KEEPING ADDITIVITY OF THE NULL IDEAL SMALL

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Abstract. We shall show that various statements are consistent with additivity of the null ideal equal to \(\aleph_1\); for example, “all branchless trees of size \(\aleph_1\) are special”, (S) conjecture and “there are only five cofinal types of directed posets of size \(\aleph_1\)”.

0. Introduction

In this paper we provide machinery for proving that a certain large class of forcings has a certain regularity property. The class in question includes posets used for

1. specializing branchless trees [S],
2. (S) conjecture [T2],
3. embedding the poset for adding \(\aleph_1\) Cohen reals into a given poset of uniform density \(\aleph_1\) [SZ],
4. classification of directed posets [T1] or transitive relations [T3] of size \(\aleph_1\),
5. other “side condition” combinatorics on \(\aleph_1\); e.g. shooting an uncountable set through a coherent sequence on \(\omega_1\) [T2].

The regularity property we obtain implies preservation of additivity of the ideal of Lebesgue null sets. As a corollary, it is consistent with ZFC set theory that additivity of the null ideal is \(\aleph_1\) and all statements obtainable through (1)–(5) above hold, that is, all branchless trees of size \(\aleph_1\) are special, (S) conjecture holds etc. Thus a definite limitation on a canonical variation of the powerful “side condition” method has been exacted for the first time.

Our notation follows the set-theoretical standard as set forth in [J]. In a forcing notion, \(p \leq q\) means “\(p\) is more informative than \(q\)”. A tree \(T\) of height \(\omega_1\) is special if there is a function \(f : T \to \omega\) with \(s < t\) in \(T\) implying \(f(s) \neq f(t)\). Trees grow upwards. (S) conjecture is the statement “every hereditarily separable Hausdorff space is hereditarily Lindelöf”. The symbol \(\mathcal{N}\) denotes the collection of Borel null sets, often confused with their Borel codes. \(H_\kappa\) is the set of all sets of hereditary cardinality \(<\kappa\). If \(N\) is an elementary submodel of \(H_\kappa\) and \(P \in N\) is a forcing, a condition \(p \in P\) is called \(N\)-master if for every dense set \(D \subset P\) in \(N\), the set \(D \cap N\) is predense below \(p\).

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1. Localization

Definition 1. Let $\mathcal{F} \subset \omega^\omega$ and let $e$ be a positive integer. The family $\mathcal{F}$ is said to be $e$-localized if there exists a function $h : \omega \to [\omega]^{<\aleph_0}$ such that

1. $|h(n)| \leq n^e$ for every integer $n$,
2. for every $f \in \mathcal{F}$ there is an integer $n \in \omega$ so that for every $m > n$, $f(m) \notin h(m)$ holds.

If $e = 1$ then the family $\mathcal{F}$ is said to be localized.

The relevance of the above definition is revealed in the result of Bartoszyński [B, BJ Section 2.3.A] saying that for a transitive model $M$ of ZFC the following are equivalent:

1. $M \cap \omega^\omega$ is localized;
2. $M \cap \omega^\omega$ is $e$-localized for some positive integer $e$;
3. the union of all Lebesgue measure zero Borel sets coded in $M$ has measure zero.

There is a natural c.c.c. forcing for making the set of ground model reals localized [Tr] and there are some preservation theorems for “the set of ground model reals is not localized” [JS], [BJ]. We shall prove that a large class of forcings preserves unlocalized families in a strong sense.

Definition 2. (1) [JS] Let $\kappa$ be a large regular cardinal and $N \prec H_\kappa$. We say that a function $f \in \omega^\omega$ is $N$-big if for every $h : \omega \to [\omega]^{<\aleph_0}$ in the model $N$ with $|h(n)| \leq n$ the set $\{m \in \omega : f(m) \notin h(m)\}$ is infinite.

