ON KORENBLUM’S MAXIMUM PRINCIPLE

WILHELM SCHWICK

(Communicated by Theodore W. Gamelin)

Abstract. If $f$ and $g$ are analytic functions in the unit disk and $\| \cdot \|$ is the Bergman norm, conditions are studied under which there exists an absolute constant $c$ such that $|f(z)| \geq |g(z)|$ for $c \leq |z| < 1$ implies $\|f\| \geq \|g\|$.

In [1] Korenblum proved the following theorem:

Consider $f,g \in A_2(D)$ such that each zero of $g$ is a zero of $f$ (according to the multiplicity) and $|f(z)| \geq |g(z)|$ for $e^{-2}/2 \leq |z| < 1$. Then

$$\int_D |f(z)|^2 dm \geq \int_D |g(z)|^2 dm.$$

In the same paper and in [3] he stated the conjecture:

There exists an absolute constant $c$ with $0 < c < 1$ such that for functions $f,g \in A_2(D)$

(*) \quad (|f(z)| \geq |g(z)| \text{ for } c \leq |z| < 1) \Rightarrow \int_D |f(z)|^2 dm \geq \int_D |g(z)|^2 dm.

That means that the conjecture could be proved for $c = e^{-2}/2$ under an additional assumption on the zeroes of $f$ and $g$. In this paper other conditions are formulated such that (*) holds and the conjecture is reduced to a weaker one. We make use of the following notation: for a meromorphic function $h$ in $D$ define $\mathcal{N}(r,h)$ as the set of poles of $h$ in $\{|z| < r\}$ and $n(r,h)$ as the number of poles of $h$ in $\{|z| < r\}$ (according to the multiplicity). We prove:

Theorem 1. For $0 < \epsilon < 1/2$ a constant $c_\epsilon > 0$ exists such that for any pair $f,g \in A_2(D)$

$$\left( |f(z)| \geq |g(z)| \text{ for } c_\epsilon \leq |z| < 1 \text{ and } \mathcal{N}\left(1 - \epsilon, \frac{f}{g}\right) \setminus \mathcal{N}\left(\epsilon, \frac{f}{g}\right) \neq \emptyset \right)$$

$$\Rightarrow \int_D |f(z)|^2 dm \geq \int_D |g(z)|^2 dm.$$
Theorem 2. For $0 < \epsilon < \frac{1}{2}$ a constant $c_\epsilon > 0$ exists such that for any pair $f, g \in A_2(\mathbb{D})$
\[
\left( |f(z)| \geq |g(z)| \text{ for } c_\epsilon \leq |z| < 1 \text{ and } n \left( 1 - \epsilon, \frac{1}{f} \right) \neq n \left( 1 - \epsilon, \frac{1}{g} \right) \right)
\]
implies that each circle $\{ |z| = \epsilon \}$, $\epsilon > 0$, has at least one zero in the annulus $\{ c_\epsilon \leq |z| < 1 \}$ according to the multiplicity (Theorem 1), or if the number of zeroes of $f$ and $g$ in a circle $\{ |z| < 1 - \epsilon_0 \}$ does not coincide (Theorem 2).

Theorem 3. There exists a constant $c > 0$ such that for any pair $f, g \in A_2(\mathbb{D})$
\[
\left( |f(z)| \geq |g(z)| \text{ for } c \leq |z| < 1 \text{ and } n \left( 1, \frac{g}{f} \right) = n \left( 1, \frac{f}{g} \right) = 1 \right)
\]
implies that each circle $\{ |z| = 1 \}$ does not coincide (Theorem 2).

Theorem 4. Let $\alpha \in \mathbb{D}$, $\alpha \neq 0$, let $A_2^\alpha(\mathbb{D}) = \{ h \in A_2(\mathbb{D}) | h(\alpha) = 0 \}$. If $0 < r_1 < \frac{1}{2}$ and $|a| < \frac{x}{4}$, then
\[
T_a : A_2^\alpha(\mathbb{D}) \rightarrow A_2^\alpha(\mathbb{D})
\]
defines a contractive operator.

