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CHARACTERIZATION OF THE DUALS OF LATTICES OF CONTINUOUS FUNCTIONS WITH RESPECT TO DISJOINTNESS PRESERVING GROUPS

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Abstract. The duals of $C_0(a,b)$ and $C[a,b]$ with respect to disjointness preserving groups are characterized. A. Plessner's result (1929) about the translation group is extended. A Wiener-Young type theorem for disjointness preserving groups is obtained.

It is assumed that the reader is familiar with the notions of a *Banach lattice*, *positive operator* and *disjointness preserving operator*. [AB], [LZ] and [MN] are good references for this material. We also assume familiarity with the theory of C_0*-semigroups and groups*. [Pa] and [Na] should provide all necessary information when needed.

Throughout $-\infty \le a < b \le +\infty$, X is either (a,b) or $[a,b]$, where by $[-\infty, b]$ we mean $(-\infty, b]$, and by $[a,+\infty]$ we mean $[a,+\infty)$.

Definition. A function $f : X \to \mathbb{C}$ is called *vanishing at infinity* if $\forall \varepsilon > 0\ \exists K \subset X$, K compact such that $\forall x \in X \setminus K\ |f(x)| < \varepsilon$.

We denote the lattice of all continuous functions $f : X \to \mathbb{C}$ vanishing at infinity by $C_0(X)$, the lattice of all bounded continuous functions $f : X \to \mathbb{C}$ by $C_b(X)$, the lattice of all continuous functions $f : X \to \mathbb{C}$ with compact support by $C_c(X)$, and the lattice of all regular complex-valued Borel measures on X with finite variation by $M(X)$.

Let $\{T(t)\}_{t\in\mathbf{R}}$ be a C_0-group on a Banach space E.

Definition. The *group dual of E with respect to* $\{T(t)\}_{t\in\mathbf{R}}$, denoted $E^{\odot}$ and pronounced E-sun, is defined in the following way:

$$E^{\odot} = \{\, u^* \in E^* : \lim_{t\to 0} \|T^*(t)u^* - u^*\| = 0\,\}.$$

[vN] is an excellent source of information about the semigroup and group duals of Banach spaces and related subjects.

Definition. A group $\{T(t)\}_{t\in\mathbf{R}}$ on a Banach lattice E is called *disjointness preserving* if $\forall t \in \mathbb{R}\ T(t)$ is a disjointness preserving operator.

Let $\{T(t)\}_{t\in\mathbf{R}}$ be a disjointness preserving C_0-group on $C_0(X)$. It follows from [Na, B-II.3.8] that $\exists \varphi : \mathbb{R}\times X \to X$ *a continuous flow* $\exists q : \mathbb{R} \to C_b(X)$ *a*

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continuous cocycle of φ such that $\forall t \in \mathbb{R}\ \forall f \in C_0(X)\ \forall x \in X\ (T(t)f)(x) = q_t(x)f(\varphi_t(x))$, where $\varphi_t = \varphi(t,\cdot)$, $q_t = q(t)$.

In the sequel we are going to use the following elementary observation.

Lemma 1. $\forall t \in \mathbb{R}\ \|q_t\| = \|T(t)\|$.

In [Na, B-II.3.21] W. Arendt characterized all continuous flows on X. For the sake of completeness we list this result here providing a somewhat more detailed proof than the original one.

Theorem 2. *φ is a continuous flow on X if and only if the following conditions are satisfied.*

1. $\exists U \subset X$, *U is the union of pairwise disjoint intervals* (a_i,b_i), $i \in I$, *where I is either finite or countable.*
2. $\exists \psi : U \to \mathbb{R}$ *such that* $\forall i \in I\ \psi_i = \psi|_{(a_i,b_i)} : (a_i,b_i) \to \mathbb{R}$ *is a homeomorphism.*
3. $\forall t \in \mathbb{R}$

$$\varphi_t(x) = \begin{cases} \psi_i^{-1}(\psi_i(x)+t) & \text{if } x \in (a_i,b_i), \\ x & \text{if } x \notin U. \end{cases}$$