2. Let $P$ be a forcing. We say that $P$ is friendly if for every $p \in P$, every large enough regular cardinal $\kappa$, every countable elementary submodel $N \prec H_\kappa$ with $p,P$ in $N$ and every $N$-big function $f \in \omega^\omega$ there is an $N$-master condition $q \leq p$ such that $q \Vdash \text{"} f \text{ is } N[G]-\text{big}\text{"}.$

The important point is that it is possible to iterate friendly forcings preserving the statement “the family of ground model reals is not localized” or equivalently, “$\bigcup (N \cap V) \notin N^n$” -Lemma 13. Obviously, a finite iteration of friendly forcings is friendly and friendliness is inherited by regular subposets. In [JS] it is proved that the random algebra as well as every $\sigma$-centered forcing is friendly. We considerably extend these results.

2. Specializing trees

The purpose of this section is to prove that the usual specializing forcing for a branchless tree of height $\omega_1$ is friendly. The technique will be of great use in the next section. For now, fix a tree $T$ of height $\omega_1$ and no branches of length $\omega_1$. There is no restriction on the size of levels of $T$.

Definition 3. (1) If $a,b$ are disjoint finite subsets of $T$ then we say that $a$ and $b$ fit together if for every $s \in a$ and $t \in b$, $s$ and $t$ are incompatible as elements of $T$.

2. The specialization forcing is $P = \{p : p$ is a finite function from $T$ to $\omega$ such that $s <_T t$ in $\text{dom}(p)$ implies $p(s) \neq p(t)\}$ ordered by reverse inclusion.

Lemma 4. Let $\{a_\alpha : \alpha \in \omega_1\}$ be a family of pairwise disjoint finite subsets of $T$. Then there is an infinite set $Y \subset \omega_1$ such that $a_\alpha : \alpha \in Y$ pairwise fit together.
Proof. The usual proof of c.c.c.-ness of $P$ [S, p. 103] shows that there are \( \alpha \neq \beta \) with \( a_\alpha, a_\beta \) fitting together. The lemma follows from the Erdős-Dushnik-Miller partition relation \( \omega_1 \to (\omega_1, \omega) \) applied to the function \( f : [\omega_1]^2 \to 2 \) defined by \( f(\alpha, \beta) = 0 \) iff \( a_\alpha, a_\beta \) fit together.

Now assume that \( \kappa \) is a large regular cardinal, \( N < H_\kappa \) is a countable submodel with \( T \in N \) and \( f \) is \( N \)-big. We shall prove that \( P \models \text{"} f \) is \( N[G]\)-big.". Then, since the forcing \( P \) is c.c.c., any condition in it is \( N \)-master and it witnesses the friendliness of \( P \) as desired.

For contradiction, let \( p \in P, n \in \omega \) and \( \dot{h} \in N \) be such that

1. \( p \models \text{"} \text{for all } m \in \omega, |\dot{h}(m)| \leq m. \text{"} \)
2. \( p \models \text{"} \text{for every } m > n, f(m) \in \dot{h}(m) \text{"}. \)

Let \( p_0 = p \cap N \in N \). By c.c.c.-ness of \( P \), by strengthening the condition \( p \) if necessary one can arrange that \( p_0 \models \text{"} \text{for all } m \in \omega, |\dot{h}(m)| \leq m. \text{"} \).

Work in \( N \). Fix an integer \( m > n \) and by a tree induction construct a tree \( X \subset <\omega(\omega + 1) \), a partition \( X = X_0 \cup X_1 \) and a function \( F : X \to P \) so that

1. \( X \) empty sequence \( 0 \) is in \( X \) and \( F(0) = p_0 \),
2. \( s \subset t \) in \( X \) implies \( F(s) \supseteq F(t) \) in \( P \),

and for each \( s \in X \) with \( \ell h(s) = i \) exactly one of the following holds:

3. either, there is a sequence \( \langle q_j : j \in \omega \rangle \) such that each \( q_j \leq F(s) \) forces in \( P \) that \( i \in \dot{h}(m) \), and moreover, the sets \( \text{dom}(q_j \setminus F(s)) : j \in \omega \) pairwise fit together. In this case, \( s \in X_0 \), the set of successors of \( s \) in \( X \) is exactly \( \{ s^<(j) : j \in \omega \} \) and \( F(s^<(j)) = q_j \),
4. or, no such sequence exists. Then \( s \in X_1 \), the only successor of \( s \) in the tree \( X \) is \( s^>(\omega) \) and \( F(s^>(\omega)) = F(s) \).