Proof of Theorem 1. Let $0 < \epsilon < \frac{1}{2}$ be given, and assume that $f_n, g_n \in A_2(\mathbb{D})$ exist with
\[
|f_n(z)| \geq |g_n(z)| \text{ for } c_n \leq |z| < 1 \text{ (} c_n \searrow 0 \text{) and}
\]
\[
\int_{\mathbb{D}} |f(z)|^2 dm_2 < \int_{\mathbb{D}} |g_n(z)|^2 dm_2.
\]
Define
\[
\varphi_n(z) = \frac{g_n(z)}{f_n(z)}, \quad z \in \mathbb{D}.
\]
Then $\varphi_n$ is meromorphic in $\mathbb{D}$ and $|\varphi_n(z)| \leq 1$ for $c_n \leq |z| < 1$. Therefore $(\varphi_n)_{n \in \mathbb{N}}$ is normal in $\{ 0 < |z| < 1 \}$ and w.l.o.g. we may assume that $\varphi_n$ tends to a limit function $\varphi$ locally uniformly in $\{ 0 < |z| < 1 \}$. $\varphi$ is analytic in $0 < |z| < 1$ and the singularity in 0 is removable. Hence we have $|\varphi(z)| \leq 1$ in $\{ |z| < 1 \}$. Then the condition
\[
\mathcal{N} \left( 1 - \epsilon, \frac{g_n}{f_n} \right) \setminus \mathcal{N} \left( \epsilon, \frac{f_n}{g_n} \right) \neq \emptyset
\]
implies that each $\varphi_n$ has at least one zero in $\epsilon \leq |z| < 1 - \epsilon$. Hence we have $|\varphi(z)| < 1$ in $|z| < 1$. If $R := \{ \frac{1}{2} \leq |z| \leq \frac{3}{4} \}$, $\eta_0 > 0$ exists such that $|\varphi(z)| \leq 1 - 2\eta_0$ for $z \in R$. Then $n_0 \in \mathbb{N}$ exists with $n_0 \cdot \eta_0 \geq 1$. Choose $n_1 \geq n_0$ with $c_n \leq \frac{1}{n_0}$ and
\[
|\frac{g_n(z)}{f_n(z)}| = |\varphi_n(z)| \leq \sqrt{1 - \eta_0} \quad \text{for } z \in R,
\]
if \( n \geq n_1 \). For \( h \in A_2(\mathbb{D}) \) with \( h(z) = \sum_{k=0}^{\infty} a_k z^k \) the function

\[
J(r, h) := \int_{|z| < r} |h(z)|^2 dm_2 = \pi \cdot \sum_{k=0}^{\infty} \frac{|a_k|^2}{k + 1} r^{2k+2} \quad (0 \leq r < 1)
\]
satisfies the inequality

\[
J(r + \rho, h) - J(r, h) \leq J(R + \rho, h) - J(R, h)
\]

for \( 0 \leq r < R < 1 \) and \( R + \rho < 1 \), since the same estimate holds for the powers \( r^{2k+2} \). For \( n \geq n_1 \) we get

\[
\int_R |f_n(z)|^2 - |g_n(z)|^2 dm_2 \geq \int_R \frac{1}{1 - \eta_0} |g_n(z)|^2 - |g_n(z)|^2 dm_2
\]

\[
> \int_R \eta_0 |g_n(z)|^2 dm_2 = \eta_0 \cdot \sum_{j=0}^{n_0-1} \int_{\frac{1}{2} + \frac{1}{n_0} < |z| < \frac{1}{2} + \frac{j + 1}{n_0}} |g_n(z)|^2 dm_2
\]

\[
\geq n_0 \cdot \eta_0 \cdot \int_{\frac{1}{2} < |z| < \frac{1}{2} + \frac{1}{n_0}} |g_n(z)|^2 dm_2 \geq \int_{|z| < \epsilon_n} |g_n(z)|^2 dm_2
\]

and therefore

\[
\int_D |f_n(z)|^2 - |g_n(z)|^2 dm_2 > 0.
\]

This is a contradiction and proves Theorem 1.

Proof of Theorem 2. We start with sequences \( f_n, g_n \) as in the proof of Theorem 1. Then the result follows as above, if we are able to show that \( |\varphi(z)| < 1 \) for \( |z| < 1 \) holds for the limit function \( \varphi \).

