PRESERVATION OF THE RANGE
UNDER PERTURBATIONS OF AN OPERATOR

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Abstract. A sufficient condition for the stability of the range of a positive operator in a Hilbert space is given. As a consequence, we get a class of additive perturbations which preserve regularity of the critical point 0 of a positive operator in a Krein space.

1. Introduction

In this note we answer the following question:

Let \( P \) and \( P_1 \) be positive selfadjoint operators in a Hilbert space. Find sufficient conditions for the relation

\[ R(P^{1/2}) = R(P_1^{1/2}) . \]

Aside from its independent interest, our result implies the preservation of some spectral properties of positive definitizable operators in a Krein space under additive perturbations. The principal feature that distinguishes spectral properties of definitizable operators from spectral properties of selfadjoint operators in Hilbert spaces is the existence of finitely many critical points of the spectral function. The spectral properties of a definitizable operator outside of an open neighborhood of its critical points are similar to spectral properties of a selfadjoint operator in a Hilbert space. This similarity extends even to a critical point, provided that the spectral function is bounded in a neighborhood of that critical point. Critical points with this property are said to be regular. Significantly different behavior occurs at critical points that are not regular. Such critical points are called singular.

For the definitions and basic spectral properties of definitizable operators see [7]; for further analysis of critical points relevant to this note see [1, 2, 4, 5]. In [1, 2, 4, 5, 8] the reader can find sufficient conditions for the preservation of the regularity of the critical point \( \infty \) under additive perturbations. In [4] the preservation of the regularity of the critical point 0 has also been studied. The basic assumption in [4] is that the unperturbed operator is similar to a selfadjoint operator in a Hilbert space. In the main result of this note (see Theorem 3 (b)) we give a class of additive perturbations which preserve regularity of the critical point 0. Here the
unperturbed operator is any positive operator with nonempty resolvent set and a regular critical point 0.

2. Notation and general assumptions

Let \((\mathcal{K}, [\cdot, \cdot])\) be a Krein space and let \(J\) be a fundamental symmetry in \(\mathcal{K}\). Let \((\cdot, \cdot)\) be the corresponding Hilbert space scalar product, \((x, y) = [Jx, y], x, y \in \mathcal{K}\). Let \(a\) and \(v\) be two symmetric forms in \(\mathcal{K}\) with domains \(\mathcal{D}(a)\) and \(\mathcal{D}(v)\), respectively. In addition assume that the form \(a\) is closed and positive. (By positive we mean \(a(x) > 0\) for all \(x \in \mathcal{D}(a), x \neq 0\).) Further assume that \(\mathcal{D}(a) \subseteq \mathcal{D}(v)\) and that there exist real numbers \(\alpha > -\frac{1}{2}\) and \(\beta\) such that

\[
\alpha \leq \frac{v(x)}{a(x)} \leq \beta \quad \text{for all} \quad x \in \mathcal{D}(a) \setminus \{0\}.
\]

Define

\[
a_1 = a + v.
\]

The form \(a_1\) is a closed symmetric form defined on \(\mathcal{D}(a_1) = \mathcal{D}(a)\) (see [6, Theorem VI.3.4]). Clearly, \(a_1\) is also positive. Let \(P\) and \(P_1\) be the positive selfadjoint operators associated in the Hilbert space \((\mathcal{K}, (\cdot, \cdot))\) with the forms \(a\) and \(a_1\), respectively. (See [6, Theorem VI.2.1].) Let \(A = JP\) and \(A_1 = JP_1\). We say that the operators \(A\) and \(A_1\) are associated with the forms \(a\) and \(a_1\) in the Krein space \((\mathcal{K}, [\cdot, \cdot])\). Note that both \(A\) and \(A_1\) are injective.

3. Results

In this section we use the notation introduced in Section 2. We also assume that all the assumptions stated in Section 2 are satisfied.

**Theorem 1.** (a) \(\mathcal{D}(P_1^{\frac{1}{2}}) = \mathcal{D}(P_1^{\frac{1}{2}})\).

(b) \(\mathcal{R}(P_1^{\frac{1}{2}}) = \mathcal{R}(P_1^{\frac{1}{2}})\).

**Proof.** (a) follows from [6, Theorem VI.2.1].

(b) From [6, Lemma VI.3.1] it follows that there exists a bounded selfadjoint operator \(C\) such that

\[
v(x, y) = (CP^{1/2}x|P^{1/2}y) \quad \text{for all} \quad x, y \in \mathcal{D}(a).\]

Moreover (1) yields that \(\sigma(C) \subseteq [\alpha, \beta]\). It follows from [6, Theorem VI.2.1] that the operator \(P_1\) is given by

\[
P_1 = P^{1/2}(I + C)P^{1/2}.
\]

The operator \(I + C\) is boundedly invertible and \(P\) is injective, hence \(P_1\) is injective.