A set \( o(X) \subset \omega \) is defined by \( i \in o(X) \) iff there exists a function \( G : X \to \omega \) such that for every sequence \( s \in X \) of length \( i \), if \( \forall j \in i s(j) > G(s \upharpoonright j) \) then \( s \in X_0 \).

Claim 5. \( |o(X)| \leq m \).

Proof. Suppose for a contradiction that there are \( m + 1 \) elements of \( |o(X)| \), enumerated in the increasing order as \( i_0 \) through \( i_m \). Pick witnesses \( G_k : k \leq m \) for \( i_k \in o(X) \). Then for every sequence \( s \in X \) of length \( i_m + 1 \) such that \( \forall j \leq i_m \forall k \leq m s(j) > G_k(s \upharpoonright j) \) (and there are plenty of these) the value \( F(s) \) as an element of \( P \) forces each one of the \( m + 1 \) distinct integers \( i_k : k \leq m \) into the set \( \dot{h}(m) \).

But this is absurd, since \( p \geq F(s) \) and \( p \models |\dot{h}(m)| \leq m. \)

Claim 6. \( \forall (m) \in o(X) \).

Proof. The proof of this fact takes place outside of the model \( N \). To define the witness \( G : X \to \omega \) for \( f(m) \in o(X) \), consider two cases:

1. \( s \in X_1 \). Then let \( G(s) = 0 \).
2. \( s \in X_0 \). By (3) above, \( \{ a_j = \text{dom}(F(s^<(j)) \setminus F(s)) : j \in \omega \} \) is a family of pairwise disjoint fitting finite subsets of the tree \( T \). There is an integer \( j_0 \) such that for every \( j > j_0 \), the sets \( a_j \) and \( \text{dom}(p \setminus p_0) \) fit together; set \( G(s) = j_0 \).

The existence of an integer \( j_0 \) as in (2) above can be demonstrated as follows. By elementarity of the model \( N \), if \( u \in N \cap T \) and \( t \in \text{dom}(p \setminus p_0) \) are compatible as elements of the tree \( T \), then necessarily \( u \triangleleft_T t \), for if \( t \triangleleft_T u \) then \( t \in N \), contradicting the definition of the condition \( p_0 \). By the mutual fitting of the \( a_j \)'s, only finitely many of the sets \( a_j \subset N \cap T \) can have nonempty intersection with the
Theorem 6. Let $X$ be valid at $\alpha$. Above, it is possible to show that $P(T_2)$ that the poset $P$ witnessing the property (2) at $\beta$.

Remark. The purpose of this section is to define the class of ideal-based forcings and to prove friendliness of elements of this class. Our scheme is designed to comprehend many side condition forcings as used in the work of S. Todorcevic [T1], [T2], [T3] and others. Let $A$ be a set of finite subsets of $\omega_1$ ordered by $\subseteq$ and let $\mathcal{I}$ be an ideal on $\omega_1$ such that the following axioms are satisfied:

(A) $\subseteq$ refines the inclusion, for each $a \in A$ and $\beta \in \omega_1$, $a \cap \beta \subseteq a$ holds and if $a, b$ are both in $A$ and $\subseteq$-compatible then $a \cup b \in A$ is their $\subseteq$-upper bound.

3. Side condition forcings

Theorem 6. Let $T$ be a tree of height $\omega_1$ without branches of height $\omega_1$. Then the standard $T$-specialization forcing is friendly.

Remark. The result becomes rather trivial if the tree $T$ is supposed to have countable levels. In such a case, it is easy to prove that every real added by the specialization forcing comes from a Cohen-generic extension. Thus the specialization forcing for $T$ must necessarily be friendly by results of [JS].