Suppose \( |\varphi(z)| \equiv 1 \). It follows that for \( n \geq n_0 \) all zeroes and poles of \( \varphi_n \) lie in \(|z| < \epsilon \) \( \cup \{ 1 - \epsilon < |z| < 1 \} \). We also have \( n(\epsilon, \varphi_n) = n(\epsilon, \frac{1}{\varphi_n}) \) for all, but finitely many \( n \in \mathbb{N} \). To see this let \( a^{(n)}_{\mu}(\mu = 1, \ldots, n(\epsilon, \frac{1}{\varphi_n})) \) be the zeroes and \( b^{(n)}(\nu = 1, \ldots, n(\epsilon, \varphi_n)) \) be the poles of \( \varphi_n \) in \(|z| < \epsilon \) \((n \geq n_0)\). W.l.o.g. we have \( \varphi_n(0) \neq 0, \infty \) \((n \in \mathbb{N})\). Otherwise choose \( |z_0| < \frac{1}{2} \) with \( \varphi_n(z_0) = 0, \infty \) for \( n \in \mathbb{N} \) and prove \( n(\frac{z}{2}, \varphi_n) = n(\frac{1}{2}, \frac{1}{\varphi_n}) \) for \( \varphi_n(z) = \varphi_n(z + z_0) \). The Poisson-Jensen formula implies

\[
\log |\varphi_n(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |\varphi_n(r e^{it})| dt - \sum_{\mu=1}^{n(\epsilon, \frac{1}{\varphi_n})} \log \frac{r_j}{|a^{(n)}_{\mu}|} + \sum_{\nu=1}^{n(\epsilon, \varphi_n)} \log \frac{r_j}{|b^{(n)}_{\nu}|}
\]

for \( \epsilon < r_1 < r_2 < 1 - \epsilon \). Then

\[
\frac{1}{2\pi} \int_0^{2\pi} \log |\varphi_n(r_2 e^{it})| - \log |\varphi_n(r_1 e^{it})| dt = \left[ n\left(\epsilon, \frac{1}{\varphi_n}\right) - n(\epsilon, \varphi_n) \right] \cdot \log \frac{r_2}{r_1}
\]

and the assertion follows, since the left-hand side tends to 0. The proposition

\[
n(1 - \epsilon, \frac{1}{f_n}) \neq n \left(1 - \epsilon, \frac{1}{g_n}\right)
\]

implies that the number of zeroes of \( f_n \) and \( g_n \) in \( \epsilon \leq |z| < 1 - \epsilon \) does not coincide, if \( n \) is sufficiently large. Then \( |\varphi(z)| < 1 \) follows as in Theorem 1. This is a contradiction and proves Theorem 2. ■
If we want to prove the existence of an absolute constant $c$ such that (*) holds under Korenblum’s additional assumption on the zeroes of $f$ and $g$ (compare [1]), it is a consequence of the classical maximum principle that we only have to study the case that $f/g$ has at least one zero in $|z| < c$. Then the assertion follows from Theorem 2.

In the following we want to reduce conjecture (*). We suppose that it does not hold and consider $f_n, g_n \in A_2(\mathbb{D})$ with

$$|f_n(z)| \geq |g_n(z)| \quad \text{for } c_n \leq |z| < 1 \quad (c_n \searrow 0)$$

and

$$\int_{\mathbb{D}} |f_n(z)|^2 dm_2 < \int_{\mathbb{D}} |g_n(z)|^2 dm_2.$$

If $\varphi_n$ and $\phi$ are defined as above, the proof of Theorem 1 implies $|\varphi(z)| \equiv 1$. W.l.o.g. we may assume that $f_n$ and $g_n$ are bounded and that all zeroes and poles of $\varphi_n$ lie in $\{|z| < \frac{1}{4}\}$ or even in a smaller circle. Otherwise replace $f_n$ and $g_n$ by

$$\tilde{f}_n(z) := f_n(z/2), \quad \tilde{g}_n(z) := g_n(z/2).$$

It is a consequence of Theorem 2 that $n(\frac{1}{4}, \varphi_n) = n(\frac{1}{4}, \varphi)$ for $n$ sufficiently large.

