ISOMORPHICALLY EXPANSIVE MAPPINGS IN $l_2$

J. GARCÍA-FALSET, A. JIMÉNEZ-MELADO, AND E. LLORÉNS-FUSTER

(Communicated by Palle E. T. Jorgensen)

Abstract. We show that for any renorming $\|\cdot\|$ of $l_2$, the well known fixed point free mappings by Kakutani, Baillon and others are not nonexpansive.

1. Introduction

Let $C$ be a subset of a Banach space $(X, \|\cdot\|)$. A mapping $T : C \rightarrow C$ is called $k$-lipschitzian ($k > 0$) if $\|T(x) - T(y)\| \leq k \|x - y\|$ for all $x, y \in C$.

Nonexpansive mappings are those which have Lipschitz constant $k = 1$. A mapping $T : C \rightarrow C$ is said to be uniformly $k$-lipschitzian if every iteration $T^n$ is $k$-lipschitzian, i.e. if for every positive integer $n$, and $x, y \in C$

$$\|T^n(x) - T^n(y)\| \leq k \|x - y\|.$$ 

We say that $X$ has the fixed point property (FPP) if every nonexpansive mapping $T : C \rightarrow C$ defined on a nonempty convex and weakly compact subset $C$ of $X$ has a fixed point.

If $T : C \rightarrow C$ is a nonexpansive mapping with respect to some norm $|\cdot|$ on $X$ equivalent to the norm $\|\cdot\|$, then it is straightforward to see that $T$ is also uniformly lipschitzian with respect to the norm $\|\cdot\|$. Hence, in order to look for a fixed point free $|\cdot|$-nonexpansive mapping we must seek this mapping among those which are uniformly lipschitzian with respect to the norm $\|\cdot\|$.

It is known that bounded closed convex subsets of $\ell_2$ have the fixed point property for nonexpansive mappings, but it is unknown whether the same is true for bounded closed convex subsets of a Banach space $X$ where $X$ is $\ell_2$ with an equivalent norm. In this paper it is shown that certain uniformly lipschitzian mappings which are known to be fixed point free in the unit ball of $\ell_2$ are also not nonexpansive relative to any renorming of $\ell_2$. This shows that one strategy for answering the above question is precluded. At the same time it shows that the class of mappings which are uniformly lipschitzian on $\ell_2$ is strictly larger than the class of mappings which are nonexpansive with respect to some equivalent norm on $\ell_2$. Theorems 1 and 2 serve mainly as vehicles for reaching this conclusion.

Received by the editors November 28, 1995 and, in revised form, March 18, 1996.
1991 Mathematics Subject Classification. Primary 47H09, 47H10.
Key words and phrases. Nonexpansive mappings, uniformly lipschitzian mappings, fixed points.
This research has been partially supported by D.G.I.C.Y.T. PB93-1177-C02-02 and D.G.I.C.Y.T. PB94-1496.
2. Preliminaries

We recall now some famous examples of uniformly lipschitzian mappings in ℓ₂. Let ∥·∥ be the usual euclidean norm on the sequence Hilbert space ℓ₂, and let B be the closed ||·|| -unit ball and let S : ℓ₂ → ℓ₂ be the right shift operator defined by

\[ S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots). \]

Then the mapping K : B → B defined by

\[ K(x) = \left( \frac{(1 - \|x\|_2)e + S(x)}{\| (1 - \|x\|_2)e + S(x) \|_2} \right), \]

where e = (1, 0, \ldots), is uniformly 2-lipschitzian and has no fixed point in B ([GKT]).

Later on, Baillon ([B]) found another example of a uniformly π/2-lipschitzian mapping without fixed point in B. This is the mapping G : B → B defined by

\[ G(x) = \begin{cases} \cos \left( \frac{\pi}{2} \|x\|_2 \right) e + \sin \left( \frac{\pi}{2} \|x\|_2 \right) \frac{S(x)}{\|x\|_2}, & x \neq 0, \\ e, & x = 0 \end{cases} \]

(for details about this map, see [T]).

Both examples are modifications of an earlier one due to Kakutani (see [GK]): for 0 < r < 1, define Kᵣ : B → B by

\[ K_r(x) = r(1 - \|x\|_2)e + S(x). \]

Then Kᵣ have the (non uniform) Lipschitz constant \(\sqrt{1 + r^2}\) and has no fixed point in B.

3. The results

The announced results will follow easily from the following lemma.

**Lemma 1.** Let \((X, \| \cdot \|)\) be a Banach space and let \(B_X, S_X\) be the closed unit ball and the unit sphere of \(X\) respectively. Suppose that \(T : B_X \to X\) is a mapping which is nonexpansive with respect to some norm \(\| \cdot \|\) on \(X\), equivalent to \(| \cdot |\). Then we have that

\[ d = \inf_{y \in S_X} |T(0) - T(y)| \leq 1. \]

**Proof.** There exist positive constants \(\alpha, \beta\) such that, for every \(v \in X\),

\[ \alpha |v| \leq \|v\| \leq \beta |v|. \]