Remark. The same technology can be used to demonstrate friendliness of a number of finite condition forcings, whose c.c.c. is proved in a certain canonical manner. For example, let $\{f_\alpha : \alpha \in \omega_1\} \subset \omega^\omega$ be a modulo finite increasing unbounded sequence of increasing functions. Let the partition $H : [\omega_1]^2 \to 2$ be defined as $H(\alpha, \beta) = 0$ if $\alpha < \beta$ and there is an integer $n \in \omega$ with $f_\beta(n) > f_\alpha(n)$. It is known [T2] that the poset $P$ of finite 0-homogeneous sets ordered by reverse inclusion is c.c.c. and destroys the unboundedness of the sequence. Using the same method as above, it is possible to show that $P$ is friendly. The following is open:

Question 7. Is OCA [T2] consistent with additivity of the null ideal equal to $\mathfrak{N}_1$?
Fact 8. The forcings for the following problems are (amended) ideal-based:

Moreover, for each \( a \in A \) there are

\( \mathcal{J} \)

(C) a \( \mathcal{J} \)-positive set \( Z \subset \omega_1 \) such that \( a \sqsubset a \cup \{ \beta \} \in A \) holds for every \( \beta \in Z \),

(D) an \( \mathcal{J} \)-large set \( Y \subset \omega_1 \) such that for every \( \beta \in Y \) the implication \( a \cap \beta \leq (a \cap \beta) \cup \{ \beta \} \in A \) holds.

The pair \( \langle A, \sqsubseteq \rangle \) is to be understood as a problematic finite-condition forcing construction for which c.c.c. cannot be proved, or which collapses \( \aleph_1 \) outright. The existence of the ideal ensures that there is a way to add a \( \sqsubseteq \)-filter which meets many dense subsets of \( \langle A, \sqsubseteq \rangle \). Fix a large regular cardinal \( \kappa \). The ideal-based forcing \( P \) derived from \( A, \sqsubseteq, \mathcal{J} \) has the following form:

\[
P = \{ f : f \text{ is a finite function from } \omega_1 \text{ to } H_\kappa, \text{ for } \alpha \in \text{dom}(f) \ f(\alpha) = \langle M_\alpha, \xi_\alpha \rangle \text{ and} \}
\]

(E) the set \( \text{body}(f) = \{ \xi_\alpha : \alpha \in \text{dom}(f) \} \) is in \( A \),

(F) every \( M_\alpha \) is a countable elementary submodel of \( H_\kappa \) containing \( A, \sqsubseteq, \mathcal{J}, f \upharpoonright \alpha \),

(G) \( \xi_\alpha \notin \bigcup (M_\alpha \cap \mathcal{J}) \).

The order on \( P \) is defined by \( f \leq g \) if \( g \subset f \) and \( \text{body}(g) \sqsubseteq \text{body}(f) \).

The forcing \( P \) adds a \( \sqsubseteq \)-filter \( \{ a \in A : a = \text{body}(p) \text{ for some } p \in G \} \) which meets all dense subsets of \( \langle A, \sqsubseteq \rangle \) which are in some sense large as measured by \( \mathcal{J} \).

 Frequently, for the sake of preservation of \( \aleph_2 \) one needs to consider an amended variation of \( P \) which has matrices of models as side conditions instead of just an \( \varepsilon \)-chain of models as above [T2]. We call such forcings \textit{amended ideal-based}: since our proofs carry over to the class of amended ideal-based forcings with only more complicated notation, we concentrate on the class of ideal-based forcings proper.

The point of course is that this class is reasonably wide; indeed, our scheme includes many of the side condition posets used in the literature. The following fact provides a by no means complete list.

Fact 8. The forcings for the following problems are (amended) ideal-based:

\begin{enumerate}
  \item[(1)] (S) conjecture [T2],
  \item[(2)] making a poset of uniform density \( \aleph_1 \) add \( \aleph_1 \) Cohen reals [SZ],
  \item[(3)] classification of transitive relations on \( \aleph_1 \) [T1], [T3],
  \item[(4)] shooting an uncountable set through a coherent sequence on \( \omega_1 \) [T2].
\end{enumerate}

Proof. We consider the case of (S) conjecture. As in [T2], it is only necessary to cope with the following problem. Let \( \omega_1 \) be equipped with topology \( \mathcal{T} \) so that the space \( (\omega_1, \mathcal{T}) \)

\begin{enumerate}
  \item[(1)] is hereditarily separable, that is, for every \( X \subset \omega_1 \) there is a countable subset \( Y \subset X \) with the same closure,
  \item[(2)] is not hereditarily Lindelöf, and it is even right separated, that is, for each \( \alpha \in \omega_1 \) there is an open set \( O_\alpha \) such that \( \alpha \in O_\alpha \) and the closure of \( O_\alpha \) is a subset of \( \alpha + 1 \).
\end{enumerate}