Define the forms \(\tilde{a}\) and \(\tilde{v}\) on \(\mathcal{D}(\tilde{a}) = \mathcal{R}(P^{1/2})\) by

\[
\tilde{a}(x, y) = (P^{-1/2}x|P^{-1/2}y), \quad x, y \in \mathcal{D}(\tilde{a}),
\]

\[
\tilde{v}(x, y) = -(C(I + C)^{-1}P^{-1/2}x|P^{-1/2}y), \quad x, y \in \mathcal{D}(\tilde{a}).
\]

The operator \(-C(I + C)^{-1} = (I + C)^{-1} - I\) is a bounded selfadjoint operator. By the spectral mapping theorem its spectrum is contained in

\[
\left[-\frac{\beta}{1 + \beta}, \frac{\alpha}{1 + \alpha}\right] \subset (-1, 1).
\]

\footnote{Note added in proof: P. Jonas has extended the results of this note to \(\alpha > -1\).}
Therefore there exists $\gamma < 1$ such that
\begin{equation}
|\tilde{v}(x)| \leq \gamma \tilde{a}(x), \quad x \in D(\tilde{a}).
\end{equation}
Define
\[ \tilde{a}_1 = \tilde{a} + \tilde{v}. \]
By [6, Theorem VI.1.33] the form $a_1$ is a closed symmetric form on $D(\tilde{a}_1) = D(\tilde{a})$.
Since $\tilde{a}$ is positive, it follows from (4) that $\tilde{a}_1$ is also positive. Let $\tilde{P}_1$ be the
associated positive selfadjoint operator. From the definition of $\tilde{a}_1$ we have
\begin{equation}
\tilde{a}_1(x, y) = ((I + C)^{-1}P^{-1/2}x|P^{-1/2}y), \quad x, y \in D(\tilde{a}).
\end{equation}
It follows from [6, Theorem VI.2.1] and (5) that the operator $\tilde{P}_1$ is given by
\begin{equation}
\tilde{P}_1 = P^{-1/2}(I + C)^{-1}P^{-1/2}.
\end{equation}
From (3) and (6) it follows that $\tilde{P}_1 = P_1^{-1}$. Therefore
\[ \mathcal{R}(P_1^{-1/2}) = D(P_1^{-1/2}) = D(P_1^{1/2}) = D(\tilde{a}_1) = D(\tilde{a}) = D(P^{-1/2}) = \mathcal{R}(P^{1/2}). \]

**Lemma 2.** Let $A = JP$ and $B = JQ$ be positive operators with nonempty resolvent
sets in the Krein space $K$. Assume that there exists $\nu > 0$ such that $\mathcal{R}(P^{\nu}) = \mathcal{R}(Q^{\nu})$.
Then the following statements are equivalent.
(a) 0 is not a singular critical point of $A$.
(b) 0 is not a singular critical point of $B$.

**Proof.** We use a “regularization” $P_\nu = P(I + P)^{-1}$ of a positive operator $P$ in a
Hilbert space. The operator $P_\nu$ is a bounded everywhere defined positive selfadjoint
operator with $\mathcal{R}(P_\nu) = \mathcal{R}(P)$. The operators $J(P_\nu)^{\nu}$ and $J(Q_\nu)^{\nu}$ are bounded
positive operators with the same range. By [3, Lemma 1.2], 0 is not a singular critical point of
$J(P_\nu)^{\nu}$ if and only if 0 is not a singular critical point of $J(Q_\nu)^{\nu}$. By [3, Lemma 1.1], 0 is not a singular critical point of $J(P_\nu)^{\nu}$ if and only if 0 is
not a singular critical point of $JP_\nu$. Therefore 0 is not a singular critical point of $JP_\nu$ if and only if 0 is not a singular critical point of $JQ_\nu$. Since the definitizable operators $A = JP$ and $JP_\nu$ have the same range, [3, Lemma 1.2] implies that 0 is
not a singular critical point of $A$ if and only if 0 is not a singular critical point of $JP_\nu$. This sequence of equivalences proves the lemma.

**Theorem 3.** Assume that the positive operators $A$ and $A_1$ have nonempty resolvent
sets.
(a) The following statements are equivalent.
(i) $\infty$ is a singular critical point of $A$.
(ii) $\infty$ is a singular critical point of $A_1$.
(b) The following statements are equivalent.
(i) 0 is a singular critical point of $A$.
(ii) 0 is a singular critical point of $A_1$.
(c) The following statements are equivalent.
(i) $A$ is similar to a selfadjoint operator in $(K, (\cdot | \cdot))$.
(ii) $A_1$ is similar to a selfadjoint operator in $(K, (\cdot | \cdot))$. 

