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PRESERVATION OF THE RANGE UNDER PERTURBATIONS OF AN OPERATOR

BRANKO ĆURGUS AND BRANKO NAJMAN

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Abstract. A sufficient condition for the stability of the range of a positive operator in a Hilbert space is given. As a consequence, we get a class of additive perturbations which preserve regularity of the critical point 0 of a positive operator in a Krein space.

1. Introduction

In this note we answer the following question:

Let P and P_1 be positive selfadjoint operators in a Hilbert space. Find sufficient conditions for the relation

$$\mathcal{R}(P^{\frac{1}{2}}) = \mathcal{R}(P_1^{\frac{1}{2}}) .$$

Aside from its independent interest, our result implies the preservation of some spectral properties of positive definitizable operators in a Krein space under additive perturbations. The principal feature that distinguishes spectral properties of definitizable operators from spectral properties of selfadjoint operators in Hilbert spaces is the existence of finitely many critical points of the spectral function. The spectral properties of a definitizable operator outside of an open neighborhood of its critical points are similar to spectral properties of a selfadjoint operator in a Hilbert space. This similarity extends even to a critical point, provided that the spectral function is bounded in a neighborhood of that critical point. Critical points with this property are said to be regular. Significantly different behavior occurs at critical points that are not regular. Such critical points are called singular.

For the definitions and basic spectral properties of definitizable operators see [7]; for further analysis of critical points relevant to this note see [1, 2, 4, 5]. In [1, 2, 4, 5, 8] the reader can find sufficient conditions for the preservation of the regularity of the critical point ∞ under additive perturbations. In [4] the preservation of the regularity of the critical point 0 has also been studied. The basic assumption in [4] is that the unperturbed operator is similar to a selfadjoint operator in a Hilbert space. In the main result of this note (see Theorem 3 (b)) we give a class of additive perturbations which preserve regularity of the critical point 0. Here the

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unperturbed operator is any positive operator with nonempty resolvent set and a regular critical point 0.

2. Notation and general assumptions

Let $(\mathcal{K}, [\,\cdot\,|\,\cdot\,])$ be a Krein space and let J be a fundamental symmetry in $\mathcal{K}$. Let $(\,\cdot\,|\,\cdot\,)$ be the corresponding Hilbert space scalar product, $(x, y) = [Jx, y],\ x, y \in \mathcal{K}$. Let a and v be two symmetric forms in $\mathcal{K}$ with domains $\mathcal{D}(a)$ and $\mathcal{D}(v)$, respectively. In addition assume that the form a is closed and positive. (By positive we mean $a(x) > 0$ for all $x \in \mathcal{D}(a),\ x \neq 0$.) Further assume that $\mathcal{D}(a) \subseteq \mathcal{D}(v)$ and that there exist real numbers $\alpha > -\frac{1}{2}$ and β such that [1]

$$\alpha \leq \frac{v(x)}{a(x)} \leq \beta \quad \text{for all} \quad x \in \mathcal{D}(a) \setminus \{0\}\ . \tag{1}$$

Define

$$a_1 = a + v\ .$$

The form a_1 is a closed symmetric form defined on $\mathcal{D}(a_1) = \mathcal{D}(a)$ (see [6, Theorem VI. 3.4]). Clearly, a_1 is also positive. Let P and P_1 be the positive selfadjoint operators associated in the Hilbert space $(\mathcal{K}, (\,\cdot\,|\,\cdot\,))$ with the forms a and a_1, respectively. (See [6, Theorem VI. 2.1].) Let $A = JP$ and $A_1 = JP_1$. We say that the operators A and A_1 are associated with the forms a and a_1 in the Krein space $(\mathcal{K}, [\,\cdot\,|\,\cdot\,])$. Note that both A and A_1 are injective.

3. Results

In this section we use the notation introduced in Section 2. We also assume that all the assumptions stated in Section 2 are satisfied.

Theorem 1. (a) $\mathcal{D}(P^{\frac{1}{2}}) = \mathcal{D}(P_1^{\frac{1}{2}})$.
(b) $\mathcal{R}(P^{\frac{1}{2}}) = \mathcal{R}(P_1^{\frac{1}{2}})$.

Proof. (a) follows from [6, Theorem VI. 2.1].

(b) From [6, Lemma VI.3.1] it follows that there exists a bounded selfadjoint operator C such that

$$v(x, y) = (CP^{1/2}x|P^{1/2}y) \quad \text{for all} \quad x,\ y \in \mathcal{D}(a)\ . \tag{2}$$

Moreover (1) yields that $\sigma(C) \subseteq [\alpha, \beta]$. It follows from [6, Theorem VI.2.1] that the operator P_1 is given by

$$P_1 = P^{1/2}(I + C)P^{1/2}\ . \tag{3}$$

The operator $I + C$ is boundedly invertible and P is injective, hence P_1 is injective.