Proof. Sufficiency. First we will establish that φ is a flow on $\mathbb{R}$. Let $x,s,t \in \mathbb{R}$. Then either $x \in U$ or $x \notin U$. If $x \notin U$, then $\varphi_t(x) = \varphi_s(x) = \varphi_{t+s}(x) = x$ and therefore $\varphi_t(\varphi_s(x)) = \varphi_{t+s}(x)$. If $x \in (a_i,b_i)$ for some $i \in I$, then

$$\varphi_t(\varphi_s(x)) = \psi_i^{-1}\left(\psi_i\left(\psi_i^{-1}(\psi_i(x)+s)\right)+t\right) = \psi_i^{-1}(\psi_i(x)+t+s) = \varphi_{t+s}(x).$$

It is also clear that $\forall x \in \mathbb{R}\ \varphi_0(x) = x$.

The next step is to prove the continuity of φ. It is fairly clear that if $x \in U$ or $x \notin \bar{U}$, then $\forall t \in \mathbb{R}\ \varphi_t(x)$ is continuous at (t,x). Suppose $x \in \bar{U} \setminus U$, $t \in \mathbb{R}$. Since $\varphi_t(x) = x$, we have to prove that

$$\varphi_s(y) \to x \text{ as } y \downarrow x,\ s \to t \qquad \text{and} \qquad \varphi_s(y) \to x \text{ as } y \uparrow x,\ s \to t.$$

We will prove only the first fact of the above two. The second one can be proved in the same manner.

Let $\varepsilon > 0$. Then there are two possibilities: either $(x,x+\varepsilon) \subset U$ or $(x,x+\varepsilon) \not\subset U$. Let us consider the second case first. Then $\exists \delta\ 0 < \delta < \varepsilon$ such that $x+\delta \notin U$. Let $x < y < x+\delta$. If $y \notin U$, then $\forall s \in \mathbb{R}\ \varphi_s(y) = y$ and therefore $x < \varphi_s(y) < x+\varepsilon$. If, on the other hand, $y \in (a_i,b_i)$ for some $i \in I$, then

$$x \le a_i < y < b_i \le x+\delta < x+\varepsilon.$$

Note that $\forall s \in \mathbb{R}\ \varphi_s(y) \in (a_i,b_i)$ and therefore $x < \varphi_s(y) < x+\varepsilon$.

Assume now $(x,x+\varepsilon) \subset U$. Then $\exists i \in I$ such that $a_i = x < x+\varepsilon \le b_i$. Since ψ_i is a homeomorphism, without losing generality we may assume that it is a decreasing function. Let

$$\delta = \psi_i^{-1}(\psi_i(x+\varepsilon)+|t|+1) - x > 0.$$

Thus, $\psi_i(x+\delta) = \psi_i(x+\varepsilon)+|t|+1$. Therefore, $\forall y \in (x,x+\delta)$, $\forall |s| < |t|+1$

$$\psi_i(y)+s > \psi_i(x+\delta) - |t| - 1 = \psi_i(x+\varepsilon).$$

It follows that $x < \varphi_s(y) < x+\varepsilon$. This concludes the proof that $\varphi_s(y) \to x$ as $y \downarrow x$, $s \to t$.

NECESSITY. First we will prove that $\forall t \in \mathbb{R}$ φ_t is a strictly increasing function. Assume this is not true. Then $\exists t \in \mathbb{R}$ $\exists x, y \in X$, $x < y$ such that $\varphi_t(x) \geq \varphi_t(y)$. Since φ is a continuous function and since $\varphi_0 = \mathrm{id}_X$, $\exists s \in (0,t]$ such that $\varphi_s(x) = \varphi_s(y)$. Contradiction with the fact that φ_s is a homeomorphism.

Let $K = \{\, x \in X : \forall t \in \mathbb{R}\ \varphi_t(x) = x \,\}$. Clearly K is a closed set. Also if $X = [a,b]$, it is easy to see that $a, b \in K$. Therefore, $U = X \setminus K$ is open in $\mathbb{R}$ and thus is a union of of pairwise disjoint intervals (a_i, b_i), $i \in I$, where I is either finite or countable.

