CONSTRUCTION OF INVARIANT CURVES
FOR SINGULAR HOLOMORPHIC VECTOR FIELDS

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Abstract. Camacho and Sad proved the existence of invariant analytic curves for germs of singular holomorphic foliations \( F \) over a two dimensional complex analytic variety \( M \). Their proof is only of existential nature. Here we provide a simple constructive proof by giving criteria to choose a singular point at each blowing-up that follows an analytic invariant curve.

Our algorithm is founded on the stability by blowing-up of the property (\( \star \)) introduced in the following definition.

Definition. Consider a singular holomorphic foliation \( F \) over a two dimensional complex analytic variety \( M \), a normal crossings divisor \( E \) over \( M \) and a point \( q \in E \). We say that the triple \( (F, E, q) \) has the property (\( \star \)) if and only if one of the following properties holds:

- (\( \star \))-1: The point \( q \) lies exactly in one irreducible component \( S \) of \( E \), which is invariant for \( F \) and the index \( i_q(F, S) \in \mathbb{Q}_{\leq 0} = \{ r \in \mathbb{Q} ; r \geq 0 \} \).
- (\( \star \))-2: The point \( q \) lies in two irreducible components \( S_+ \) and \( S_- \) of \( E \) (call this point a “corner”), both are invariant curves and there is a real number \( a > 0 \) such that:
  \[
  i_q(F, S_+) \in \mathbb{Q}_{\leq -a} = \{ r \in \mathbb{Q} ; r \leq -a \},
  i_q(F, S_-) \notin \mathbb{Q}_{\geq -1/a} = \{ r \in \mathbb{Q} ; r \geq -1/a \}.
  \]
- (\( \star \))-3: The point \( q \) lies exactly in one irreducible component \( S \) of \( E \), it is a nonsingular point of \( F \) and \( S \) is transversal to \( F \) at \( q \).

(The definition and basic properties of the index can be found in [1]).

Remark. If we have the property (\( \star \))-2, then \( q \) is not a simple (irreducible) singularity. If we have either (\( \star \))-1 and \( q \) is a simple singularity or (\( \star \))-3, then there is a nonsingular analytic invariant curve \( \Gamma \) through \( q \) transversal to \( E \).

Theorem. Assume that \( (F, E, q) \) satisfies either (\( \star \))-1 or (\( \star \))-2. Consider the blowing-up \( \pi : M' \to M \) at the point \( q \). Let \( F' \) be the strict transform of \( F \) by \( \pi \). Put \( D = \pi^{-1}(q) \) and \( E' = \pi^{-1}(E) \). Then there is a point \( q' \in D \) such that \( (F', E', q') \) satisfies the property (\( \star \)).
Proof. If $\pi$ is a dicritical blowing-up we immediately get a point $q' \in D$ such that $(F', E', q')$ satisfies the property $(\ast)$-3.

Assume that $\pi$ is non dicritical and hence $D$ is an invariant curve for $F'$.

Consider first the case that $(F, E, q)$ satisfies $(\ast)$-1. Let $S'$ be the strict transform of $S$ by $\pi$ and put $\{q'\} = D \cap S'$. Let $p_1, \ldots, p_s$ be the singularities of $F'$ on $D \setminus \{q'\}$. Suppose that $(F', E', p_i)$ does not satisfy $(\ast)$-1 for any $i = 1, \ldots, s$. Then we have that

$$i_q(F', D) = 1 - \sum_{i=1}^{s} i_{p_i}(F', D) \in \mathbb{Q}_{(-2)}.$$ 

Since $i_q(F', S') = i_q(F, S) - 1 \notin \mathbb{Q}_{(-1)}$, then $(F', E', q')$ satisfies $(\ast)$-2.

Consider now the case that $(F, E, q)$ satisfies $(\ast)$-2. Let $S'_+$ and $S'_-$ be the strict transforms by $\pi$ of $S_+$ and $S_-$ respectively. Let $p_1, \ldots, p_s$ be the singularities of $F'$ on $D \setminus \{q_+, q_-\}$, where $q_+ = D \cap S'_+$ and $q_- = D \cap S'_-$. If $(F', E', p_i)$ does not satisfy $(\ast)$-1 for $i = 1, \ldots, s$, then

$$i_{q_+}(F', D) + i_{q_-}(F', D) = 1 - \sum_{i=1}^{s} i_{p_i}(F', D) \in \mathbb{Q}_{(-1)}.$$ 

Since $(F, E, q)$ satisfies the property $(\ast)$-2 we have that

$$i_{q_+}(F', S'_+) = i_q(F, S_+) - 1 \in \mathbb{Q}_{(-1)};$$
$$i_{q_-}(F', S'_-) = i_q(F, S_-) - 1 \notin \mathbb{Q}_{(-2)}.$$ 

If $(F', E', q_+)$ does not satisfy $(\ast)$-2 then $i_{q_+}(F', D) \in \mathbb{Q}_{(-2)}$. By (1) $i_{q_-}(F', D) \in \mathbb{Q}_{(-2)}$ and thus $(F', E', q_-)$ satisfies $(\ast)$-2.

To get an analytic invariant curve $\Gamma$ for $F$ at $q$ we proceed as follows. After the blowing-up with center $q$ we take a point $p_1$ in the exceptional divisor $E_1$ with the property $(\mathbb{A})$: if the blowing-up is dicritical we get $(\ast)$-3 and the algorithm stops, otherwise we get $(\ast)$-1 since the sum of the indices is $-1$. Repeat. By reduction of singularities, in a finite number of steps we get either $(\ast)$-3 or an irreducible singularity satisfying $(\ast)$. This gives an analytic invariant curve $\Gamma'$ transversal to the divisor that projects over $\Gamma$.

References


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