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# ISOMORPHICALLY EXPANSIVE MAPPINGS IN $l_2$

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Abstract. We show that for any renorming $\|\cdot\|$ of $\ell_2$, the well known fixed point free mappings by Kakutani, Baillon and others are not nonexpansive.

## 1. Introduction

Let $C$ be a subset of a Banach space $(X, \|\cdot\|)$. A mapping $T : C \to C$ is called *k-lipschitzian* $(k > 0)$ if $\| T(x) - T(y) \| \leq k \| x - y \|$ for all $x, y \in C$.

*Nonexpansive mappings* are those which have Lipschitz constant $k = 1$. A mapping $T : C \to C$ is said to be *uniformly k-lipschitzian* if every iteration $T^n$ is $k$-lipschitzian, i.e. if for every positive integer $n$, and $x, y \in C$

$$\| T^n(x) - T^n(y) \| \leq k \| x - y \| .$$

We say that $X$ has the fixed point property (FPP) if every nonexpansive mapping $T : C \to C$ defined on a nonempty convex and weakly compact subset $C$ of $X$ has a fixed point.

If $T : C \to C$ is a nonexpansive mapping with respect to some norm $|\cdot|$ on $X$ equivalent to the norm $\|\cdot\|$, then it is straightforward to see that $T$ is also uniformly lipschitzian with respect to the norm $\|\cdot\|$. Hence, in order to look for a fixed point free $|\cdot|$-nonexpansive mapping we must seek this mapping among those which are uniformly lipschitzian with respect to the norm $\|\cdot\|$.

It is known that bounded closed convex subsets of $\ell_2$ have the fixed point property for nonexpansive mappings, but it is unknown whether the same is true for bounded closed convex subsets of a Banach space $X$ where $X$ is $\ell_2$ with an equivalent norm. In this paper it is shown that certain uniformly lipschitzian mappings which are known to be fixed point free in the unit ball of $\ell_2$ are also not nonexpansive relative to any renorming of $\ell_2$. This shows that one strategy for answering the above question is precluded. At the same time it shows that the class of mappings which are uniformly lipschitzian on $\ell_2$ is strictly larger than the class of mappings which are nonexpansive with respect to some equivalent norm on $\ell_2$. Theorems 1 and 2 serve mainly as vehicles for reaching this conclusion.

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## 2. Preliminaries

We recall now some famous examples of uniformly lipschitzian mappings in $\ell_2$.

Let $\|\cdot\|_2$ be the usual euclidean norm on the sequence Hilbert space $\ell_2$, let $B$ be the closed $\|\cdot\|_2$-unit ball and let $S : \ell_2 \to \ell_2$ be the right shift operator defined by

$$S(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$

Then the mapping $K : B \to B$ defined by

$$K(x) = \frac{(1 - \|x\|_2)e + S(x)}{\| (1 - \|x\|_2)e + S(x) \|_2},$$

where $e = (1, 0, \dots)$, is *uniformly*-2-lipschitzian and has no fixed point in $B$ ([GKT]).

Later on, Baillon ([B]) found another example of a uniformly $\frac{\pi}{2}$-lipschitzian mapping without fixed point in $B$. This is the mapping $G : B \to B$ defined by

$$G(x) = \begin{cases} \cos\left(\frac{\pi}{2}\|x\|_2\right) e + \sin\left(\frac{\pi}{2}\|x\|_2\right) \frac{S(x)}{\|x\|_2}, & x \neq 0, \\ e, & x = 0 \end{cases}$$

(for details about this map, see [T]).

Both examples are modifications of an earlier one due to Kakutani (see [GK]): for $0 < r < 1$, define $K_r : B \to B$ by

$$K_r(x) = r(1 - \|x\|_2)e + S(x).$$

Then $K_r$ have the (non uniform) Lipschitz constant $\sqrt{1 + r^2}$ and has no fixed point in $B$.

## 3. The results

The announced results will follow easily from the following lemma.

**Lemma 1.** *Let $(X, |\cdot|)$ be a Banach space and let $B_X, S_X$ be the closed unit ball and the unit sphere of $X$ respectively. Suppose that $T : B_X \to X$ is a mapping which is nonexpansive with respect to some norm $\|\cdot\|$ on $X$, equivalent to $|\cdot|$. Then we have that*

$$d = \inf_{y \in S_X} |T(0) - T(y)| \leq 1\,.$$

*Proof.* There exist positive constants $\alpha, \beta$ such that, for every $v \in X$,

$$\alpha|v| \leq \|v\| \leq \beta|v|.$$

Since $T$ is $\|\cdot\|$-nonexpansive, for any $y \in S_X$ we have that

$$\alpha|T(0) - T(y)| \leq \|T(0) - T(y)\| \leq \|y\| \leq \beta$$

and then

$$\alpha d \leq \|y\| \leq \beta$$

for all $y \in S_X$. Thus, for every $v \in X$ we have that

$$\alpha d|v| \leq \|v\| \leq \beta|v|.$$

Indeed, an induction argument shows that

$$\alpha d^n|v| \leq \|v\| \leq \beta|v|$$

for every positive integer $n$ and every $v \in X$, which is impossible unless $d \leq 1$. □