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Proof. (a) Let $P$ and $P_1$ be the positive selfadjoint operators associated in the Hilbert space $(\mathcal{K}, (\cdot | \cdot))$ with $a$ and $a_1$, respectively. It follows from Theorem 1 that $\mathcal{D}(P^{1/2}) = \mathcal{D}(P_{1}^{1/2})$. Clearly, $P = JA$ and $P_1 = JA_1$. The equivalence follows from [1, Corollary 3.6].

(b) It follows from Theorem 1 that $\mathcal{R}(P^{1/2}) = \mathcal{R}(P_{1}^{1/2})$. The equivalence follows from Lemma 2.

(c) follows from (a) and (b).

Assume that $\alpha$ and $\beta$ satisfy (1) and $\alpha < \beta$. Define

$$\kappa^+ = \begin{cases} -\frac{1}{2\alpha} & \text{if } \alpha < 0, \\ +\infty & \text{if } \alpha \geq 0, \end{cases}$$

$$\kappa^- = \begin{cases} -\infty & \text{if } \beta \leq 0, \\ -\frac{1}{2\beta} & \text{if } \beta > 0. \end{cases}$$

Note that $\kappa^- < 0 < \kappa^+$. A simple calculation yields the following lemma.

Lemma 4. (a) Let $\kappa \in (2\kappa^-, 2\kappa^+)$. Then $a_{\kappa}$ is also a closed positive symmetric form on $\mathcal{D}(a_{\kappa}) = \mathcal{D}(a)$.

(b) Let $\kappa \in (\kappa^-, \kappa^+)$. Then the forms $\kappa v$ and $\alpha v$ satisfy (1) with some $\alpha', \beta'$ such that $-\frac{1}{2} < \alpha' < \beta'$.

Let $P_{\kappa}$ be the positive selfadjoint operator associated in $(\mathcal{K}, (\cdot | \cdot))$ with $a_{\kappa}$.

Corollary 5. Let $\kappa \in (\kappa^-, \kappa^+)$. Then

$$\mathcal{D}(P_{\kappa}^{1/2}) = \mathcal{D}(P_{1}^{1/2}) = \mathcal{D}(a), \quad \mathcal{R}(P_{\kappa}^{1/2}) = \mathcal{R}(P_{1}^{1/2}).$$

We need the following result, which is due to P. Jonas (personal communication).

Proposition 6. Assume that the operator $A$ has nonempty resolvent set and that $\infty$ is not a singular critical point of $A$. Assume that the form $v$ is $\alpha$-relatively form bounded with the relative bound less than 1. Then the resolvent set of the selfadjoint operator $A_1$ is nonempty.

Note that $v$ is $\alpha$-relatively form bounded with the relative bound less than 1 if (1) holds with $\alpha > -1, \beta < 1$. Another sufficient condition is $v(x) = (Vx|x)$ with a bounded selfadjoint operator $V$. It is also sufficient that the operator $C$ in (2) is compact in $\mathcal{H}$ or if it is bounded in $\mathcal{H}$ with the norm smaller than 1.

Let $A_{\kappa} = JP_{\kappa}$ be the positive selfadjoint operator associated with $a_{\kappa}$ in $(\mathcal{K}, (\cdot | \cdot))$.

Corollary 7. Assume that the operator $A$ has nonempty resolvent set and that $\infty$ is not a singular critical point of $A$. There exist real numbers $\kappa_{1}^{\pm}$ such that $\kappa^- < \kappa_{1}^- < 0 < \kappa_{1}^+ < \kappa^+$ and that $A_{\kappa}$ has nonempty resolvent set for $\kappa_{1}^- < \kappa < \kappa_{1}^+$. If $A_{\kappa}$ has nonempty resolvent set, then $\infty$ is not a singular critical point of $A_{\kappa}$ and the following statements are equivalent.

(i) $0$ is not a singular critical point of $A$.

(ii) $0$ is not a singular critical point of $A_{\kappa}$.

(iii) $A$ is similar to a selfadjoint operator in $(\mathcal{K}, (\cdot | \cdot))$.

(iv) $A_{\kappa}$ is similar to a selfadjoint operator in $(\mathcal{K}, (\cdot | \cdot))$.

Proof. The first statement follows from Proposition 6 and Corollary 5, and the equivalences follow from Theorem 3 (b), (c).
Remark 8. An explicit formula for $\kappa_1^\pm$ in terms of $\alpha$ and $\beta$ is easily deduced. We omit it, since P. Jonas has proved a more precise version of Proposition 6 which gives better estimates for $\kappa_1^\pm$.

References


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