We wish to violate the hereditary separability of the space \( (\omega_1, \mathcal{T}) \) by introducing an uncountable discrete subset to it. The forcing for doing that [T2] can be cast as an ideal-based forcing derived from \( A = [\omega_1]^{<\aleph_0}, a \sqsubseteq b \) just in case \( a \subset b \) and for every \( \xi \in (b \setminus a) \) and every \( \zeta \in a \) \( \xi \notin O_\zeta \); furthermore, \( \mathcal{J} = \{ X \subset \omega_1 : \text{the closure of } X \text{ is countable} \} \). It is not difficult to check the axioms (A) through (D) in the definition of ideal-based. (B) follows from hereditary separability and (D) from \( \mathcal{J} \)-smallness of every \( O_\alpha \)
The intended uncountable discrete set will be \( \bigcup \{ \text{body}(f) : f \in G \} \), where \( G \subset P \) is a generic filter.

**Theorem 9.** Any ideal-based forcing \( P \) is proper and friendly.

**Proof.** Let \( A, \subseteq, \mathcal{I}, \kappa \) be the parameters from which \( P \) is defined. To prove the properness, let \( p_0 \in P, \lambda \) be a large regular cardinal, let \( N \prec H_\lambda \) be a countable elementary submodel with \( p_0, A \subseteq, \mathcal{I}, \kappa \in N, \) and let \( \delta = N \cap \omega_1. \) By (C) there is a countable ordinal \( \xi \) such that \( \xi \notin (\mathcal{I} \cap N) \) and \( \text{body}(p_0) \subseteq \text{body}(p_0) \cup \{ \xi \} \in A. \)

Let \( p_1 = p_0 \cup \{ \langle \delta, (N \cap H_\kappa, \xi) \rangle \}. \) Obviously, \( p_1 \leq p_0 \) and we shall show that \( p_1 \) is a master condition for the model \( N. \) Thus, for any dense set \( D \) of \( P \) which happens to be in \( N, \) the set \( D \cap N \) must be proved predense below \( p_1. \) Fix \( p_2 \leq p_1 \) and a dense set \( D \in N; \) we shall produce conditions \( p_5 \leq p_2 \) and \( q \in D \cap N \) with \( p_5 \leq q, \) completing the proof of properness. By strengthening \( p_2 \) if necessary, it can be assumed that there is an element of \( D \) above \( p_2. \)

Let \( p_3 = p_2 \cap N. \) Obviously \( p_3 \in P \cap N \) and \( p_2 \leq p_3, \) by (A). The whole point of the proof is to find a way of carefully extending \( p_3 \) within \( N \) while preserving compatibility with \( p_2. \) Let \( k = \| p_2 \| \) and \( \xi_0 \cdots \xi_{k-1} \) enumerate \( \text{body}(p_2) \setminus \text{body}(p_3) \) in the increasing order. By induction on \( l < k \) define sets \( S(t)(l) \subset \omega_1 \) for all \( t \in \omega_1 \) simultaneously by

1. \( S(t)(0) = \{ \xi \in \omega_1 \setminus \text{max(\text{body}(p_3) \cup \text{rng}(t))} : \exists p_4 \leq p_3 \) such that \( p_4 \) has an element of \( D \) above it and \( \text{body}(p_3) \cup \text{rng}(t) \subseteq \text{body}(p_4) = \text{body}(p_3) \cup \text{rng}(t) \cup \{ \xi \} \in A. \}
2. \( S(t)(l+1) = \{ \xi \in \omega_1 \setminus \text{max(\text{body}(p_3) \cup \text{rng}(t))} : \text{body}(p_3) \cup \text{rng}(t) \subseteq \text{body}(p_4) \cup \text{rng}(t) \cup \{ \xi \} \in A \) and the set \( S(t \setminus \langle \xi \rangle)(l) \) is \( \mathcal{I} \)-positive.\]