Let $i \in I$, $x \in (a_i, b_i)$, $\beta(t) = \varphi_t(x)$. We claim that β is an injection. Suppose this is false, i.e. $\exists r, s \in \mathbb{R}$ such that $\varphi_s(x) = \varphi_r(x)$ which means that $\varphi_{s-r}(x) = x$. Scaling by t, if necessary, we may assume that $\varphi_1(x) = x$. Let $y = \varphi_{1/2}(x)$. It follows from the definition of a flow that $\varphi_{1/2}(y) = x$. Since as we showed above $\varphi_{1/2}$ is a strictly increasing function, we conclude that $x = y$.

Using a similar argument, we can show that $\forall n \in \mathbb{N}$ $\varphi_{1/2^n}(x) = x$ and thus $\forall m \in \mathbb{Z}$ $\varphi_{m/2^n}(x) = x$. Since numbers $\{m/2^n\}_{n \in \mathbb{N},\, m \in \mathbb{Z}}$ are dense in $\mathbb{R}$ and since φ is a continuous flow, it follows that $\forall t \in \mathbb{R}$ $\varphi_t(x) = x$, i.e. $x \in K$. Contradiction with $x \in (a_i, b_i)$. Thus, β is an injection.

Our next claim is that β maps $\mathbb{R}$ onto (a_i, b_i). Suppose $\varphi_t(x) \notin (a_i, b_i)$. Then $\exists s \in (0,t]$ such that $y = \varphi_s(x) \in K$. From the definition of K it follows that $\varphi_{-s}(y) = y$ and thus $x = y \in K$. Contradiction with $x \in (a_i, b_i)$. We conclude that the image of β is contained in (a_i, b_i). Thus, $\beta : \mathbb{R} \to (c,d)$ is a homeomorphism and $(c,d) \subset (a_i, b_i)$. Without loss of generality we may assume that β is an increasing function. Therefore, $\beta(t) \to d$ as $t \to +\infty$. Let $s \in \mathbb{R}$. Then

$$\varphi_s(d) = \varphi_s(\lim_{t \to +\infty} \varphi_t(x)) = \lim_{t \to +\infty} \varphi_s(\varphi_t(x)) = \lim_{t \to +\infty} \varphi_{t+s}(x) = d.$$

Hence, $d \in K$ and thus $d = b_i$. Analogously $c = a_i$. This establishes the claim.

Finally, let $\psi_i = \beta^{-1}$, $y \in (a_i, b_i)$, $s = \psi_i(y)$, i.e. $y = \varphi_s(x)$. Then

$$\varphi_t(y) = \varphi_t(\varphi_s(x)) = \varphi_{t+s}(x) = \psi_i^{-1}(s+t) = \psi_i^{-1}(\psi_i(y) + t).$$

□

Remark. If φ is a continuous flow, then $\forall n \in \mathbb{Z}$ $\varphi_n = \varphi_1^n$, i.e. φ_n is the nth iterant of the function φ_1. Theorem 2 shows that the functions $\{\varphi_t\}_{t \in \mathbf{R}}$ are the *continuous iterants of* φ_1 *in the sense of Abel*, i.e. there exists a function ψ such that φ_1 satisfies the *Abel equation* $\psi(\varphi_1(x)) - \psi(x) = 1$ (see [Ku, Ch.VII]), and consequently $\forall t \in \mathbb{R}$ $\forall x \in X$ $\psi(\varphi_t(x)) - \psi(x) = t$.

Suppose μ is a nonnegative Borel measure on X, $t \in \mathbb{R}$. $\forall F \subset X$, F Borel define $\mu_t(F) = \mu(\varphi_{-t}(F))$. By [DS, III.10.8] μ_t is a nonnegative Borel measure on X and $\forall F \subset X$, F Borel $\forall f \in C_0(X)$

$$(*) \qquad \int_F f \, d\mu_t = \int_{\varphi_{-t}(F)} f \circ \varphi_t \, d\mu.$$

It can also be easily seen that if $\mu \in M(X)$, then so is μ_t and the above equality holds as well.

