ON THE GENERALIZED STEPANOV THEOREM

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Abstract. The generalized Stepanov theorem is derived from the Alexandrov theorem on the twice differentiability of convex functions. A parabolic version of the generalized Stepanov theorem is also proved.

In the first part of this note we provide a new proof of the generalized Stepanov theorem. This classical result is due to Calderón and Zygmund [3] (see also Oliver [12]), but is usually associated with Stepanov’s name because it generalizes the Stepanov theorem (see e.g. [6]). The result we prove below (Theorem 1) constitutes a special case of a general theorem in [3]. Recently this particular version found applications in proving the twice differentiability a.e. of viscosity solutions of elliptic partial differential equations, see [11], [14] and [2]. The only complete proof of the generalized Stepanov theorem the authors are aware of is contained in [3], where Whitney’s extension theorem is used. In this note the generalized Stepanov theorem will be proved by means of the Aleksandrov theorem on the twice differentiability of convex functions [1]; see also [5], [9], [10] or the appendix in [4] for more modern treatments.

In the second part of this note we show how to modify our proof to obtain a parabolic version of the generalized Stepanov theorem (Theorem 3). A result of this type is needed to prove the differentiability a.e. twice in x and once in t of viscosity solutions of parabolic equations. To the best of the authors’ knowledge this result is original, though some relevant arguments appear in [16].

In the first part of this note we will use the notation |·| and ⟨·, ·⟩ to stand for the Euclidean norm and inner-product in \( \mathbb{R}^n \), and \( B_r(x) \) will denote the open ball in \( \mathbb{R}^n \) of radius \( r \) centered at \( x \). Given a measurable set \( A \) in an Euclidean space, \( |A| \) will denote its Lebesgue measure.

Recall some notation from [3] (see also [17]). Let \( u: \Omega \rightarrow \mathbb{R}, \Omega \subset \mathbb{R}^n \), be bounded and \( x \in \Omega \). We say that \( u \in T^2_* (x) \) (\( u \in T^2_* (x) \), resp.) if there exists an affine function \( P_x \) (a quadratic function \( Q_x \)) such that

\[
\sup_{y \in B_r(x) \cap \Omega} |u(y) - P_x(y)| \leq O(r^2)
\]

\[
\left( \sup_{y \in B_r(x) \cap \Omega} |u(y) - Q_x(y)| \leq o(r^2) \text{ as } r \downarrow 0, \text{ resp.}. \right.
\]

Observe that \( u \in T^2_* (x) \) if and only if \( u \) possesses a second order Taylor series expansion at \( x \) whose remainder behaves like \( o(r^2) \). If this is the case we will say...
that $u$ is twice differentiable at $x$. On the other hand, $u \in T^2_\infty(x)$ is equivalent to saying that $u$ can be enclosed between two paraboloids meeting at $x$. In particular, if $\Omega \subset \mathbb{R}^n$ is open and $u \in T^2_\infty(x)$, then $u$ is differentiable at $x$ and $P_x(y) = u(x) + (Du(x), y - x)$.

**Theorem 1** (Calderón-Zygmund [3]). Let $\Omega \subset \mathbb{R}^n$ be open and bounded and suppose that $u : \bar{\Omega} \to \mathbb{R}$ is bounded. If $u \in T^2_\infty(x)$ for a.e. $x \in \Omega$ then $u \in T^2_\infty(x)$ for a.e. $x \in \Omega$.

**Proof.** By the assumption for a.e. $x \in \Omega$ there are $p_x \in \mathbb{R}^n$ and $M_x \geq 0$ such that

$$|u(y) - u(x) - \langle p_x, y - x \rangle| \leq M_x |y - x|^2$$

for all $y \in \Omega$; note that $p_x$ is uniquely determined and we can assume that $M_x$ is the smallest with this property. It follows that $M_x$ is well defined and finite a.e., moreover, the mapping $x \mapsto M_x$ is measurable. For $M = 1, 2, \ldots$ put

$$\Omega_M = \{x \in \Omega : M_x \leq M\};$$

then every $\Omega_M$ is measurable and $\bigcup_{M=1}^\infty \Omega_M$ is of full measure in $\Omega$. Therefore it is enough to show that for every $M$

$$u \in T^2_\infty(x) \text{ for a.e. } x \in \Omega_M.$$ 

From now on let $M$ be fixed. Note that for every $x \in \Omega_M$

$$u(y) - \langle p_x, y \rangle \leq u(x) - \langle p_x, x \rangle + M|y - x|^2$$