Define $a^{(n)}_\mu, b^{(n)}_\nu$ as above. If we consider

$$f_n \left(\frac{a^{(n)}_1 - b^{(n)}_1}{|a^{(n)}_1 - b^{(n)}_1|} \cdot z + b^{(n)}_1\right) \quad \text{and} \quad g_n \left(\frac{a^{(n)}_1 - b^{(n)}_1}{|a^{(n)}_1 - b^{(n)}_1|} \cdot z + b^{(n)}_1\right)$$

instead of $f_n$ and $g_n$, we can assume that each $g_n$ has a positive zero and that 0 is a zero of each $f_n$. For $m(n) = n(\frac{1}{4}, \varphi_n)$ and

$$A_n(z) = \prod_{\mu=1}^{m(n)} \frac{z - a^{(n)}_\mu}{1 - \bar{a}^{(n)}_\mu z}, \quad B_n(z) = \prod_{\nu=1}^{m(n)} \frac{z - b^{(n)}_\nu}{1 - \bar{b}^{(n)}_\nu z}$$

we have the representations

$$g_n(z) = h_n(z) \cdot A_n(z), \quad f_n(z) = h_n(z) \cdot B_n(z) \cdot e^{\psi_n(z)},$$

where $h_n$ and $\psi_n$ are analytic in $\mathbb{D}$. W.l.o.g. we may suppose $|a^{(n)}_\mu| < c_n, |b^{(n)}_\nu| < c_n$ for $\mu, \nu = 1, \ldots, m(n)$. The proposition gives

$$\left|\frac{B_n(z)}{A_n(z)}\right| \cdot e^{\Re \psi_n(z)} \geq 1$$

for $c_n \leq |z| < 1$ and the maximum principle implies $\Re \psi_n(z) > 0$ in $|z| < 1$, since $|\frac{B_n(z)}{A_n(z)}| \to 1$ for $|z| \to 1$. Then we have (compare [4], p. 140)

$$\Re \psi_n(0) \cdot \frac{1 + |z|}{1 - |z|} \geq \Re \psi_n(z) \geq \Re \psi_n(0) \cdot \frac{1 - |z|}{1 + |z|}$$

for $|z| < 1$ and therefore

$$\Re \psi_n(0) \geq \frac{1 - c_n}{1 + c_n} \cdot \log \max_{|\zeta| = c_n} \left|\frac{A_n(\zeta)}{B_n(\zeta)}\right|.$$
for $|z| < 1$. The method in the proof of Theorem 2 to show that the numbers of zeroes and poles of $\varphi_n$ coincide applied to the quotient $A_n/B_n$ with $r_1 = r < 1$ and $r_2 = 1$ gives

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{A_n}{B_n}(re^{it}) \right| dt = 0$$

and therefore

$$\max_{|\zeta| = r} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right| > 1$$

for $c_n \leq r < 1$. Hence

$$|f_n(z)| = |h_n(z)| \cdot |B_n(z)| \cdot e^{\text{Re} \, \psi_n(z)}$$

$$\geq |h_n(z)| \cdot |B_n(z)| \cdot \max_{|\zeta| = c_n} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right|$$

$$\geq |h_n(z)| \cdot |B_n(z)| \cdot \max_{|\zeta| = c_n} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right|^\frac{1-|z|}{1-|c_n|}$$

for $|z| \leq \frac{1}{2}$. If $0 < r_0 < \frac{1}{2}$ is a fixed radius and $c_n < r_0$ Hadamard’s three circle theorem implies

$$\max_{|\zeta| = c_n} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right| \geq \max_{|\zeta| = r_0} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right|^\frac{\log(1/|c_n|)}{\log(1/|r_0|)}.$$