Suppose $\mu \in M(X)$, $t \in \mathbb{R}$. $\forall F \subset X$, F Borel define

$$\mu_t'(F) = \int_F q_t \circ \varphi_{-t} \, d\mu_t.$$

Since $q_t \in C_b(X)$, [DS, III.10.4] implies that $\mu_t' \in M(X)$ and $\forall F \subset X$, F Borel $\forall f \in C_0(X)$

$$(**) \qquad \int\limits_F f\,d\mu_t' = \int\limits_F q_t \circ \varphi_{-t} \cdot f\,d\mu_t = \int\limits_{\varphi_{-t}(F)} q_t \cdot f \circ \varphi_t\,d\mu.$$

Lemma 3. $\forall t \in \mathbb{R}\ \forall \mu \in M(X)\ \mu_t' = T^*(t)\mu$.

Proof. It follows from above that $\forall t \in \mathbb{R}\ \mu_t' \in M(X)$ and $\forall f \in C_0(X)$

$$\int\limits_X T(t)f\,d\mu = \int\limits_X q_t \cdot f \circ \varphi_t\,d\mu = \int\limits_X f\,d\mu_t'.$$

□

Lemma 4. *If* $\mu \in M(X)$ *and* $\mu|_U = 0$*, then* $\mu \in C_0(X)^{\odot}$ *with respect to* $\{T(t)\}_{t\in\mathbf{R}}$.

Proof. Suppose $t \in \mathbb{R}$, $F \subset X$ is Borel. Then

$$\mu_t'(F) = \int\limits_X q_t \chi_{\varphi_{-t}(F)}\,d\mu.$$

Since $\mu|_U = 0$, $\mu(\varphi_{-t}(F) \cap U) = 0$ and therefore

$$\mu_t'(F) = \int\limits_X q_t \chi_{\varphi_{-t}(F)\setminus U}\,d\mu.$$

Since φ_{-t} is a bijection and since $\varphi_{-t}(U) = U$, $\varphi_{-t}(F) \setminus U = \varphi_{-t}(F \setminus U)$. Also since $\varphi_{-t}|_{X\setminus U} = \mathrm{id}_{X\setminus U}$, $\varphi_{-t}(F \setminus U) = F \setminus U$. We obtain:

$$\mu_t'(F) = \int\limits_X q_t \chi_{F\setminus U}\,d\mu = \int\limits_X q_t \chi_F\,d\mu$$

since $\mu(F \cap U) = 0$. Thus,

$$\mu_t'(F) - \mu(F) = \int\limits_X (q_t - 1)\chi_F\,d\mu.$$

Let $\mathcal{F}$ be the set of all partitions $\{F_j\}$ of X. Then $\forall t \in \mathbb{R}$

$$\begin{aligned} \|\mu_t' - \mu\| &= \sup_{\{F_j\}\in\mathcal{F}} \sum_{j=1}^{\infty} |(\mu_t' - \mu)(F_j)| \\ &\le \sup_{\{F_j\}\in\mathcal{F}} \sum_{j=1}^{\infty} \int\limits_X |q_t - 1|\chi_{F_j}\,d\mu = \int\limits_X |q_t - 1|\,d\mu. \end{aligned}$$

Since $\{T(t)\}_{t\in\mathbf{R}}$ is a C_0-group, it is locally bounded. Hence, by Lemma 1 $\exists D > 0$ such that $\forall |t| \le 1\ \|q_t\| \le D$ and therefore $\|q_t - 1\| \le D + 1$. Since by the definition of a cocycle $q_0 = 1$ and since $(D+1)\chi_X \in L^1(X, \mu)$, it follows from the Dominated Convergence Theorem that

$$\int\limits_X |q_t - 1|\,d\mu \to 0 \quad \text{as} \quad t \to 0.$$

Hence, $\|\mu_t' - \mu\| \to 0$ as $t \to 0$. Since by Lemma 3 $\mu_t' = T^*(t)\mu$, it follows that $\mu \in C_0(X)^{\odot}$. □

Lemma 5. *Suppose* $i \in I$, $\alpha \in L^1((a_i, b_i), d\psi)$. $\forall F$ *Borel,* $F \subset X$ *define*

$$\mu(F) = \int_{a_i}^{b_i} \alpha \chi_F \, d\psi,$$

where ψ *is as in Theorem* 2. *Then* $\mu \in C_0(X)^{\odot}$ *with respect to* $\{T(t)\}_{t\in\mathbf{R}}$.