for all $y \in \Omega$, or

$$\hat{u}(y) \leq \hat{u}(x) + \langle q_x, y - x \rangle$$

for all $y \in \Omega$, where $\hat{u} = u - M|\cdot|^2$ and $q_x = p_x - 2Mx$. Denoting by $\hat{u}$ the upper concave envelope of $u$ on $\Omega$, that is,

$$\hat{u}(x) = \inf\{p(x) : p \text{ is affine and } p \geq u \text{ on } \Omega\},$$

we obtain that $\hat{\hat{u}} = \hat{u}$ on $\Omega_M$, or using the notation in [7], $\Omega_M \subset \Gamma$, where $\Gamma = \Gamma_\hat{u} = \{\hat{\hat{u}} = \hat{u}\}$ is the upper contact set of $\hat{u}$ on $\Omega$. From the Aleksandrov theorem $\hat{u}$ is twice differentiable a.e., that is, there exists $F \subset \Omega$ of full measure such that $\hat{\hat{u}} \in T^2_\infty(x)$ for every $x \in F$. Note that $Du = D\hat{u}$ on $\Omega_M \cap F$, which yields

(1) $$|\hat{u}(y) - \hat{u}(y)| \leq O(|y - x|^2) \text{ for every } x \in \Omega_M \cap F.$$ 

We will show that (1) implies that

(2) $$|\hat{u}(y) - \check{u}(y)| \leq o(|y - x|^2) \text{ as } y \to x \text{ for a.e. } x \in \Omega_M.$$ 

Put $v = \hat{u} - \check{u}$ and for $N = 1, 2, \ldots$ let

$$\Omega_{M,N} = \{x \in \Omega_M : |v(y)| \leq N|y - x|^2 \text{ for all } y \in \Omega\}.$$ 

To prove (2) it is enough to show that for every $N$

(3) $$|v(y)| \leq o(|y - x|^2) \text{ as } y \to x \text{ for a.e. } x \in \Omega_{M,N}.$$ 

We will show that this holds for any point of density of $\Omega_{M,N}$. So let $x_0 \in \Omega_{M,N}$ be a point of density and let $1 > \epsilon > 0$. Then for all sufficiently small $r$, say $r < \delta$, where $B_\delta(x_0) \subset \Omega$,

(4) $$\frac{|B_r(x_0) \setminus \Omega_{M,N}|}{|B_r(x_0)|} < \epsilon^n.$$
Suppose that \( y \in B_{\delta(1-\varepsilon)}(x_0) \) and let \( r = |y - x_0|/(1-\varepsilon) < \delta \). It follows that \( B_r(y) \subset B_r(x_0) \) and from (4) \( B_r(y) \cap \Omega_{M,N} \neq \emptyset \), say \( x_1 \in B_r(y) \cap \Omega_{M,N} \). Then

\[
|v(y)| \leq N|y - x_1|^2 < N\epsilon^2r^2 = \frac{N\epsilon}{(1-\epsilon)^2}|y - x_0|^2,
\]

and (3), and consequently (2), follows.

To finish the proof of the theorem it is enough to remark that if \( \tilde{u} \in t^2_\infty(x) \) and \( |\tilde{u}(y) - \tilde{u}(y)| \leq o(|y - x|^2) \) as \( y \to x \), then the Taylor expansion for \( \tilde{u} \) works for \( u \), and thus \( \tilde{u} \in t^2_\infty(x) \), and consequently \( u \in t^2_\infty(x) \).

**Remark 2.** Under the assumptions of Theorem 1 we proved that for a.e. \( x_0 \in \Omega \) there exist \( p(x_0) \in \mathbb{R}^n \) and a symmetric \( n \times n \) matrix \( A(x_0) \) such that

\[
(5) \quad u(y) = u(x_0) + \langle p(x_0), y - x_0 \rangle + \frac{1}{2} \langle A(x_0)(y - x_0), y - x_0 \rangle + o(|y - x_0|^2) \quad \text{as} \quad y \to x_0.
\]

Clearly \( u \) is then differentiable at every such point \( x_0 \) with \( Du(x_0) = p(x_0) \). A natural question arises whether \( A(x) \) is the derivative of \( Du(x) \). Denoting \( F_1 = \{ x \in \Omega : Du(x) \text{ exists} \} \) and \( F_2 = \{ x \in \Omega : u \in t^2_\infty(x) \} \subset F_1 \), we would like to find out whether for a.e. \( x_0 \in F_2 \)

\[
(6) \quad Du(y) = Du(x_0) + \langle A(x_0), y - x_0 \rangle + o(|y - x_0|) \quad \text{as} \quad F_1 \ni y \to x_0.
\]

