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ON GENERA OF SMOOTH CURVES IN HIGHER DIMENSIONAL VARIETIES

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Abstract. We prove that for any smooth projective variety X of dimension ≥ 3, there exists an integer $g_0 = g_0(X)$, such that for any integer $g \geq g_0$, there exists a smooth curve C in X with $g(C) = g$.

Introduction

It's a very elementary fact that, for any integer $g \geq 0$, there exists a smooth curve $C \subset \mathbb{P}^3$ of genus g. It is interesting to ask what the analogous situation is when projective space is replaced by an arbitrary smooth projective variety X of dimension ≥ 3. It can happen of course that a given variety X contains no curves of small genus. For example, abelian varieties contain no rational curves, and Clemens [1] has shown that on a generic hypersurface of degree d in $\mathbb{P}^n$, there are no curves of genus $\leq \frac{d-2n+1}{2}$. So the natural question is whether all sufficiently large genera are realized by smooth curves. Our main result states that this is indeed the case.

Theorem 1. *Let X be a smooth projective variety of dimension $n \geq 3$. Then there exists an integer $g_0 = g_0(X)$ such that for any integer $g \geq g_0$, there exists a smooth curve $C \subset X$ of genus g.*

We also prove an analogous statement for the geometric genera of nodal curves on surfaces.

The paper is organized as follows. In section 1, we prove a lemma that a certain type of function will represent all large enough integers. In section 2, we construct a smooth surface S in any smooth projective variety of dim ≥ 3 such that $Pic(S)$ is very large. We choose divisors in S which give us various genera. In section 3, we prove the analogous statement for surfaces that any large enough integer can be realized as the geometric genus of some nodal curves on a given surface.

We work over the complex number $\mathbb{C}$.

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1. A numerical lemma

Lemma 1. *Let*

$$f(x_1, \dots, x_{18}) = \sum_{i=1}^{9} a x_i^2 + \sum_{i=10}^{18} b x_i^2 + \sum_{i=1}^{18} c x_i,$$

where $a, b \in \mathbb{N}$, $c \in \mathbb{Z}$, and $|a-b| = 1$ or 2. Then there exists an integer $m_0 = m_0(a,b)$ such that for any even integer $m \geq m_0$, $f(x_1, \dots, x_{18}) = m$ has an integral solution.

Proof. 1. Suppose $(a,b) = 1$:

There exists an $n_0 \in \mathbb{Z}$ such that for any integer $n \geq n_0$, $n = ar + bs$ for some $r, s \in \mathbb{N}$. Furthermore, every positive integer is the sum of four squares. So

$$n = a \sum_{i=1}^{4} \bar{x}_i^2 + b \sum_{i=5}^{8} \bar{x}_i^2$$

for some $\bar{x}_1, \dots, \bar{x}_8 \in \mathbb{Z}$. Hence for any even integer $m \geq 2n_0$,

$$m = f(\bar{x}_1, \dots, \bar{x}_4, -\bar{x}_1, \dots, -\bar{x}_4, 0, \bar{x}_5, \dots, \bar{x}_8, -\bar{x}_5, \dots, -\bar{x}_8, 0)$$

for some $\bar{x}_1, \dots, \bar{x}_8 \in \mathbb{Z}$.

2. Suppose $(a,b) = 2$:

(i) Suppose $m \equiv 0 \pmod 4$:

We write $\frac{m}{4}$ as a combination of $\frac{a}{2}$ and $\frac{b}{2}$ for $m \gg 0$. So similiarly,

$$m = f(\bar{x}_1, \dots, \bar{x}_4, -\bar{x}_1, \dots, -\bar{x}_4, 0, \bar{x}_5, \dots, \bar{x}_8, -\bar{x}_5, \dots, -\bar{x}_8, 0)$$

for some $\bar{x}_1, \dots, \bar{x}_8 \in \mathbb{Z}$.