Proof. $d\psi$ is either a nonnegative or nonpositive measure on (a_i, b_i). Without loss of generality we may assume it is nonnegative. Suppose $(c, d) \subset (a_i, b_i)$. Since ψ_i is a continuous function, $d\psi(c, d) = \psi(d) - \psi(c)$. Then $\forall t \in \mathbb{R}$

$$\begin{aligned}(d\psi)_t(c, d) &= d\psi(\varphi_{-t}(c), \varphi_{-t}(d)) = \psi(\varphi_{-t}(d)) - \psi(\varphi_{-t}(c)) \\ &= \psi(d) - t - \psi(c) + t = \psi(d) - \psi(c) = d\psi(c, d).\end{aligned}$$

It follows that $\forall G \subset (a_i, b_i)$, G open $(d\psi)_t(G) = d\psi(G)$. [Ru, 2.18] implies that both $d\psi$ and $(d\psi)_t$ are regular. Thus, $\forall F \subset (a_i, b_i)$, F Borel $(d\psi)_t(F) = d\psi(F)$ which means that $\forall t \in \mathbb{R}$ $(d\psi)_t = d\psi$ on (a_i, b_i).

Let $F \subset X$ be Borel, $t \in \mathbb{R}$. Then by $(*)$ and $(**)$

$$\begin{aligned}(\mu_t' - \mu)(F) &= \int_{a_i}^{b_i} q_t \alpha \chi_{\varphi_{-t}(F)} \, d\psi - \int_{a_i}^{b_i} \alpha \chi_F \, d\psi \\ &= \int_{a_i}^{b_i} q_t \circ \varphi_{-t} \cdot \alpha \circ \varphi_{-t} \cdot \chi_{\varphi_{-t}(F)} \circ \varphi_{-t} \, d\psi_t - \int_{a_i}^{b_i} \alpha \chi_F \, d\psi \\ &= \int_{a_i}^{b_i} (q_t \circ \varphi_{-t} \cdot \alpha \circ \varphi_{-t} - \alpha) \chi_F \, d\psi\end{aligned}$$

since $(d\psi)_t = d\psi$ on (a_i, b_i), $\chi_{\varphi_{-t}(F)} \circ \varphi_{-t} = \chi_F$. Using the argument similar to the one in the proof of Lemma 4, we conclude that $\forall t \in \mathbb{R}$

$$\|\mu_t' - \mu\| \le \int_{a_i}^{b_i} |q_t \circ \varphi_{-t} \cdot \alpha \circ \varphi_{-t} - \alpha| \, d\psi.$$

We have shown in the proof of Lemma 4 that $\exists D > 1$ such that $\forall |t| \le 1$ $\|q_t\| \le D$. Let $\varepsilon > 0$. Since $\alpha \in L^1((a_i, b_i), d\psi)$, $\exists g \in C_c(a_i, b_i)$ such that

$$\int_{a_i}^{b_i} |\alpha - g| \, d\psi < \frac{\varepsilon}{3D} < \frac{\varepsilon}{3}.$$

Since $(d\psi)_t = d\psi$ on (a_i, b_i), it follows from $(*)$ that $\forall |t| \le 1$

$$\int_{a_i}^{b_i} |q_t \circ \varphi_{-t} \cdot \alpha \circ \varphi_{-t} - q_t \circ \varphi_{-t} \cdot g \circ \varphi_{-t}| \, d\psi = \int_{a_i}^{b_i} |q_t \alpha - q_t g| \, d\psi \le D \frac{\varepsilon}{3D} = \frac{\varepsilon}{3}.$$

Let $K = \operatorname{supp} g$, $K' = \varphi(K \times [-1, 1]) \subset (a_i, b_i)$. Since φ is a continuous flow, K' is compact. Suppose $x \in (a_i, b_i) \setminus K'$. It means that $\forall y \in K$ $\forall |t| \le 1$ $x \ne \varphi_{-t}(y)$

which implies that $\varphi_t(x) \neq y$. Hence, $\varphi_t(x) \notin K$ and $g(\varphi_t(x)) = 0$. It follows that $\forall |t| \leq 1$ $\operatorname{supp}(g \circ \varphi_t - g) \subset K'$. Therefore, $\forall |t| \leq 1$

$$|q_t \circ \varphi_{-t} \cdot g \circ \varphi_{-t} - g| \leq \|g\|(D+1)\chi_{K'}.$$