By the \( C^2 \) version of the Aleksandrov theorem (see e.g. [10] or [4]) convex functions have this property, and therefore the proof of Theorem 1 shows that (6) holds in the approximate sense for a.e. \( x_0 \in \Omega \). That is, there exists \( F_3 \subset F_2 \subset \Omega \) of full measure such that for every \( x_0 \in F_3 \) and \( \epsilon > 0 \) the set

\[
\{ y \in F_1 : |Du(y) - Du(x_0) - \langle A(x_0), y - x_0 \rangle| < \epsilon|y - x_0| \}
\]

has density 1 at \( x_0 \). In general, to claim (6) stronger assumptions on \( u \) are required; see e.g. Theorem 3.5.7 in [17].

We would like to emphasize that this paper is concerned with pointwise derivatives and in general in our setting one doesn’t expect the existence of generalized derivatives. However, if \( u \in t^2_\infty(x) \) for all \( x \in \Omega \) with \( p \) and \( A \) as in (5) belonging to \( L^p(\Omega) \), \( 1 \leq p < \infty \), then \( u \in W^{2,p}(\Omega) \); see Theorem 3.9.5 in [17].

A modification of our approach leads to a proof of a parabolic version of the generalized Stepanov theorem. We are concerned with real-valued functions on \( \mathbb{R}^{n+1} \). We will write points in \( \mathbb{R}^{n+1} \) as \( (x,t) \), where \( x \in \mathbb{R}^n \) and \( t \in \mathbb{R} \). Given \( (y,s), (x,t) \in \mathbb{R}^{n+1} \), define their parabolic distance \( d \) according to

\[
d((y,s), (x,t)) = \sqrt{|x-y|^2 + |t-s|}
\]

and their one-sided parabolic distance \( d_\infty \) by

\[
d_\infty((y,s), (x,t)) = \begin{cases} d((y,s), (x,t)) & \text{if } s \leq t, \\
+\infty & \text{otherwise.} \end{cases}
\]

Let \( u : Q \to \mathbb{R} \), \( Q \subset \mathbb{R}^{n+1} \), be bounded and \( (x_0,t_0) \in Q \). We say that \( u \in T^{2,1}_\infty(x_0,t_0) \) \((u \in t^{2,1}_\infty(x_0,t_0), \text{resp.})\) if there exists an affine function \( P_{x_0,t_0} \) of variable
x (a quadratic in x and affine in t function $Q_{x_0,t_0}$) such that

$$|u(y,s) - P_{x_0,t_0}(y,s)| \leq O_2(d_t((y,s),(x_0,t_0))) \quad \text{for} \quad (y,s) \in Q$$

$$(u(y,s) - Q_{x_0,t_0}(y,s)) \leq o(d_t((y,s),(x_0,t_0))) \quad \text{as} \quad Q \ni (y,s) \to (x_0,t_0), \quad \text{resp.}.$$  

Note that in the definition of $t_{21}^\infty(x_0,t_0)$ an appropriate inequality holds for s both larger and smaller than $t_0$, while in the definition of $T_{21}^\infty(x_0,t_0)$ only the values $s \leq t_0$ matter. $u \in t_{21}^\infty(x_0,t_0)$ roughly corresponds to the differentiability of $u$ at $(x_0,t_0)$, twice in $x$, once in $t$.

**Theorem 3.** Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $T > 0$ and suppose that $u: \overline{Q} \to \mathbb{R}$ is bounded, where $Q = \Omega \times (0,T)$. If $u \in T_{21}^\infty(x,t)$ for a.e. $(x,t) \in Q$ then $u \in t_{21}^\infty(x,t)$ for a.e. $(x,t) \in Q$.

For $(x,t) \in \mathbb{R}^{n+1}$ and $r > 0$ put

$$P_r(x,t) = \{(y,s) \in \mathbb{R}^{n+1} : d((x,t),(y,s)) < r\},$$

$$Q_r(x,t) = \{(y,s) \in \mathbb{R}^{n+1} : d_{\infty}((x,t),(y,s)) < r\}.$$

The following proposition will be used in the proof of Theorem 3. It follows in a standard way from a version of the covering theorem of Vitali, which employs $Q_r$'s instead of the Euclidean balls; see e.g. Remark I.3.1 in [8].