(ii) Suppose $m \equiv 2 \pmod 4$:

Then we have $a + b \equiv 2 \pmod 4$ and $m - a - b \equiv 0 \pmod 4$. Hence for $m \gg 0$

$$m - a - b = f(\bar{x}_1, \dots, \bar{x}_4, -\bar{x}_1, \dots, -\bar{x}_4, 0, \bar{x}_5, \dots, \bar{x}_8, -\bar{x}_5, \dots, -\bar{x}_8, 0),$$

$$m = f(\bar{x}_1, \dots, \bar{x}_4, -\bar{x}_1, \dots, -\bar{x}_4, 1, \bar{x}_5, \dots, \bar{x}_8, -\bar{x}_5, \dots, -\bar{x}_8, -1).$$

for some $\bar{x}_1, \dots, \bar{x}_8 \in \mathbb{Z}$. This proves the lemma. $\square$

Remark 1. From the above proof, $\sum_{i=1}^{4} a\bar{x}_i^2 \leq m$ and $\sum_{i=5}^{8} b\bar{x}_i^2 \leq m$. So $\bar{x}_i \leq \sqrt{m}$. We can modify Lemma 1 as:

Given f, for all even integer $m \gg 0$, m can be represented by $f(x_1, \dots, x_{18})$ with $|x_i| \leq \sqrt{m}$ for all i.

2. Main theorem

It's enough to prove the theorem when dim $X = 3$. So let X be a smooth projective threefold. We first construct curves C_1 and $C_1' \subset X$ (C_1 smooth but C_1' singular) whose arithmetic genera differ by one. Let $H_i (i = 1 \dots 5)$ denote linearly equivalent very ample divisors in X. Consider

$$C_1 = (H_1 \cap H_2) \cup (H_3 \cap H_4),\ C_1' = (H_1 \cap H_2) \cup (H_3 \cap H_5).$$

Choose H_i's properly so that C_1 is smooth but disconnected (with $H_1 \cap H_2$ and $H_3 \cap H_4$ as its two components), and C_1' is connected and has a simple node at $H_1 \cap H_2 \cap H_3 \cap H_5$. Such C_1 and C_1' have the same intersections with divisors in X, and their arithmetic genera differ by one. By moving the very ample divisors, we can construct more curves, $C_1, \dots, C_{18}$, such that

1. They all are disjoint.
2. For all divisors $D \subset X$, $\forall\ 1 \leq i, j \leq 18$,
$$C_i \cdot D = C_j \cdot D.$$
3. $p_a(C_i) = p_a(C_j) + 1$, $\forall\ 1 \leq i \leq 9$, $10 \leq j \leq 18$.

Fix next a very ample divisor H in X and a smooth surface $S \in |nH|$ $(n \gg 0)$ containing all curves constructed above. Computing on S for any of our curves C:
$$2p_a(C) - 2 = K_X \cdot C + nH \cdot C + C \cdot_S C.$$
Let
$$C_i \cdot C_i = -a, \text{ for all } 1 \leq i \leq 9;$$
then
$$C_i \cdot C_i = -(a+2), \text{ for all } 10 \leq i \leq 18.$$
Since we chose $S \in |nH|$ with $n \gg 0$, we may assume that $a > 0$.

Next, let H_S denote the restriction of the very ample divisor H in X to S. Let
$$D_{m,x_1,\dots,x_{18}} \equiv K_S + mH_S + \sum_{i=1}^{18} x_i C_i,$$
$$D_m \equiv K_S + mH_S.$$

The plan is this: We show that for $|x_i|$ small compared to m, the linear series $|D_{m,x_1,\dots,x_{18}}|$ is very ample. By using Lemma 1, we show that by choosing suitable x_i we can find curves in their linear series of all possible genera sufficiently close to $p_a(D_m)$. Then we let m vary.

Lemma 2. *For any $e > 0$, there exists positive integer $m_1 = m_1(e)$ such that $|D_{m,x_1,\dots,x_{18}}|$ is very ample $\forall(m \geq m_1, |x_i| \leq e\sqrt{m})$. In particular, there exist smooth curves in $|D_{m,x_1,\dots,x_{18}}|$.*

Proof. We prove the very ampleness of adjoint linear series by Reider's Theorem [2]. We need to show that for any irreducible curve $\Gamma \subset S$,
$$(D_{m,x_1,\dots,x_{18}} - K_S) \cdot \Gamma \geq 3$$
and
$$(D_{m,x_1,\dots,x_{18}} - K_S)^2 \geq 10.$$
We can find an integer t such that $|tH_S - \sum_{i=1}^{18} \epsilon_i C_i|$ are very ample for any ϵ_i=1, 0, or -1, and hence $|tH_S - \sum_{i=1}^{18} \epsilon_i C_i|$ has positive intersection with any effective curve. If we pick m such that $m > 18te\sqrt{m} \geq 18t|x_i|$, easy computation shows that these are indeed the case. $\square$