Since $\chi_{K'} \in L^1((a_i, b_i), d\psi)$, the Dominated Convergence Theorem implies that $\exists \delta > 0$ such that $\forall |t| < \delta$

$$\int\limits_{a_i}^{b_i} |q_t \circ \varphi_{-t} \cdot g \circ \varphi_{-t} - g| \, d\psi < \frac{\varepsilon}{3}.$$

Hence,

$$\begin{aligned}
&\int\limits_{a_i}^{b_i} |q_t \circ \varphi_{-t} \cdot \alpha \circ \varphi_{-t} - \alpha| \, d\psi \\
\leq &\int\limits_{a_i}^{b_i} |q_t \circ \varphi_{-t} \cdot \alpha \circ \varphi_{-t} - q_t \circ \varphi_{-t} \cdot g \circ \varphi_{-t}| \, d\psi \\
+ &\int\limits_{a_i}^{b_i} |q_t \circ \varphi_{-t} \cdot g \circ \varphi_{-t} - g| \, d\psi + \int\limits_{a_i}^{b_i} |\alpha - g| \, d\psi \\
< &\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Gluing all pieces together, we will obtain that $\forall |t| < \delta$ $\|\mu_t' - \mu\| < \varepsilon$. Thus, $\lim\limits_{t \to 0} \mu_t' = \mu$ and $\mu \in C_0(X)^{\odot}$ with respect to $\{T(t)\}_{t \in \mathbf{R}}$. □

Lemma 6. $C_0(X)^{\odot} \subset M(X \setminus U) \oplus L^1(U, d\psi)$.

Proof. Suppose $\mu \in M(X)$. Then

$$\mu = \nu + \sum_{i \in I} \mu_i,$$

where $\nu|_U = 0$ and $\forall i \in I$ $\mu_i|_{(X)\setminus(a_i,b_i)} = 0$. Let A be the infinitesimal generator of $\{T(t)\}_{t \in \mathbf{R}}$, ω_0 the growth bound of $\{T(t)\}_{t \in \mathbf{R}}$, $\lambda > \omega_0$. Since by [vN, 1.3.1] $C_0(X)^{\odot} = \overline{D(A^*)}$, it suffices to prove that $R(\lambda, A^*)\nu \in M(X \setminus U)$ and $\forall i \in I$ $R(\lambda, A^*)\mu_i \in L^1(U, d\psi)$.

Let $f \in C_0(X)$. Then it follows from [Pa, 1.5.4] that $\forall x \in \mathbb{R}$

$$(R(\lambda, A)f)(x) = \int\limits_0^{+\infty} e^{-\lambda t} q_t(x) f(\varphi_t(x)) \, dt.$$

Hence, since $\nu|_U = 0$ and $\varphi_t|_{X \setminus U} = \mathrm{id}_{X \setminus U}$

$$\begin{aligned}
&\langle R(\lambda, A^*)\nu, f\rangle = \langle \nu, R(\lambda, A)f\rangle \\
= &\int\limits_X d\nu(x) \int\limits_0^{+\infty} e^{-\lambda t} q_t(x) f(x) \, dt = \int\limits_X f(x) H_\lambda(x) \, d\nu(x),
\end{aligned}$$

where

$$H_\lambda(x) = \int_0^{+\infty} h_\lambda(t,x)\,dt, \qquad h_\lambda(t,x) = e^{-\lambda t} q_t(x).$$

Let $\varepsilon = (\lambda - \omega_0)/2$. Since $\{T(t)\}_{t\in\mathbf{R}}$ is a C_0-group, it is locally bounded. Hence, by Lemma 1 $\exists B > 0$ such that $\forall t \in \mathbb{R}$ $\|q_t\| \le Be^{t(\omega_0+\varepsilon)}$, and $\forall x \in X$

$$|H_\lambda(x)| \le \int_0^{+\infty} |h_\lambda(t,x)|\,dt \le B \int_0^{+\infty} e^{-\varepsilon t}\,dt = \frac{B}{\varepsilon}.$$

Thus, if $\forall F \subset X$, F Borel we define

$$\xi(F) = \int_F H_\lambda(x)\,d\nu(x),$$

then by [DS, III.10.4] $\xi \in M(X)$ and $\xi = R(\lambda, A^*)\nu$. Also since $\nu|_U = 0$, $\xi|_U = 0$ as well.

