ISOMORPHICALLY EXPANSIVE MAPPINGS IN $\ell_2$

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Abstract. We show that for any renorming $\|\cdot\|$ of $\ell_2$, the well known fixed point free mappings by Kakutani, Baillon and others are not nonexpansive.

1. Introduction

Let $C$ be a subset of a Banach space $(X, \|\cdot\|)$. A mapping $T : C \to C$ is called $k$-lipschitzian ($k > 0$) if $\|T(x) - T(y)\| \leq k \|x - y\|$ for all $x, y \in C$.

Nonexpansive mappings are those which have Lipschitz constant $k = 1$. A mapping $T : C \to C$ is said to be uniformly $k$-lipschitzian if every iteration $T^n$ is $k$-lipschitzian, i.e. if for every positive integer $n$, and $x, y \in C$

$$\|T^n(x) - T^n(y)\| \leq k \|x - y\|.$$ We say that $X$ has the fixed point property (FPP) if every nonexpansive mapping $T : C \to C$ defined on a nonempty convex and weakly compact subset $C$ of $X$ has a fixed point.

If $T : C \to C$ is a nonexpansive mapping with respect to some norm $|\cdot|$ on $X$ equivalent to the norm $\|\cdot\|$, then it is straightforward to see that $T$ is also uniformly lipschitzian with respect to the norm $\|\cdot\|$. Hence, in order to look for a fixed point free $|\cdot|$-nonexpansive mapping we must seek this mapping among those which are uniformly lipschitzian with respect to the norm $\|\cdot\|$.

It is known that bounded closed convex subsets of $\ell_2$ have the fixed point property for nonexpansive mappings, but it is unknown whether the same is true for bounded closed convex subsets of a Banach space $X$ where $X$ is $\ell_2$ with an equivalent norm.

In this paper it is shown that certain uniformly lipschitzian mappings which are known to be fixed point free in the unit ball of $\ell_2$ are also not nonexpansive relative to any renorming of $\ell_2$. This shows that one strategy for answering the above question is precluded. At the same time it shows that the class of mappings which are uniformly lipschitzian on $\ell_2$ is strictly larger than the class of mappings which are nonexpansive with respect to some equivalent norm on $\ell_2$. Theorems 1 and 2 serve mainly as vehicles for reaching this conclusion.

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2. Preliminaries

We recall now some famous examples of uniformly lipschitzian mappings in $\ell_2$.

Let $\|\cdot\|_2$ be the usual euclidean norm on the sequence Hilbert space $\ell_2$, let $B$ be the closed $\|\cdot\|_2$-unit ball and let $S : \ell_2 \to \ell_2$ be the right shift operator defined by $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$.

Then the mapping $K : B \to B$ defined by

$$K(x) = \frac{(1 - \|x\|_2)e + S(x)}{\| (1 - \|x\|_2)e + S(x) \|_2},$$

where $e = (1, 0, \ldots)$, is uniformly 2-lipschitzian and has no fixed point in $B$ (GKT).

Later on, Baillon ([B]) found another example of a uniformly $\pi$-lipschitzian mapping without fixed point in $B$. This is the mapping $G : B \to B$ defined by

$$G(x) = \begin{cases} 
\cos \left( \frac{\pi}{2} \|x\|_2 \right) e + \sin \left( \frac{\pi}{2} \|x\|_2 \right) S(x) \|x\|_2, & x \neq 0, \\
e, & x = 0
\end{cases}$$

(for details about this map, see [T]).

Both examples are modifications of an earlier one due to Kakutani (see [GK]): for $0 < r < 1$, define $K_r : B \to B$ by

$$K_r(x) = r(1 - \|x\|_2)e + S(x).$$

Then $K_r$ have the (non uniform) Lipschitz constant $\sqrt{1 + r^2}$ and has no fixed point in $B$.

3. The results

The announced results will follow easily from the following lemma.

**Lemma 1.** Let $(X, \|\cdot\|)$ be a Banach space and let $B_X, S_X$ be the closed unit ball and the unit sphere of $X$ respectively. Suppose that $T : B_X \to X$ is a mapping which is nonexpansive with respect to some norm $\|\cdot\|$ on $X$, equivalent to $\|\cdot\|$. Then we have that