If $n$ is sufficiently large, we then have

$$|f_n(z)| \geq |h_n(z)| \cdot |B_n(z)| \cdot \max_{|\zeta| = r_0} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right|$$

for $|z| \leq \frac{1}{2}$ and

$$|h_n(z)| \cdot |B_n(z)| \cdot \max_{|\zeta| = r_0} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right| \geq |g_n(z)|$$

for $r_0 \leq |z| \leq \frac{1}{2}$. Define

$$\tilde{f}_n(z) = h_n \left( \frac{z}{2} \right) \cdot B_n \left( \frac{z}{2} \right) \cdot \max_{|\zeta| = r_0} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right|$$

and

$$\tilde{g}_n(z) = g_n \left( \frac{z}{2} \right)$$

for $|z| < 1$ and $0 < r_0 < 1$. Then $|\tilde{f}_n(z)| \geq |\tilde{g}_n(z)|$ for $r_0 \leq |z| < 1$ (n sufficiently large) and

$$\int_D |\tilde{f}_n(z)|^2 dm_2 = \int_D |h_n(\frac{z}{2})B_n(\frac{z}{2})|^2 \cdot \max_{|\zeta| = \frac{r_0}{2}} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right|^2 \cdot \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right| dm_2$$

$$= 4 \int_{|z| < \frac{1}{2}} \left| h_n(z)B_n(z) \right|^2 \cdot \max_{|\zeta| = \frac{r_0}{2}} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right|^2 dm_2$$

$$\leq 4 \int_{|z| < \frac{1}{2}} |f_n(z)|^2 dm_2 < 4 \int_{|z| < \frac{1}{2}} |g_n(z)|^2 dm_2 = \int_D |\tilde{g}_n(z)|^2 dm_2.$$
products. If \( A(z) \) and \( B(z) \) have the same number of zeroes in \( \mathbb{D} \) and if each zero of \( A(z) \) or \( B(z) \) lies in \( \{ |z| < r_0 \} \), then

\[
\int_{|z| < \frac{1}{2}} |h(z) \cdot B(z)|^2 \cdot \max_{|\zeta| = r_0} \left| \frac{A(\zeta)}{B(\zeta)} \right|^2 \, dm_2 \geq \int_{|z| < \frac{1}{2}} |h(z) \cdot A(z)|^2 \, dm_2.
\]

The proof of Theorem 2 yields that the zeroes of the Blaschke products may even be assumed to be contained in a smaller circle.

**Proof of Theorem 3.** The above considerations starting with sequences \( f_n \) and \( g_n \) such that (**) does not hold and ending with conjecture (**∗∗) show that Theorem 3 follows, if we are able to solve (**) in the case \( A(z) = \frac{z-a}{1-\bar{a}z} \) \((a > 0)\) and \( B(z) = z \).

Then we have to show the existence of \( 0 < r_0 < \frac{1}{2} \) such that

\[
\int_{|z| < \frac{1}{2}} |h(z)|^2 |z|^2 \cdot \max_{|\zeta| = r_0} \left| \frac{\zeta - a}{1-\bar{a}\zeta} \right|^2 \, dm_2 \geq \int_{|z| < \frac{1}{2}} |h(z)|^2 \left| \frac{z-a}{1-\bar{a}z} \right|^2 \, dm_2
\]

for all \( h \) analytic in \( \mathbb{D} \), respectively

\[
\int_{|z| < \frac{1}{2}} |h(z)|^2 |z|^2 \cdot \max_{|\zeta| = r_0} \left| \frac{\zeta - a}{1-\bar{a}\zeta} \right|^2 \, dm_2 \geq \int_{|z| < \frac{1}{2}} |h(z)|^2 \left| \frac{z-a}{1-\bar{a}z} \right|^2 \, dm_2.
\]

For \( |z| \leq \frac{1}{2} \) and \( 0 < r_0 < \frac{1}{4} \) we get

\[
\max_{|\zeta| = r_0} \left| \frac{\zeta - a}{1-\bar{a}\zeta} \right| \geq \max_{|\zeta| = a} \left| \frac{\zeta - a}{1-\bar{a}\zeta} \right| = \frac{1 + |a|^2}{1 + |a|^2} = \frac{1 + |a|^2}{1 + 1 |a|^2} = \max_{|\zeta| = \frac{1}{2}} \left| \frac{\zeta - a}{1-\bar{a}\zeta} \right|.
\]

Hence it suffices to prove the existence of \( 0 < r_1 < \frac{1}{2} \) such that

\[
\int_{|z| < \frac{1}{2}} |h(z)|^2 |z|^2 \cdot \max_{|\zeta| = r_1} \left| \frac{\zeta - a}{1-\bar{a}\zeta} \right|^2 \, dm_2 \geq \int_{|z| < \frac{1}{2}} |h(z)|^2 \left| \frac{z-a}{1-\bar{a}z} \right|^2 \, dm_2
\]

for all \( h \) analytic in \( \mathbb{D} \). This inequality, transformed to the unit circle, follows from the next lemma.