Define the forms $\tilde{a}$ and $\tilde{v}$ on $\mathcal{D}(\tilde{a}) = \mathcal{R}(P^{1/2})$ by

$$\begin{aligned} \tilde{a}(x, y) &= (P^{-1/2}x|P^{-1/2}y), \quad x, y \in \mathcal{D}(\tilde{a}), \\ \tilde{v}(x, y) &= -(C(I + C)^{-1}P^{-1/2}x|P^{-1/2}y), \quad x, y \in \mathcal{D}(\tilde{a})\ . \end{aligned}$$

The operator $-C(I + C)^{-1} = (I + C)^{-1} - I$ is a bounded selfadjoint operator. By the spectral mapping theorem its spectrum is contained in

$$\left[-\frac{\beta}{1+\beta}, -\frac{\alpha}{1+\alpha}\right] \subset (-1, 1)\ .$$

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[1] Note added in proof: P. Jonas has extended the results of this note to $\alpha > -1$.

Therefore there exists $\gamma < 1$ such that

$$|\tilde{v}(x)| \leq \gamma \tilde{a}(x), \quad x \in \mathcal{D}(\tilde{a}) \ . \tag{4}$$

Define

$$\tilde{a}_1 = \tilde{a} + \tilde{v} \ .$$

By [6, Theorem VI.1.33] the form a_1 is a closed symmetric form on $\mathcal{D}(\tilde{a}_1) = \mathcal{D}(\tilde{a})$. Since $\tilde{a}$ is positive, it follows from (4) that $\tilde{a}_1$ is also positive. Let $\tilde{P}_1$ be the associated positive selfadjoint operator. From the definition of $\tilde{a}_1$ we have

$$\tilde{a}_1(x,y) = ((I+C)^{-1}P^{-1/2}x|P^{-1/2}y), \quad x, y \in \mathcal{D}(\tilde{a}) \ . \tag{5}$$

It follows from [6, Theorem VI.2.1] and (5) that the operator $\tilde{P}_1$ is given by

$$\tilde{P}_1 = P^{-1/2}(I+C)^{-1}P^{-1/2} \ . \tag{6}$$

From (3) and (6) it follows that $\tilde{P}_1 = P_1^{-1}$. Therefore

$$\mathcal{R}(P_1^{1/2}) = \mathcal{D}(P_1^{-1/2}) = \mathcal{D}(\tilde{P}_1^{1/2}) = \mathcal{D}(\tilde{a}_1) = \mathcal{D}(\tilde{a}) = \mathcal{D}(P^{-1/2}) = \mathcal{R}(P^{1/2}) \ .$$

□

Lemma 2. *Let $A = JP$ and $B = JQ$ be positive operators with nonempty resolvent sets in the Krein space $\mathcal{K}$. Assume that there exists $\nu > 0$ such that $\mathcal{R}(P^\nu) = \mathcal{R}(Q^\nu)$. Then the following statements are equivalent.*

(a) 0 *is not a singular critical point of A.*
(b) 0 *is not a singular critical point of B.*

Proof. We use a "regularization" $P_r = P(I+P)^{-1}$ of a positive operator P in a Hilbert space. The operator P_r is a bounded everywhere defined positive selfadjoint operator with $\mathcal{R}(P_r) = \mathcal{R}(P)$. The operators $J(P_r)^\nu$ and $J(Q_r)^\nu$ are bounded positive operators with the same range. By [3, Lemma 1.2], 0 is not a singular critical point of $J(P_r)^\nu$ if and only if 0 is not a singular critical point of $J(Q_r)^\nu$. By [3, Lemma 1.1], 0 is not a singular critical point of $J(P_r)^\nu$ if and only if 0 is not a singular critical point of JP_r. Therefore 0 is not a singular critical point of JP_r if and only if 0 is not a singular critical point of JQ_r. Since the definitizable operators $A = JP$ and JP_r have the same range, [3, Lemma 1.2] implies that 0 is not a singular critical point of A if and only if 0 is not a singular critical point of JP_r. This sequence of equivalences proves the lemma. □

Theorem 3. *Assume that the positive operators A and A_1 have nonempty resolvent sets.*

(a) *The following statements are equivalent.*
  (i) ∞ *is a singular critical point of A.*
  (ii) ∞ *is a singular critical point of A_1.*
(b) *The following statements are equivalent.*
  (i) 0 *is a singular critical point of A.*
  (ii) 0 *is a singular critical point of A_1.*
(c) *The following statements are equivalent.*
  (i) *A is similar to a selfadjoint operator in $(\mathcal{K}, (\,\cdot\,|\,\cdot\,))$.*
  (ii) *A_1 is similar to a selfadjoint operator in $(\mathcal{K}, (\,\cdot\,|\,\cdot\,))$.*

Proof. (a) Let P and P_1 be the positive selfadjoint operators associated in the Hilbert space $(\mathcal{K}, (\,\cdot\,|\,\cdot\,))$ with a and a_1, respectively. It follows from Theorem 1 that $\mathcal{D}(P^{1/2}) = \mathcal{D}(P_1^{1/2})$. Clearly, $P = JA$ and $P_1 = JA_1$. The equivalence follows from [1, Corollary 3.6].