Since \(T\) is \(\| \cdot \|\)-nonexpansive, for any \(y \in S_X\) we have that

\[ \alpha |T(0) - T(y)| \leq \|T(0) - T(y)\| \leq \|y\| \leq \beta \]

and then

\[ \alpha d \leq \|y\| \leq \beta \]

for all \(y \in S_X\). Thus, for every \(v \in X\) we have that

\[ \alpha d |v| \leq \|v\| \leq \beta |v|. \]

Indeed, an induction argument shows that

\[ \alpha d^n |v| \leq \|v\| \leq \beta |v| \]

for every positive integer \(n\) and every \(v \in X\), which is impossible unless \(d \leq 1. \)

\[ \square \]
The above lemma will allow us to give a fixed point result. Recall that a sequence \((X_n)\) of finite dimensional subspaces of a Banach space \(X\) is called a \textit{Schauder finite dimensional decomposition} (FDD) of \(X\), if every \(x \in X\) has a unique representation of the form \(x = \sum x_i\) with \(x_i \in X_i\) for every \(i \in \mathbb{N}\). If \(X\) is a Banach space with a FDD \((X_n)\), and \(x \in X\), \(\text{supp}(x)\) denotes, as usual, the set of positive integers \(i\) such that \(x_i \neq 0\). M. A. Khamsi ([K]) defined, for a Banach space \(X\) with a FDD, the coefficient \(\beta_p(X)\) in the following way: For \(p \in [1, \infty)\), \(\beta_p(X)\) is the infimum of the set of numbers \(\lambda\) such that

\[ (\|x\|^p + \|y\|^p)^{1/p} \leq \lambda \|x + y\| \]

for every \(x, y \in X\) which verify \(\text{supp}(x) < \text{supp}(y)\), that is, \(\max[\text{supp}(x)] < \min[\text{supp}(y)]\).

For example, \(\beta_p(l_p) = 1\) (\(1 \leq p < \infty\)), and \(\beta_2(J) = 1\), where \(J\) is the James space which consists of all sequences \(x = (x_n) \in c_0\) such that

\[ \|x\| := \sup\{(x_{p_1} - x_{p_2})^2 + \ldots + (x_{p_n-1} - x_{p_n})^2\}^{1/2} \]

is finite (the supremum is taken for every \(n\) and for every finite increasing sequence of positive integers \((p_i)\)).

Now, for such Banach spaces we can state the following

**Theorem 1.** Let \((X, \| \cdot \|)\) be a Banach space with a FDD such that \(\beta_p(X) = 1\) for some \(p \in [1, \infty)\). Let \(B_X, S_X\) be the closed unit ball and the unit sphere of \(X\), respectively, and let \(T : B_X \to X\) be a mapping such that

a) \(T(S_X) \subset S_X\) and
b) \(\text{supp}(T(0)) < \text{supp}(v)\) for every \(v \in T(S_X)\).

If \(T\) is nonexpansive with respect to some equivalent norm on \(X\) then \(T\) has a fixed point. Specifically, \(T(0) = 0\).

**Proof.** Since \(\beta_p(X) = 1\), conditions a) and b) imply that

\[ |T(0)|^p + |T(y)|^p \leq |T(0) - T(y)|^p \]

for all \(y \in S_X\), and then we have that

\[ d = \inf_{y \in S_X} |T(0) - T(y)| \geq |T(0)|^p + 1 \]

This inequality, when combined with Lemma 1, gives that

\[ 1 \geq |T(0)|^p + 1 \]

and consequently \(T(0) = 0\). \(\square\)

A similar result can be stated in the setting of Hilbert space if we replace condition b) by: \(T(0)\) is orthogonal to the set \(T(S_X)\). Under this assumption we would obtain that

\[ \|T(0)\|^2 + \|T(y)\|^2 = \|T(0) - T(y)\|^2 \]

for every \(y \in S_X\), from which the same conclusion follows.

**Theorem 2.** Let \((X, \| \cdot \|_2)\) be a Hilbert space. Let \(B_X, S_X\) be the closed unit ball and the unit sphere of \(X\), respectively, and let \(T : B_X \to X\) be a mapping such that

a) \(T(S_X) \subset S_X\).

b) \(T(0)\) is orthogonal to the set \(T(S_X)\).

If \(T\) is nonexpansive with respect to some equivalent norm on \(X\) then \(T\) has a fixed point. Specifically, \(T(0) = 0\).
Suppose now that $T : B \rightarrow B$ is any of the mappings $K$, $G$ or $K_r$ which we refer to in the introduction. It is clear that $T$ verifies the conditions (a) and (b) of Theorem 2 and that $T(0) \neq 0$. This proves the following corollary.

**Corollary 1.** Let $T : B \rightarrow B$ be any of the mappings $K$, $G$ or $K_r$ and let $\| \cdot \|$ be any norm on $\ell_2$, equivalent to $\| \cdot \|_2$. Then $T$ is not $\| \cdot \|$-nonexpansive.

We wish to thank the referee for his valuable comments.

**References**


(J. García-Falset and E. Lloréns-Fuster) Departamento de Análisis Matemático, Facultad de Matemáticas, Universitat de València, 46100 Burjassot, Valencia, Spain

E-mail address: jesus.garcia@uv.es

E-mail address: enrique.llorens@uv.es

(A. Jiménez-Melado) Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain

E-mail address: melado@ccuma.sci.uma.es