**Lemma 5.** If \( h \in A_D(\mathbb{D}) \), \( 0 < r_1 < \frac{1}{2} \) and \( 0 < a < \frac{r_1}{4} \), then

\[
\int_D |h(z)|^2 |z|^2 \cdot \max_{|\zeta| = r_1} \left| \frac{\zeta - a}{1-\bar{a}\zeta} \right|^2 \, dm_2 \geq \int_D |h(z)|^2 \left| \frac{z-a}{1-\bar{a}z} \right|^2 \, dm_2
\]

**Proof.** We first show

\[
(1) \quad \int_D |z|^2 \cdot |z|^2 \cdot \max_{|\zeta| = r_1} \left| \frac{\zeta - a}{1-\bar{a}\zeta} \right|^2 \, dm_2 \geq \int_D |z|^2 \cdot |z|^2 \frac{1+a}{1+a} \, dm_2
\]

for \( 0 < r_1 < \frac{1}{2} \) and \( k \in \mathbb{N}_0 \). Then (1) is equivalent to

\[
\frac{\pi}{k+2} \left( \frac{1+a}{1+a \cdot r_1} \right)^2 \geq \frac{\pi}{k+1+\frac{1-a}{1+a}}.
\]
respectively
\[
\left( \frac{1}{r_1} - r_1 \right) \left( 2 + a \left( \frac{1}{r_1} + r_1 \right) \right) \cdot (k + 2) \geq \left( 1 + \frac{a}{r_1} \right)^2 \frac{2}{1 + a}.
\]
But this is obviously true if \( \frac{1}{r_1} - r_1 \geq 1 \), especially for \( r_1 \leq \frac{1}{2} \).

The now we get
\[
\int_D |h(z)|^2 |z|^2 \max_{|\zeta| = r_1} \left| \frac{\zeta - a}{1 - \overline{a} \zeta} \right|^2 dm_2 \geq \int_D \frac{|h(z)|^2 |z|^2 (1 + a)}{1 - \overline{a} \zeta} \cdot \frac{2}{1 + a} dm_2
\]
for \( h \in A_2(\mathbb{D}) \). If we apply [2], Lemma 4 with \( \gamma = \frac{1 - a}{1 + a} \) we have
\[
\int_D |h(z)|^2 |z|^2 \frac{1 + a}{1 - \overline{a} \zeta} dm_2 \geq \int_D |h(z)|^2 \left| \frac{1 - a}{1 - \overline{a} \zeta} \right|^2 \frac{1 + a}{1 - \overline{a} \zeta} \cdot \frac{2}{1 + a} dm_2
\]
which implies the assertion. \( \square \)

**Proof of Theorem 4.** Consider \( h \in A_0^0(\mathbb{D}) \). Then
\[
\int_D |h(z)|^2 dm_2 = \left( \frac{1 + a \cdot r_1}{1 + \frac{a}{r_1}} \right)^2 \cdot \int_D \frac{|h(z)|^2 |z|^2}{1 - \overline{a} \zeta} \cdot \frac{1 + a}{1 - \overline{a} \zeta} \cdot \frac{2}{1 + a} dm_2 \geq \left( \frac{1 + a \cdot r_1}{1 + \frac{a}{r_1}} \right)^2 \cdot \int_D \frac{|h(z)|^2 |z|^2}{1 - \overline{a} \zeta} \cdot \frac{1 + a}{1 - \overline{a} \zeta} \cdot \frac{2}{1 + a} dm_2 = \int_D |T_a(h)(z)|^2 dm_2. \square
\]

**References**


Fachbereich Mathematik, Universität Dortmund, 44221 Dortmund 50, Germany