Let $i \in I$. Without loss of generality we may assume that ψ is nondecreasing on (a_i, b_i). Then

$$\langle R(\lambda, A^*)\mu_i, f\rangle = \langle \mu_i, R(\lambda, A)f\rangle = \int_{a_i}^{b_i} d\mu_i(x) \int_0^{+\infty} e^{-\lambda t} q_t(x) f(\psi_i^{-1}(\psi(x)+t))\,dt.$$

Suppose $t = \psi(s) - \psi(x)$. Then by [DS, III.10.8]

$$\int_{a_i}^{b_i} d\mu_i(x) \int_0^{+\infty} e^{-\lambda t} q_t(x) f(\psi_i^{-1}(\psi(x)+t))\,dt$$
$$= \int_{a_i}^{b_i} d\mu_i(x) \int_x^{b_i} h_\lambda(\psi(s)-\psi(x), x) f(s)\,d\psi(s).$$

Applying Fubini's Theorem, we will get:

$$\int_{a_i}^{b_i} d\mu_i(x) \int_x^{b_i} h_\lambda(\psi(s)-\psi(x), x) f(s)\,d\psi(s)$$
$$= \int_{a_i}^{b_i} f(s)\,d\psi(s) \int_{a_i}^{s} h_\lambda(\psi(s)-\psi(x), x)\,d\mu_i(x)$$
$$= \int_{a_i}^{b_i} F_i(s) f(s)\,d\psi(s),$$

where

$$F_i(s) = \int_{a_i}^{s} h_\lambda(\psi(s)-\psi(x), x)\,d\mu_i(x).$$

We need to show that $F_i \in L^1((a_i, b_i), d\psi)$. Again using Fubini's Theorem and [DS, III.10.8], we will obtain:

$$\begin{aligned} &\int_{a_i}^{b_i} |F_i(s)|\, d\psi(s) \le \int_{a_i}^{b_i} d\psi(s) \int_{a_i}^{s} |h_\lambda(\psi(s) - \psi(x), x)|\, d|\mu_i|(x) \\ &= \int_{a_i}^{b_i} d|\mu_i|(x) \int_{x}^{b_i} |h_\lambda(\psi(s) - \psi(x), x)|\, d\psi(s) = \int_{a_i}^{b_i} d|\mu_i|(x) \int_{0}^{+\infty} |h_\lambda(t, x)|\, dt \\ &\le \frac{B}{\varepsilon} |\mu_i|(a_i, b_i) < +\infty \end{aligned}$$

since $\forall x \in X \int_0^{+\infty} |h_\lambda(t, x)|\, dt \le B/\varepsilon$.

Hence, $\forall i \in I$ $R(\lambda, A^*)\mu_i = \nu_i$, where $\forall F \subset \mathbb{R}$, F Borel

$$\nu_i(F) = \int_{a_i}^{b_i} F_i \chi_F \, d\psi,$$

which means that ν_i can be associated with a function from $L^1((a_i, b_i), d\psi)$. ☐

Theorem 7. *Let $\{T(t)\}_{t\in\mathbf{R}}$ be a disjointness preserving C_0-group on $C_0(X)$. Then $\exists U \subset X$, U is the union of pairwise disjoint intervals (a_i, b_i), $i \in I$, where I is either finite or countable and $\exists \psi : U \to \mathbb{R}$ such that $\forall i \in I$ $\psi_i = \psi|_{(a_i,b_i)} : (a_i, b_i) \to \mathbb{R}$ is a homeomorphism and the corresponding group dual $C_0(X)^\odot = M(X \setminus U) \oplus L^1(U, d\psi)$.*

Proof. Follows from Lemmas 4, 5 and 6. ☐

Remark. The above theorem generalizes the well-known result of A. Plessner ([Pl]) that if $f : \mathbb{R} \to \mathbb{C}$ and $\mathrm{Var}_{\mathbf{R}}[f] < +\infty$, then f is absolutely continuous if and only if $\mathrm{Var}_{\mathbf{R}}[f(\cdot + t) - f(\cdot)] \to 0$ as $t \to 0$.