**Claim 10.** The set \( S(\langle i \rangle)(k-1) \) is \( \mathcal{I} \)-positive.

**Proof.** Note that the system \( \{ S(t)(l) : t \in \omega_1, l < k \} \) belongs to all the models mentioned in \( p_2 \) above \( \delta \) since it is in \( N \cap H_{\kappa_2}. \) The claim will be proved by contradiction. If \( S(\langle i \rangle)(k-1) \) were an element of \( \mathcal{I}, \) by induction on \( l < k \) one could show that \( \xi_l \notin S(\langle i \rangle : l' < l)(k-1-l). \) But this is a contradiction to the case (1) of the definition of the system \( \{ S(t)(l) : t \in \omega_1, l < k \}, \) since \( \xi_{k-1} \in S(\langle i \rangle : l < k - 1)(l) \) as witnessed by the condition \( p_2. \)

Now by induction on \( l < k \) build \( \zeta_l, T_l, X_l \) so that

1. for \( l \leq k, \) \( a_l = \langle \zeta_{l'} : l' < l \rangle \) is an increasing sequence of countable ordinals larger than \( \text{max(\text{body}(p_3))} \) in \( N, \)
2. \( T_l \subset N \) is an \( \mathcal{I} \)-positive countable subset of the \( \mathcal{I} \)-positive set \( S(a_l)(k-l-1), \) by (B),
3. \( \text{body}(p_2) \) and \( \text{body}(p_3) \cup \text{rng}(a_l) \) are \( \mathcal{I} \)-compatible and an \( \mathcal{I} \)-large set \( X_l \subset \omega_1 \) is a witness to (D) for \( \text{body}(p_2) \cup \text{rng}(a_l), \)
4. \( \zeta_l \in T_l \cap X_l. \)

By the construction, \( a_k \in N, \) \( \text{body}(p_2) \) and \( \text{body}(p_3) \cup \text{rng}(a_k) \) are \( \mathcal{I} \)-compatible and moreover, there is a condition \( p_4 \leq p_3 \) such that there is an element of \( D \) above it and \( \text{body}(p_4) = \text{body}(p_3) \cup \text{rng}(a_k). \) By the elementarity of the model \( N, \) there are such \( p_4 \) and \( q \in D \) above it already in \( N. \) By the definition of the forcing \( P, \) \( p_5 = p_4 \cup p_2 \) is a lower bound of \( p_4 \) and \( p_2 \) and has \( q \in D \cap N \) above it as desired.

The friendliness of \( P \) is proved by a trick similar to the one in Section 2. Let us adopt the framework from the proof of properness of \( P, \) in particular, choose
$p_0, N \ldots$ and the master condition $p_1 \leq p_0$ for the model $N$ as constructed above. Let $f \in N$ be an $N$-big function. We shall show that $p_1 \models \text{“} f \text{ is } N[G]-\text{big}\text{”}$. 

For contradiction, let $p_2 \leq p_1, h \in N$ and $n \in \omega$ be such that 

1. $p_2 \models \text{“} \text{for all integers } m, |h(m)| \leq m\text{”,}$ 
2. $p_2 \models \text{“} \text{for all integers } m > n, f(m) \in h(m)\text{”}.\

Let $p_3 = p_2 \cap N$. Then $p_3 \geq p_2$ is in $N$ and by strengthening the condition $p_2$ if necessary we may assume that $p_3 \models \text{“} \text{for all integers } m, |h(m)| \leq m\text{”}$. Let $k = |p_2 \setminus p_3| \geq 1$.

Work in $N$. Fix an integer $m > n$. By a tree induction construct a tree $X \subset \langle \omega, N \rangle$, its subset $X_0$ and functions $F : X \to P, T : X_0 \to V$ so that: 

1. the empty sequence $0$ is in $X$ and $F(0) = p_3$, 
2. for every $s \subset t$ both in $X$, $F(t) \leq F(s)$ holds in $P$. 

Moreover, at each $s \in X$, exactly one of the two following cases will hold.