Proof of Main Theorem. Let
$$g_m = p_a(D_m),$$
$$S_m = \{p_a(D_{m,x_1,\dots,x_{18}}) | m, x_i \in \mathbb{Z}\}.$$
Note that
$$p_a(D_{m,x_1,\dots,x_{18}}) = g_m - \frac{1}{2} f(x_1, \dots, x_{18}),$$

where

$$f(x_1,\dots,x_{18})=\sum_{i=1}^{9}(a+2)x_i^2+\sum_{i=10}^{18}ax_i^2+\sum_{i=1}^{18}m(H\cdot C_i)x_i$$

satisfies the condition of Lemma 1.

We first show that for any integer $z\le g_{m-1}$, z will be in S_m. This can be done by Lemma 1 and letting $m\gg 0$ because

$$g_m-z\ge g_m-g_{m-1}=mH_S^2+o(1).$$

Secondly , we want to show that for any integer $z, g_{m-2}\le z\le g_{m-1}$, z is actually genus of some smooth curves if $m\gg 0$. Choose e so that $e^2m\ge 2(g_m-g_{m-2}), \forall m\gg 0$. Hence for any integer $z, g_{m-2}\le z\le g_{m-1}$, $z=p_a(D_{m,x_1,\dots,x_{18}})$ for some $|x_i|\le\sqrt{2(g_m-z)}\le e\sqrt{m}$ [Remark 1]. By Lemma 2, such z can be realized as genus of some smooth curves.

Finally, let m vary. Every integer g lies inside some interval $[g_{m-2},g_{m-1}]$. This completes the proof. □

3. Variants, and example

We can't expect to realize all large genera by smooth curves on a general surface. The best thing we can hope is allowing curves to have some simple nodes.

Theorem 2. *Let S be a smooth surface. Then there exists an integer $g_0=g_0(S)$ such that for any integer $g\ge g_0$, there exist nodal curves with geometric genus $=g$.*

Proof. Let H be a very ample divisor on S. Let

$$g_m=p_a(K_S+mH),$$

$$d_m=g_m-g_{m-1}.$$

It is enough to show that for $m\gg 0$, and $\forall r$, $1\le r\le d_m$, there exists a nodal curve $C\in|K_S+mH|$ with exactly r simple nodes.

Let $C_1\in|[\frac{m}{4}]H|$ be a smooth curve in S. Let $\Sigma=\{x_1,\dots,x_r\}$ be r distinct points in C_1. Now consider the blowup of S along Σ

$$f:\tilde S=Bl_\Sigma(S)\to S.$$

Let E_i's denote the exceptional divisors over x_i's. Let

$$L_{m,r}=mf^*H-3(E_1+\ldots+E_r).$$

Then

$$K_{\tilde S}+L_{m,r}\equiv f^*(K_S+mH)-2(E_1+\ldots+E_r),$$

$$L_{m,r}\equiv(m-3[\frac{m}{4}])f^*H+3\tilde C_1,$$

where $\tilde C_1$ denotes the proper transformation of C_1.

Computation shows that, for m large enough, $L_{m,r}$ has both self-intersection and intersection with effective curves positive enough. So $|K_{\tilde S}+L_{m,r}|$ is very ample by Reider's Theorem. Then we can pick a curve $\tilde C\in|K_{\tilde S}+L_{m,r}|$ such that $\tilde C$ is smooth and $\tilde C$ intersects E_i at 2 distinct points $\forall\ 1\le i\le r$. Hence $C:=f_*(\tilde C)$ is a nodal curve in $|K_S+mH|$ with r simple nodes at $x_1,\dots,x_r$. This completes the proof. We left the details to the readers. □

Turning back to the higher dimensional case, we conclude with an elementary example to show that the least integer g_0 appearing in Theorem 1 can be arbitrarily large:

Example. Let $X = C_1 \times C_2 \times C_3$, where C_i's are smooth projective curves. There are natural projections p_i's to each C_i's. For any smooth irreducible curve C in X, not all the projections of C are points. Hence we have:

$$g(C) \geq \min(g(C_1), g(C_2), g(C_3)).$$

Hence there is no uniform g_0 for all projective varieties of same dimension.

References

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