**Proposition 4.** Let $A \subset \mathbb{R}^{n+1}$ be measurable. Then

$$\lim_{r \to 0} \frac{|Q_r(x,t) \setminus A|}{|Q_r(x,t)|} = 0 \quad \text{for a.e.} \quad (x,t) \in A.$$  

**Proof of Theorem 3.** The proof of Theorem 3 parallels that of Theorem 1. By assumption for a.e. $(x,t) \in Q$ there exist $p_{x,t} \in \mathbb{R}^n$ and $M_{x,t} \geq 0$ such that

$$|u(y,s) - u(x,t) - \langle p_{x,t}, y-x \rangle| \leq M_{x,t}(|y-x|^2 + t - s) \quad \text{for all} \quad y \in \Omega, \quad s \in [0,t].$$

As before the mapping $(x,t) \mapsto M_{x,t}$ is measurable and putting for $M = 1, 2, \ldots$

$$Q_M = \{(x,t) \in Q : M_{x,t} \leq M\}$$

gives that $\bigcup_{M=1}^\infty Q_M$ is of full measure in $Q$. Fix $M$ and define $\tilde{u}(x,t) = u(x,t) - M(|x|^2 - t)$; it follows that for every $(x,t) \in Q_M$

$$\tilde{u}(y,s) \leq \tilde{u}(x,t) + \langle q, y-x \rangle \quad \text{for all} \quad y \in \Omega \quad \text{and} \quad s \in [0,t],$$

with an appropriate $q \in \mathbb{R}^n$. In the parabolic context the upper concave envelope $\hat{u}$ of given function $\tilde{u}: Q \to \mathbb{R}$ is defined by (see [13] or [15])

$$\hat{u} = \inf\{v : v \geq \tilde{u} \text{ on } Q, \quad v \text{ concave in } x \text{ and increasing in } t\},$$

and thus (7) shows that $\hat{u} = \tilde{u}$ on $Q_M$. A parabolic version of the Aleksandrov theorem (see Theorem 1, Appendix 2 in [9]) guarantees that there exists $F \subset Q$ of full measure such that $\hat{u} \in t_{21}^\infty(x,t)$ for every $(x,t) \in F$. It follows that

$$\tilde{u}(y,s) - \hat{u}(y,s) \leq O_2(d_t((y,s),(x,t))) \quad \text{for every} \quad (x,t) \in Q_M \cap F.$$  

We will show that (8) implies that

$$|\hat{u}(y,s) - \hat{u}(y,s)| \leq o(d_t((y,s),(x,t))) \quad \text{for a.e.} \quad (x,t) \in Q_M,$$
which will give the result as in the proof of Theorem 1. Put \( v = \hat{u} - \tilde{u} \) and for 
\( N = 1, 2, \ldots \) let

\[
Q_{M,N} = \{ (x, t) \in Q_M : |v(y, s)| \leq N d_\kappa^2((y, s), (x, t)) \text{ for every } (y, s) \in Q \},
\]

and suppose that \((x_0, t_0) \in Q_{M,N}\) is such that

\[
\lim_{r \to 0} \frac{|Q_r(x_0, t_0) \setminus Q_{M,N}|}{|Q_r(x_0, t_0)|} = 0;
\]

by Proposition 4 a.e. \((x_0, t_0) \in Q_{M,N}\) will do. Let \( 0 < \epsilon < 1 \). For all sufficiently
small \( r \), say \( r < \delta \),

\[
\frac{|Q_r(x_0, t_0) \setminus Q_{M,N}|}{|Q_r(x_0, t_0)|} < \epsilon^{n+2}.
\]

Suppose that \((y, s) \in P_{\delta(1-\epsilon)}(x_0, t_0)\) and let \( r = d((y, s), (x_0, t_0))/(1 - \epsilon) < \delta \).
It follows that \( P_r(y, s) \subset P_r(x_0, t_0) \) and from (10) \( Q_r(y, s) \cap Q_{M,N} \neq \emptyset \), say
\( (x_1, t_1) \in Q_r(y, s) \cap Q_{M,N} \). In particular \( t_1 \geq s \) and therefore

\[
|v(y, s)| \leq N((y - x_1)^2 + t_1 - s) < N \epsilon^2 \epsilon^2 = \epsilon \frac{N \epsilon}{(1 - \epsilon)^2} d_\kappa^2((y, s), (x_0, t_0)).
\]

Thus (9) is proved for a.e. \((x, t) \in Q_{M,N}\) for every \( N \), and consequently for a.e.
\((x, t) \in Q_M\).

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