TWO POINT SET EXTENSIONS—
A COUNTEREXAMPLE

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Abstract. We show that there exist Cantor sets in the circle that are not extendable to sets that meet every line in the plane in exactly two points. This result solves a problem that was formulated by R. D. Mauldin.

A planar set is called a two point set if every line intersects the set in exactly two points and a partial two point set if every line intersects the set in at most two points. Circles and their subsets are obvious examples of partial two point sets. In [1] and [3], Problem 1070, Dan Mauldin asks the question whether every compact zero-dimensional partial two point set can be extended to a two point set. We show that the answer is no. Let $S^1$ stand for the unit circle in the plane centered at the origin $O$.

Proposition. There exists a Cantor set in $S^1$ that is not contained in a two point set.

Proof. Let $\lambda$ be the linear Lebesgue measure on smooth planar curves (lines and circles in our case). Select a dense open subset $U$ of $S^1$ such that $\lambda(U) \leq 1$ and $C = S^1 \setminus U$ is a Cantor set. Let $x$ be a point in the plane with norm $|x| \geq 2$. Consider as in Figure 1 the two tangent lines to the circle through $x$. The tangent points $P$ and $Q$ divide $S^1$ into two open arcs $A$ and $B$. The open line segment $L$ is perpendicular to the line through $x$ and $O$. Since $|x| \geq 2$ we have $\lambda(L) \geq 2/\sqrt{3} > 1$. Let $p_A$ and $p_B$ be the radial projections with respect to $x$ of $A$ respectively $B$ onto $L$. Note that both projections are contractions which implies that the image of each interval in $A$ or $B$ is an interval in $L$ of shorter length. This means that

$$\lambda(p_A(U \cap A) \cup p_B(U \cap B)) \leq \lambda(p_A(U \cap A)) + \lambda(p_B(U \cap B)) \leq \lambda(U \cap A) + \lambda(U \cap B) = \lambda(U) \leq 1 < \lambda(L).$$

If we pick a $y$ in $L \setminus (p_A(U \cap A) \cup p_B(U \cap B))$ then the line through $x$ and $y$ intersects $C$ in two points.

Let $\ell$ be a line in the plane with distance at least 2 towards the origin. If $C$ is contained in a two point set $D$ then $D \cap \ell$ consists of two points. Pick an $x \in D \cap \ell$ and note that there is a line through $x$ that intersects $C$ in two points and hence it intersects $D$ in three points. 

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Remark. Observe that if $\lambda(U)$ approaches 0 then we can choose $x$ closer to the circle. In fact, we can show that if $C$ is a set in $S^1$ with $\lambda(S^1 \setminus C) \leq 1$ then each point whose distance towards $S^1$ is at least $\lambda(S^1 \setminus C)$ lies on a line that meets $C$ in two points. (For points $x$ inside the circle we apply a similar measure argument to the antipodal map $p_x : S^1 \to S^1$ with respect to $x$, which has the property $\lambda(p_x(U)) \leq \frac{1+|x|}{1-|x|} \lambda(U).$)

The referee informed us that our result also follows from a theorem that was announced by Dan Mauldin at the 1995 BEST conference in Boise, Idaho. Mauldin’s theorem [2], which was obtained independently, is more general than our proposition.

REFERENCES

2. R. D. Mauldin, On sets which meet each line in exactly two points, in preparation.

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