), and by \( U \) the open unit ball of \( C(K) \). Recall that for example every connected compact manifold is this kind of space \( K \).

### 3.4. Theorem.

Let \( r > 0 \) and let \( F : rU \to C(K) \) be a uniformly \((K, rU)\)-integral holomorphic mapping. There exist linear operators \( A \in L(C(K), \ell_\infty(K)) \) and \( B \in L(E, C(K)) \), where \( E \subset \ell_\infty(K) \) is the closed linear span of \( H(A(U)) \), such that

\[
F(f) = BH(Af) = B \left( \frac{1}{1-Af} \right)
\]

for all \( f \in U \).

Proof. 1°. We first consider the case \( K = I^3 \). One easily verifies that the mapping \( G : U \to C(I^3), G(x) := F(rx) \), is uniformly \((K, U)\)-integral. We apply Theorem 2.1 to write \( G = B_G \circ \psi_G \), where \( B_G \in L(C(I^3)) \),

\[
\psi_G(f) = f_0 + \sum_{n=1}^\infty \prod_{k=1}^n \psi_n^{(k)} f
\]

and \( ||\psi_n^{(k)}|| \leq 1 \) for all \( n \) and \( k \) (see (2.1)). We get the representation \( F = B_G \circ \psi \), where

\[
\psi(f) = f_0 + \sum_{n=1}^\infty r^{-n} \prod_{k=1}^n \psi_n^{(k)} f.
\]
For all \( n \in \mathbb{N} \), the polarization formula ([D2], Theorem 1.5) implies the following equality for all complex numbers \( \lambda_k, k = 1, \ldots, n \):

\[
\prod_{k=1}^{n} \lambda_k = \frac{1}{n! 2^n} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_n (\sum_{k=1}^{n} \varepsilon_k \lambda_k)^n.
\]

Applying this we get

\[
\psi(f) = f_0 + \sum_{n \in \mathbb{N}} r^{-n} 2^{-n}(n!)^{-1} n^n \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_n \left( n^{-1} \sum_{k=1}^{n} \varepsilon_k \psi_n^{(k)}(f) \right)^n
\]

\[
= f_0 + \sum_{n \in \mathbb{N}} e^{-n}(n!)^{-1} n^n \sum_{\varepsilon_j = \pm 1} (2r/e)^{-n} \varepsilon_1 \cdots \varepsilon_n \left( n^{-1} \sum_{k=1}^{n} \varepsilon_k \psi_n^{(k)}(f) \right)^n.
\]

We have

\[
e^{-n}(n!)^{-1} n^n \leq 1,
\]

since \((n!)^{-1} n^n\) is the \(n\)th term in the Taylor series of \(e^n\). Hence (3.16) yields a representation for \(\psi\) which satisfies the assumptions of Lemma 3.3; in particular, \(A_n^{(j)} := n^{-1} \sum_{k=1}^{n} \varepsilon_k \psi_n^{(k)}\). Let \(A\) and \(B_1\) be as in Lemma 3.3. Setting

\[
B = B_G B_1
\]

gives the desired factorization of \(F\) in the case \(K = I^3\).

2. Let \(K\) be arbitrary. We first choose continuous surjections \(\varphi : K \to I^3\) and \(\varrho : I^3 \to K\) with regular averaging operators. Concerning \(\varrho\), it is enough to take a retraction \(I^3 \to I\) and compose it with a continuous surjection \(I \to K\) having a regular averaging operator (see [H], Theorem 2.2., or [T3], Theorem 2.) The map \(\varphi\) can be found by composing a retraction from \(K\) onto a subspace homeomorphic to \(I\) (recall that \(I\) is an absolute retract space), with a continuous surjection \(I \to I^3\) having a regular averaging operator. We denote a contractive left inverse of \(\varrho \circ \varphi\) by \(\tilde{\varrho}\). Taking a (usually discontinuous) right inverse \(\varphi^{-1}\) of \(\varphi\) one can define a contractive left inverse \(\tilde{\varphi}\) for \(\varphi \circ \varrho\) by \(\tilde{\varphi} = \varphi^{-1} \in L(\ell_\infty(K), \ell_\infty(I^3))\).