$$d = \inf_{y \in S_X} |T(0) - T(y)| \leq 1.$$

**Proof.** There exist positive constants $\alpha, \beta$ such that, for every $v \in X$,

$$\alpha |v| \leq \|v\| \leq \beta |v|.$$

Since $T$ is $\|\cdot\|$-nonexpansive, for any $y \in S_X$ we have that

$$\alpha |T(0) - T(y)| \leq \|T(0) - T(y)\| \leq \|y\| \leq \beta$$

and then

$$\alpha d \leq \|y\| \leq \beta$$

for all $y \in S_X$. Thus, for every $v \in X$ we have that

$$\alpha d |v| \leq \|v\| \leq \beta |v|.$$}

Indeed, an induction argument shows that

$$\alpha d^n |v| \leq \|v\| \leq \beta |v|$$

for every positive integer $n$ and every $v \in X$, which is impossible unless $d \leq 1$. \qed
The above lemma will allow us to give a fixed point result. Recall that a sequence $(X_n)$ of finite dimensional subspaces of a Banach space $X$ is called a Schauder finite dimensional decomposition (FDD) of $X$, if every $x \in X$ has a unique representation of the form $x = \sum x_i$ with $x_i \in X_i$ for every $i \in \mathbb{N}$. If $X$ is a Banach space with a FDD $(X_n)$, and $x, y \in X$, $\text{supp}(x)$ denotes, as usual, the set of positive integers $i$ such that $x_i \neq 0$. M. A. Khamsi ([K]) defined, for a Banach space $X$ with a FDD, the coefficient $\beta_p(X)$ in the following way: For $p \in [1, \infty)$, $\beta_p(X)$ is the infimum of the set of numbers $\lambda$ such that 

$$
(||x||^p + ||y||^p)^{1/p} \leq \lambda ||x + y||
$$

for every $x, y \in X$ which verify $\text{supp}(x) < \text{supp}(y)$, that is, $\max[\text{supp}(x)] < \min[\text{supp}(y)]$.

For example, $\beta_p(l_p) = 1$ $(1 \leq p < \infty)$, and $\beta_2(J) = 1$, where $J$ is the James space which consists of all sequences $x = (x_n) \in c_0$ such that 

$$
||x|| := \sup\{(x_{p_1} - x_{p_2})^2 + \ldots + (x_{p_{n-1}} - x_{p_n})^2\}^{1/2}
$$

is finite (the supremum is taken for every $n$ and for every finite increasing sequence of positive integers $(p_i)$).

Now, for such Banach spaces we can state the following

**Theorem 1.** Let $(X, \cdot, \cdot)$ be a Banach space with a FDD such that $\beta_p(X) = 1$ for some $p \in [1, \infty)$. Let $B_X, S_X$ be the closed unit ball and the unit sphere of $X$, respectively, and let $T : B_X \to X$ be a mapping such that

a) $T(S_X) \subset S_X$ and 

b) $\text{supp}(T(0)) < \text{supp}(v)$ for every $v \in T(S_X)$.

If $T$ is nonexpansive with respect to some equivalent norm on $X$ then $T$ has a fixed point. Specifically, $T(0) = 0$.

**Proof.** Since $\beta_p(X) = 1$, conditions a) and b) imply that 

$$
|T(0)|^p + |T(y)|^p \leq |T(0) - T(y)|^p
$$

for all $y \in S_X$, and then we have that 

$$
d = \inf_{y \in S_X} |T(0) - T(y)| \geq \left[|T(0)|^p + 1\right]^{1/p}.
$$

This inequality, when combined with Lemma 1, gives that 

$$
1 \geq \left[|T(0)|^p + 1\right]^{1/p}
$$

and consequently $T(0) = 0$. 

A similar result can be stated in the setting of Hilbert space if we replace condition b) by: $T(0)$ is orthogonal to the set $T(S_X)$. Under this assumption we would obtain that 

$$
||T(0)||_2^2 + ||T(y)||_2^2 = ||T(0) - T(y)||_2^2
$$

for every $y \in S_X$, from which the same conclusion follows.

**Theorem 2.** Let $(X, \cdot, \cdot_2)$ be a Hilbert space. Let $B_X, S_X$ be the closed unit ball and the unit sphere of $X$, respectively, and let $T : B_X \to X$ be a mapping such that

a) $T(S_X) \subset S_X$.

b) $T(0)$ is orthogonal to the set $T(S_X)$.

If $T$ is nonexpansive with respect to some equivalent norm on $X$ then $T$ has a fixed point. Specifically, $T(0) = 0$. 

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Suppose now that $T : B \to B$ is any of the mappings $K$, $G$ or $K_r$ which we refer to in the introduction. It is clear that $T$ verifies the conditions (a) and (b) of Theorem 2 and that $T(0) \neq 0$. This proves the following corollary.

**Corollary 1.** Let $T : B \to B$ be any of the mappings $K$, $G$ or $K_r$ and let $\| \cdot \|$ be any norm on $\ell_2$, equivalent to $\| \cdot \|_2$. Then $T$ is not $\| \cdot \|$-nonexpansive.

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**References**