The following theorem generalizes the result of N. Wiener and R. C. Young ([WY]) about the behavior of measures on $\mathbb{R}$ under translation. We are going to use the following notation: if S is a subset of a Banach lattice E, then S^d will denote its disjoint complement in E.

Theorem 8. *Let $\{T(t)\}_{t\in\mathbf{R}}$ be a disjointness preserving C_0-group on $C_0(X)$. Then $\forall \mu \in M(X)$*

$$\limsup_{t\to 0} \|T^*(t)\mu - \mu\| = 2\|\mu_d\|,$$

where μ_d is the component of μ in $C_0(X)^{\odot d}$.

Proof. Suppose that φ and q are the flow and the cocycle associated with $\{T(t)\}_{t\in\mathbf{R}}$. Then it is not difficult to see that $|T(t)|$ is also disjointness preserving with the flow φ and cocycle $|q|$. Thus, by Theorem 7 both groups $\{T(t)\}_{t\in\mathbf{R}}$ and $\{|T(t)|\}_{t\in\mathbf{R}}$ have the same $C_0(X)^\odot$.

Let m be the Lebesgue measure on $\mathbb{R}$. Since $\{|T(t)|\}_{t\in\mathbf{R}}$ is a positive C_0-group, it follows from [dP, 2.3] that $\forall \mu$ in $C_0(X)^{\odot d}$ $|T(t)|^*\mu \perp \mu$ m-a.e. on $\mathbb{R}$. It follows from [MN, 3.1.21] that both $T^*(t)$ and $|T(t)|^*$ are disjointness preserving, and therefore by [AB, 8.6] $|T^*(t)\mu| = |T^*(t)||\mu|$ and $|T(t)|^*|\mu| = ||T(t)|^*\mu|$. Since by [MN, 3.1.21] $|T^*(t)| = |T(t)|^*$, we obtain:

$$|T^*(t)\mu| = |T^*(t)||\mu| = |T(t)|^*|\mu| = ||T(t)|^*\mu|.$$

Therefore, $|T(t)^*\mu| \wedge |\mu| = ||T(t)|^*\mu| \wedge |\mu| = 0$ which implies that $T(t)^*\mu \perp \mu$ m-a.e. on $\mathbb{R}$. The desired equality can now be obtained by mimicking the argument on p. 108 of [dP]. $\square$

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References

[AB] C. D. Aliprantis and O. Burkinshaw, *Positive operators*, Academic Press, Orlando, FL, 1985. MR **87h:**47086

[dP] B. de Pagter, *A Wiener-Young type theorems for dual semigroups*, Positive Operators and Semigroups on Banach Lattices (C. B. Huijsmans and W. A. J. Luxemburg, eds.), Kluwer Academic Publisher, Dordrecht, The Netherlands, 1992, pp. 101–109. MR **93j:**47059

[DS] N. Dunford and T. Schwartz, *Linear operators*, Interscience Publishers, Inc., New York, 1958. MR **22:**8302

[Ku] M. Kuczma, *Functional equations in a single variable*, Polish Acad. Sci. Monograph in Math., vol. 46, Polish Acad. Sci., Warsaw, 1968. MR **37:**4441

[LZ] W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces* I, North-Holland, Amsterdam, 1971. MR **58:**23483

[MN] P. Meyer-Nieberg, *Banach lattices*, Springer-Verlag, Berlin, 1991. MR **93f:**46025

[Na] R. Nagel (ed.), *One-parameter semigroups of positive operators*, Lecture Notes in Math., vol. 1184, Springer-Verlag, Berlin, 1984. MR **88i:**47022

[Pa] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, Berlin, 1983. MR **85g:**47061

[Pl] A. Plessner, *Eine Kennzeichnung der totalstetigen Funktionen*, J. Reine Angew. Math. **60** (1929), 26–32.

[Ru] W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1987. MR **88k:**00002

[vN] J. van Neerven, *The adjoint of a semigroup of linear operators*, Springer-Verlag, Berlin, 1992. MR **94j:**47059

[WY] N. Wiener and R. C. Young, *The total variation of $g(x+h) - g(x)$*, Trans. Amer. Math. Soc. **33** (1935), 327–340.

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