**Case 1.** There is a tree $T(s)$ on $\leq^k \omega_1$ such that 

(a) the empty sequence is in $T(s)$ and $T$ consists of increasing sequences of ordinals above $\max(\text{body}(F(s)))$, 

(b) for every $t \in T(s)$ of length $< k$ the set $\{ \zeta \in \omega_1 : t \prec (\zeta) \in T(s) \}$ is countable and $3$-positive; moreover for each $\zeta$ in this set, $\text{body}(F(s)) \cup \text{rng}(t) \subseteq \text{body}(F(s)) \cup \text{rng}(t) \cup \{ \zeta \} \in A$, 

(c) for every $t \in T(s)$ of length $k$ (i.e. a terminal node) there is a condition $q_t \leq F(s)$ in $P$ such that $\text{body}(q_t) = \text{body}(F(s)) \cup \text{rng}(t)$ and $q_t \models \text{“} |\text{length}(s) \in h(m)|\text{”}$.\

In this case, let $s \in X_0$, $T(s)$ will be the value of $T$ at $s$ and $s \prec(\langle \rangle)$ be an $s$-large set and so it is possible to find a sequence $u \in X$ of length $i$ such that $F(u) \models \text{“} |\text{length}(s) \in h(m)|\text{”}$.

**Case 2.** No such tree exists. Then $s \notin X_0$, the only successor of the sequence $s$ in $X$ is $s \prec (\langle \rangle)$ and $F(s) = F(s \prec (\langle \rangle))$.

This completes the inductive definition of the tree $X$ in $N$. Define a set $o(X) \subset \omega$ by $i \in o(X)$ if there exists a collection $\{ A(s,t) : s \in X_0, t \in T(s) \text{ is not a terminal node} \}$ of $3$-large sets so that for each sequence $u \in X$ of length $i$ if (*) below holds for all $j < i$ then $u \in X_0$.

(*) If $u \upharpoonright j \in X_0$ then $F(u \upharpoonright (j+1)) = q_t$ for some terminal node $t \in T(u \upharpoonright j)$ such that $\forall l < k \text{ } t(l+1) \in A(u \upharpoonright j, t \upharpoonright l)$.

**Claim 11.** $|o(X)| \leq m$.

**Proof.** For contradiction, suppose that there are $m+1$ elements of $o(X)$ enumerated in the increasing order as $i_0$ through $i_m$. Pick witnesses $\{ A_i(s,t) : s \in X_0, t \in T(s) \text{ is not a terminal node} \}$ for $i_0 \in o(X)$ and define $\{ A(s,t) : s \in X_0, t \in T(s) \text{ is not a terminal node} \}$ by $A(s,t) = \bigcap_{i \leq m} A_i(s,t)$. Now each $A(s,t)$ is still a $3$-large set and so it is possible to find a sequence $u \in X$ of length $i_m + 1$ such that (*) holds for each $j < l\text{th}(u)$. But then, from the construction of $X$ and $F$ it follows that $F(u) \models \text{“} \{i_0, \ldots, i_m\} \subset \text{h}(m)\text{”}$, which is absurd since $F(u) \leq p_3$ and $p_3 \models \text{“} |\text{h}(m)| \leq m\text{”}$. 

**Claim 12.** $f(m) \in o(X)$.

**Proof.** It is necessary to define the witness collection; our candidate lies outside of $N$. For $s \in X_0$ and $t \in T(s)$ there will be two cases:
(1) Either \textit{body}(p_2) and \textit{body}(F(s)) \cup \textit{rng}(t) are \subseteq-compatible. In such a case let \( A(s,t) \) be an \( \mathcal{L} \)-large witness to (D) for \( \text{body}(p_2) \cup \text{body}(F(s)) \cup \text{rng}(t) \).

(2) Otherwise let \( A(s,t) = \omega_1 \).

Now suppose that a sequence \( u \in X \) of length \( f(m) \) satisfies (*) for all \( j < f(m) \). By the definition of \( A(s,t) \) and the tree \( X \), necessarily \textit{body}(F(u)) and \textit{body}(p_2) are \subseteq-compatible and therefore \( p_2 \) and \( F(u) \) are compatible conditions in \( P \). The proof of \( u \in X_0 \) is essentially a repetition of the proof of the properness of \( P \) with the condition \( p_2 \) replaced with \( F(u) \cup p_2 \) and the phrase “element of \( D \) above it” replaced with “forces \( f(m) \) into \( h(m) \)”. \hfill \Box