(b) It follows from Theorem 1 that $\mathcal{R}(P^{1/2}) = \mathcal{R}(P_1^{1/2})$. The equivalence follows from Lemma 2.

(c) follows from (a) and (b). □

Assume that α and β satisfy (1) and $\alpha < \beta$. Define

$$\kappa^+ = \begin{cases} -\frac{1}{2\alpha} & \text{if } \alpha < 0, \\ +\infty & \text{if } \alpha \geq 0, \end{cases}$$

$$\kappa^- = \begin{cases} -\infty & \text{if } \beta \leq 0, \\ -\frac{1}{2\beta} & \text{if } \beta > 0. \end{cases}$$

Note that $\kappa^- < 0 < \kappa^+$. A simple calculation yields the following lemma.

Lemma 4. (a) *Let $\kappa \in (2\kappa^-, 2\kappa^+)$. Then a_κ is also a closed positive symmetric form on $\mathcal{D}(a_\kappa) = \mathcal{D}(a)$.*

(b) *Let $\kappa \in (\kappa^-, \kappa^+)$. Then the forms κv and a satisfy* (1) *with some α', β' such that $-\frac{1}{2} < \alpha' < \beta'$.*

Let P_κ be the positive selfadjoint operator associated in $(\mathcal{K}, (\,\cdot\,|\,\cdot\,))$ with a_κ.

Corollary 5. *Let $\kappa \in (\kappa^-, \kappa^+)$. Then*

$$\mathcal{D}(P_\kappa^{\frac{1}{2}}) = \mathcal{D}(P_1^{\frac{1}{2}}) = \mathcal{D}(a)\ ,\ \mathcal{R}(P_\kappa^{\frac{1}{2}}) = \mathcal{R}(P^{\frac{1}{2}}).$$

We need the following result, which is due to P. Jonas (personal communication).

Proposition 6. *Assume that the operator A has nonempty resolvent set and that ∞ is not a singular critical point of A. Assume that the form v is a-relatively form bounded with the relative bound less than* 1. *Then the resolvent set of the selfadjoint operator A_1 is nonempty.*

Note that v is a-relatively form bounded with the relative bound less than 1 if (1) holds with $\alpha > -1$, $\beta < 1$. Another sufficient condition is $v(x) = (Vx|x)$ with a bounded selfadjoint operator V. It is also sufficient that the operator C in (2) is compact in $\mathcal{H}$ or if it is bounded in $\mathcal{H}$ with the norm smaller than 1.

Let $A_\kappa = JP_\kappa$ be the positive selfadjoint operator associated with a_κ in $(\mathcal{K}, [\,\cdot\,|\,\cdot\,])$.

Corollary 7. *Assume that the operator A has nonempty resolvent set and that ∞ is not a singular critical point of A. There exist real numbers $\kappa_1^\pm$ such that $\kappa^- < \kappa_1^- < 0 < \kappa_1^+ < \kappa^+$ and that A_κ has nonempty resolvent set for $\kappa_1^- < \kappa < \kappa_1^+$. If A_κ has nonempty resolvent set, then ∞ is not a singular critical point of A_κ and the following statements are equivalent.*

(i) 0 *is not a singular critical point of A.*
(ii) 0 *is not a singular critical point of A_κ.*
(iii) *A is similar to a selfadjoint operator in $(\mathcal{K}, (\,\cdot\,|\,\cdot\,))$.*
(iv) *A_κ is similar to a selfadjoint operator in $(\mathcal{K}, (\,\cdot\,|\,\cdot\,))$.*

Proof. The first statement follows from Proposition 6 and Corollary 5, and the equivalences follow from Theorem 3 (b), (c). □

Remark 8. An explicit formula for $\kappa_1^{\pm}$ in terms of α and β is easily deduced. We omit it, since P. Jonas has proved a more precise version of Proposition 6 which gives better estimates for $\kappa_1^{\pm}$.

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Department of Mathematics, Western Washington University, Bellingham, Washington 98225

E-mail address: `curgus@cc.wwu.edu`

Department of Mathematics, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia

E-mail address: `najman@cromath.math.hr`