If \(F\) is as in the assumption, it follows in a straightforward way from Definition 1.1 that \(G := \varrho \circ F \circ \tilde{\varrho}\) is a uniformly \((I^3, V)\)-integral holomorphic mapping \(V \to C(I^3)\), where \(V \subset C(I^3)\) is the open unit ball. By part 1 of the proof we find bounded linear operators \(A_G\) and \(B_G\) as in (3.13) such that

\[
G(f) = B_G H(A_G f)
\]

for \(f \in V\). We get for \(f \in U\)

\[
F(f) = \tilde{\varrho} \varrho \circ F \circ \tilde{\varrho} \circ \varrho f = \tilde{\varrho} B_G H(\varrho \varrho f)
\]

\[
= \hat{\varrho} B_G \hat{\varphi} \circ H(\varrho \varrho f) = \hat{\varrho} B_G H(\varrho A_G \varrho f),
\]

so that setting \(B = \tilde{\varrho} B_G \hat{\varphi}\) and \(A := \varrho A_G \varrho\) yields the result. We leave the details to the reader. \(\square\)

3.5. Remarks. 1°. In the case \(K = I^3\) the elements of \(A(C(K))\) can be discontinuous only in the subset \(\{1\} \times I^2\) of the boundary of \(I^3\). In the case of general \(K\) the discontinuity may be more serious; it depends on the choice of the mapping \(\varphi\) above.
The explanation for the discontinuity lies in the coefficients $\varepsilon_k$ in the polarization formula (see (3.16)): they cause the space $A(C(K))$ to necessarily contain functions which oscillate infinitely often in $K$ with a constant amplitude. The space $C(K)$ does not contain such elements.

$2^\circ$. By the extension property of the space $\ell_{\infty}(K)$ it is always possible to extend $B$ as a bounded operator $\ell_{\infty}(K) \to \ell_{\infty}(K)$. (See [LT1], Proposition 2.6.2(iii).) In some cases it is possible to extend $B$ even as a bounded operator $E + C(K) \to C(K)$ where $E + C(K)$ is considered as a closed subspace of $\ell_{\infty}(K)$; see Theorem 3.6 below.

$3^\circ$. There is a natural explanation for the constant $r > \varepsilon > 1$. It comes (modulo an $\varepsilon > 0$) basically from the fact that we cannot avoid the use of the polarization formula in (3.16). A related fact is that the “uniformly integral norm” (1.3) somehow measures the symmetric multilinear mappings in the Taylor series of the given $F$, whereas the representation (3.13) is more like a “polynomial of an infinite degree”; compare to [D2], Theorem 1.7.

3.6. Theorem. Let $r$, $K$, $U$ and $F$ be as in Theorem 3.4. Assume that for every $n \in \mathbb{N}$ the linear operator

$$
T(F, n) \in L(C(K^n), C(K)) \quad \text{and} \quad (T(F, n)f)(t) = \langle f, \mu(F, n, t) \rangle \quad \text{for } t \in K
$$

where $\mu(F, n, t)$ is as in Definition 1.1, $1^\circ$, is compact. Then the operator $B$ of Theorem 3.4 can be extended to an element of $L(E + C(K), C(K))$ or $L(\ell_{\infty}(K), C(K))$, where $E + C(K)$ is considered as a closed subspace of $\ell_{\infty}(K)$.

Proof. The assumption (3.19) implies that $B_F$ in the universal mapping Theorem 2.1 is compact. (The reader has to verify this from the proof of Theorem 2.1. of [T2], especially (2.4) and (2.5) there.) Hence, also the operator $B$ is compact, see (3.12) and (3.17) etc. The result follows now from the extension properties of $L_{\infty}$–spaces for compact operators, see [LT], Theorem II.5.25.2.

References


Department of Mathematics, P.O. Box 4 (Hallituskatu 15), FIN-00014 University of Helsinki, Finland

E-mail address: Jari.Taskinen@Helsinki.Fi