The above lemma will allow us to give a fixed point result. Recall that a sequence $(X_n)$ of finite dimensional subspaces of a Banach space $X$ is called a *Schauder finite dimensional decomposition* (FDD) of $X$, if every $x \in X$ has a unique representation of the form $x = \sum x_i$ with $x_i \in X_i$ for every $i \in \mathbb{N}$. If $X$ is a Banach space with a FDD $(X_n)$, and $x \in X$, $supp(x)$ denotes, as usual, the set of positive integers $i$ such that $x_i \neq 0$. M. A. Khamsi ([K]) defined, for a Banach space $X$ with a FDD, the coefficient $\beta_p(X)$ in the following way: For $p \in [1, \infty)$, $\beta_p(X)$ is the infimum of the set of numbers $\lambda$ such that

$$(\|x\|^p + \|y\|^p)^{1/p} \leq \lambda \|x + y\|$$

for every $x, y \in X$ which verify $supp(x) < supp(y)$, that is, $\max[supp(x)] < \min[supp(y)]$.

For example, $\beta_p(l_p) = 1$ $(1 \leq p < \infty)$, and $\beta_2(J) = 1$, where $J$ is the James space which consists of all sequences $x = (x_n) \in c_0$ such that

$$\|x\| := \sup\{(x_{p_1} - x_{p_2})^2 + \ldots + (x_{p_{n-1}} - x_{p_n})^2\}^{1/2}$$

is finite (the supremum is taken for every $n$ and for every finite increasing sequence of positive integers $(p_i)$).

Now, for such Banach spaces we can state the following

**Theorem 1.** *Let $(X, |\cdot|)$ be a Banach space with a FDD such that $\beta_p(X) = 1$ for some $p \in [1, \infty)$. Let $B_X, S_X$ be the closed unit ball and the unit sphere of $X$, respectively, and let $T : B_X \to X$ be a mapping such that*

a) *$T(S_X) \subset S_X$ and*

b) *$supp[T(0)] < supp(v)$ for every $v \in T(S_X)$.*

*If $T$ is nonexpansive with respect to some equivalent norm on $X$ then $T$ has a fixed point. Specifically, $T(0) = 0$.*

*Proof.* Since $\beta_p(X) = 1$, conditions a) and b) imply that

$$|T(0)|^p + |T(y)|^p \leq |T(0) - T(y)|^p$$

for all $y \in S_X$, and then we have that

$$d = \inf_{y \in S_X} |T(0) - T(y)| \geq \left[|T(0)|^p + 1\right]^{1/p}.$$

This inequality, when combined with Lemma 1, gives that

$$1 \geq \left[|T(0)|^p + 1\right]^{1/p}$$

and consequently $T(0) = 0$. □

A similar result can be stated in the setting of Hilbert space if we replace condition b) by: $T(0)$ is orthogonal to the set $T(S_X)$. Under this assumption we would obtain that

$$\|T(0)\|_2^2 + \|T(y)\|_2^2 = \|T(0) - T(y)\|_2^2$$

for every $y \in S_X$, from which the same conclusion follows.

**Theorem 2.** *Let $(X, \|\cdot\|_2)$ be a Hilbert space. Let $B_X, S_X$ be the closed unit ball and the unit sphere of $X$, respectively, and let $T : B_X \to X$ be a mapping such that*

a) *$T(S_X) \subset S_X$.*

b) *$T(0)$ is orthogonal to the set $T(S_X)$.*

*If $T$ is nonexpansive with respect to some equivalent norm on $X$ then $T$ has a fixed point. Specifically, $T(0) = 0$.*

Suppose now that $T : B \to B$ is any of the mappings $K$, $G$ or $K_r$ which we refer to in the introduction. It is clear that $T$ verifies the conditions (a) and (b) of Theorem 2 and that $T(0) \neq 0$. This proves the following corollary.

**Corollary 1.** *Let $T : B \to B$ be any of the mappings $K$, $G$ or $K_r$ and let $\| \cdot \|$ be any norm on $\ell_2$, equivalent to $\| \cdot \|_2$. Then $T$ is not $\| \cdot \|$-nonexpansive.*

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