TWO POINT SET EXTENSIONS—
A COUNTEREXAMPLE

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Abstract. We show that there exist Cantor sets in the circle that are not extendable to sets that meet every line in the plane in exactly two points. This result solves a problem that was formulated by R. D. Mauldin.

A planar set is called a two point set if every line intersects the set in exactly two points and a partial two point set if every line intersects the set in at most two points. Circles and their subsets are obvious examples of partial two point sets. In [1] and [3], Problem 1070, Dan Mauldin asks the question whether every compact zero-dimensional partial two point set can be extended to a two point set. We show that the answer is no. Let \( S^1 \) stand for the unit circle in the plane centered at the origin \( O \).

Proposition. There exists a Cantor set in \( S^1 \) that is not contained in a two point set.

Proof. Let \( \lambda \) be the linear Lebesgue measure on smooth planar curves (lines and circles in our case). Select a dense open subset \( U \) of \( S^1 \) such that \( \lambda(U) \leq 1 \) and \( C = S^1 \setminus U \) is a Cantor set. Let \( x \) be a point in the plane with norm \( |x| \geq 2 \). Consider as in Figure 1 the two tangent lines to the circle through \( x \). The tangent points \( P \) and \( Q \) divide \( S^1 \) into two open arcs \( A \) and \( B \). The open line segment \( L \) is perpendicular to the line through \( x \) and \( O \). Since \( |x| \geq 2 \) we have \( \lambda(L) \geq 2/\sqrt{3} > 1 \).

Let \( p_A \) and \( p_B \) be the radial projections with respect to \( x \) of \( A \) and \( B \) onto \( L \). Note that both projections are contractions which implies that the image of each interval in \( A \) or \( B \) is an interval in \( L \) of shorter length. This means that

\[
\lambda(p_A(U \cap A) \cup p_B(U \cap B)) \leq \lambda(p_A(U \cap A)) + \lambda(p_B(U \cap B)) \\
\leq \lambda(U \cap A) + \lambda(U \cap B) = \lambda(U) \leq 1 < \lambda(L).
\]

If we pick a \( y \) in \( L \setminus (p_A(U \cap A) \cup p_B(U \cap B)) \) then the line through \( x \) and \( y \) intersects \( C \) in two points.

Let \( \ell \) be a line in the plane with distance at least 2 towards the origin. If \( C \) is contained in a two point set \( D \) then \( D \cap \ell \) consists of two points. Pick an \( x \in D \cap \ell \) and note that there is a line through \( x \) that intersects \( C \) in two points and hence it intersects \( D \) in three points. \( \square \)
Remark. Observe that if $\lambda(U)$ approaches 0 then we can choose $x$ closer to the circle. In fact, we can show that if $C$ is a set in $S^1$ with $\lambda(S^1 \setminus C) \leq 1$ then each point whose distance towards $S^1$ is at least $\lambda(S^1 \setminus C)$ lies on a line that meets $C$ in two points. (For points $x$ inside the circle we apply a similar measure argument to the antipodal map $p_x : S^1 \to S^1$ with respect to $x$, which has the property $\lambda(p_x(U)) \leq \frac{1+|x|}{1-|x|} \lambda(U)$.)

The referee informed us that our result also follows from a theorem that was announced by Dan Mauldin at the 1995 BEST conference in Boise, Idaho. Mauldin's theorem [2], which was obtained independently, is more general than our proposition.

References

2. R. D. Mauldin, On sets which meet each line in exactly two points, in preparation.

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