Now the definition of \( X, o(X) \) was uniform for integers \( m \geq n \). Thus within the model \( N \) there is a sequence \( X_m, o(X_m) : m > n \) such that \( |o(X_m)| \leq m \) and \( f(m) \in o(X_m) \) for every \( m > n \). But then the sequence \( o(X_m) : m > n \) in the model \( N \), understood as a function of \( m \), contradicts \( N \)-bigness of the function \( f \). \hfill \Box

It should be remarked that while ideal-based forcings preserve \textit{add}(\( N \)), they can add dominating functions. If \( \langle f_\alpha : \alpha \in \omega_1 \rangle \subseteq \omega \) is a modulo finite increasing unbounded sequence of increasing functions then it is possible to derive an \( S \)-space from it [T2]. Then the ideal-based forcing killing that space adds a function which modulo finite dominates all \( f_\alpha : \alpha \in \omega_1 \).

4. Conclusion

At last, we are in a position to construct some interesting models of set theory with the additivity of the null ideal equal to \( \aleph_1 \). The classical iteration vehicle gives

\textbf{Lemma 13.} Let \( \langle P_\alpha : \alpha \leq \theta, \check{Q}_\alpha : \alpha < \theta \rangle \) be a countable support iteration of forcings such that \( P_\alpha \models \check{Q}_\alpha \text{ is friendly} \) for each \( \alpha < \theta \). Then \( P_\theta \models \text{“the union of the null sets coded in the ground model is not null”} \).

It seems likely that in fact \( P_\theta \) is a friendly forcing, but we have no argument for that.

\textbf{Proof.} By induction on \( \beta \leq \theta \) we shall demonstrate that \( P_\beta \models \text{“the union of the null sets coded in the ground model is not null”} \). The limit step is handled by [BJ, Theorem 6.3.41]. For the successor step, assume that \( P_\beta \models \text{“the union of the null sets coded in the ground model is not null”} \); we shall prove the same statement for \( P_{\beta+1} \). Choose a generic filter \( G \subseteq P_\beta \) and work in \( V[G] \). It is enough to show that \( Q_\beta \models \text{“} V \cap \omega \text{ is not localized”} \). For contradiction, suppose \( q \in Q_\beta, \check{h} \) are such that \( q \models Q_\beta \text{“for all } n \in \omega, |\check{h}(n)| \leq n \text{ and } \check{h} \text{ localizes } V \cap \omega \)”. Choose a large regular cardinal \( \kappa \) and a countable elementary submodel \( N \prec H_\kappa \) with \( q, Q_\beta, \check{h} \in N \).

\textbf{Claim 14.} There is an \( N \)-big function \( f \in V \cap \omega \).

\textbf{Proof.} By an easy bookkeeping argument, there is a function \( k : \omega \rightarrow [\omega]^{< \aleph_0} \) such that \( \forall n \in \omega |k(n)| \leq n^2 \) and for every function \( l \in N \) with \( l : \omega \rightarrow [\omega]^{< \aleph_0} \), \( \forall n \in \omega |l(n)| \leq n \) there is an integer \( m_0 \) such that for every \( m > m_0 \), \( l(m) \subseteq k(m) \). By the induction hypothesis, \( V \cap \omega \) is not 2-localized and therefore there is a function \( f \in V \cap \omega \) such that the set \( \{ m \in \omega : f(m) \notin k(m) \} \) is infinite. Obviously, the function \( f \) is \( N \)-big. \hfill \Box

The postulated friendly master condition \( r \leq q \) for \( N, f \) contradicts the assumption \( q \models \text{“} \exists m > n \text{ } f(m) \in \check{h}(m) \text{”} \). \hfill \Box
Therefore, starting from a model of the Continuum Hypothesis, any sufficiently
generic iteration of length $\omega_2$ of proper, $\aleph_2$-p.i.c. [S, pg. 262] and friendly forcings
will provide for $\models "\text{add}(N) = \aleph_1, c = \aleph_2,$ all branchless trees of size $\aleph_1$ are special,
($S$) conjecture holds, every poset of uniform density $\aleph_1$ adds $\aleph_1$ Cohen reals etc.”
In the construction, it is necessary to use amended ideal-based forcings in order to
ensure $\aleph_2$-c.c. of the resulting iteration. The standard bookkeeping arguments are
left to the reader.

References


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