

CELLULAR FILTRATION OF K-THEORY AND DETERMINANTS OF C^* -ALGEBRAS

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ABSTRACT. In this note, we will disprove the following conjecture raised by Exel–Loring: Let A be a C^* -algebra with trace τ and let $\det : U_\infty \rightarrow \mathbb{R}/\tau_*(K_0(A))$ be a determinant associated to τ . If $\phi_t : C(S^3) \rightarrow A$ ($0 \leq t \leq 1$) is a continuous family of homomorphisms and $b \in C(S^3) \otimes M_2$ is the canonical matrix valued function on S^3 which represents the Bott element in $K_1(C(S^3))$, then $\det(\phi_0(b)) = \det(\phi_1(b))$. It should be noticed that the conjecture has been proved by Exel–Loring for the case that ϕ_t is a smooth family of homomorphisms.

Let b_n , or simply b , denote the element of $K_*(C(S^n))$ ($n \geq 1$) which has the Chern character equal to the fundamental class (i.e., the canonical generator of $H^n(S^n)$). We call b the Bott element. In [ExL], Exel–Loring tried to extend the concept of cohomology of topological spaces (commutative C^* -algebras) to general C^* -algebras and define a filtration of $K_*(A)$ of any C^* -algebra A as the following:

Definition 1. Let A be a unital C^* -algebra. The spherical filtration $F_*K_*(A)$ of $K_*(A)$ is defined by

$$\begin{aligned} F_0K_0(A) &= K_0(A) , \\ F_1K_1(A) &= K_1(A) , \\ F_nK_n(A) &= \{x \in K_n(A); x = \phi_*(b) \text{ for some } \phi \in \text{Hom}(C(S^n), M_k \otimes A)\} . \end{aligned}$$

Exel–Loring proved that the cohomology $H^n(X) \otimes \mathbb{Q}$ with rational coefficient of finite CW complex X can be recovered from the filtration of $K_*(C(X))$ by

$$H^n(X) \otimes \mathbb{Q} = \frac{(F_nK_*(C(X))) \otimes \mathbb{Q}}{(F_{n+2}K_*(C(X))) \otimes \mathbb{Q}} .$$

Also, they defined

$$H_n(A) = \frac{F_nK_*(A)}{F_{n+2}K_*(A)} .$$

It is easy to see that, for any C^* -algebra A ,

$$F_2K_*(A) \subset \bigcap_{\tau \in TA} \text{Ker } \tau_*$$

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where TA is the set of traces on A , and the map $\tau_* : K_0(A) \rightarrow \mathbb{R}$ is induced by τ . In [EL], it is proved that for many AF algebras A without finite quotient,

$$F_2K_*(A) = \bigcap_{\tau \in TA} \text{Ker } \tau_*.$$

It can also be proved, by the the strategy used in the present paper, that for many AF algebras and other real rank zero C^* -algebras A ,

$$F_2K_*(A) = F_{2n}K_*(A) \quad \text{for } n \geq 1.$$

It seems that, in non-commutative C^* -algebras, the only efficient tool to calculate the filtration $F_{2n}K_0(A)$ of $K_0(A)$ is the traces. A natural problem is to find some analogue tool for computing the filtration $F_{2n+1}K_1(A)$ of $K_1(A)$. Motivated by this problem, Exel and Loring were trying to consider the determinants of C^* -algebras as a tool and relate homomorphisms to determinants. In order for the determinants to be an efficient tool, they proposed some conjectures. And based on these conjectures, they proved several theorems. We briefly review some definitions and results in [ExL].

We use $U_k(A)$ to denote the set of unitaries in $M_k(A)$ and denote $U_\infty(A) \triangleq \bigcup_{k=1}^\infty U_k(A)$ by embedding

$$a \rightarrow \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

Considering S^3 as the unit sphere in \mathbb{C}^2 , let $b \in U_2(C(S^3))$ be defined by

$$b(m, n) = \begin{pmatrix} m & -\bar{n} \\ n & \bar{m} \end{pmatrix} \in U_2(C(S^3)) \quad \text{for } (m, n) \in S^3.$$

It is known that b represents the generator in $K_1(C(S^3))$ (i.e., the Bott element).

Let A be a C^* -algebra with a trace τ , and the map $\tau_* : K_0(A) \rightarrow \mathbb{R}$ be induced by τ . Let

$$q : \mathbb{R} \longrightarrow \mathbb{R}/\tau_*(K_0(A))$$

denote the canonical quotient map.

Definition 2. We say that a group homomorphism

$$\det : U_\infty(A) \longrightarrow \mathbb{R}/\tau_*(K_0(A))$$

is a determinant function associated to the trace τ on the C^* -algebra A if

$$\det(e^{2\pi i h}) = q(\tau(h)), \quad h \in M_n(A).$$

We will also say that \det is a determinant associated to (A, τ) .

A determinant associated to a trace is not unique, except on the connected component of the identity. However, determinants always exist, and any two determinants are related in a simple way (see §7 of [Ex]).

In [ExL], several conjectures are proposed. One of them is the following:

Conjecture 1 (§7.5 in [ExL]). Suppose that \det is a determinant associated to (A, τ) . If $\phi : C(S^3) \rightarrow A$ is a homomorphism, then $\det(\phi(b))$ depends only on $\phi_*(b) \in K_1(A)$. In particular, if $\phi_*(b) = 0$, then $\det(\phi(b)) = 0$.

In [ExL], this conjecture was proved for the case of $A = M_k(C(X))$. (See 7.4 of [ExL].)

In [ExL], the following conjecture was also proposed which is weaker than the above one.

Conjecture 2 (§7.6 in [ExL]). Suppose that \det is a determinant associated to (A, τ) . If $\phi_t : C(S^3) \rightarrow A$ is a continuous path of homomorphisms, then $\det \phi_0(b) = \det(\phi_1(b))$.

This conjecture was proved for the case of $\phi_t : C(S^3) \rightarrow A$ being a smooth path of homomorphisms (i.e., the two maps $t \mapsto \phi_t(m), t \mapsto \phi_t(n)$ are smooth) in [ExL].

Applying Conjecture 2, Exel and Loring proved several results about cellular cohomology.

In this note, we will construct a counterexample for Conjecture 2, and therefore disprove both conjectures. The result reveals some significant differences between non commutative topology and commutative topology—it is known that a continuous homotopy of two maps from a topological space X to a smooth manifold M can always be replaced by a smooth path of homotopy of these two maps, i.e., by a homotopy $F_t : X \times [0, 1] \rightarrow M$ which is smooth with respect to the parameter t . The proof of our result involves some deep classification theory of C^* -algebras of Elliott in [E].

Our counterexample is a Bunce-Dedden algebra constructed as follows. Set

$$D = \{e^{2\pi \frac{k}{2^n} i}, k, n \text{ are arbitrary integers}\}.$$

The D is a subgroup of $S^1 \subset \mathbb{C}$. Let α be the action of D on $C(S^1)$ by rotation, i.e.,

$$\alpha_{e^{2\pi \frac{k}{2^n} i}}(f)(z) = f(e^{2\pi \frac{k}{2^n} i} \cdot z)$$

for any $f \in C(S^1)$ and $z \in S^1$. Bunce-Dedden algebra BD is defined by

$$BD = C(S^1) \rtimes_{\alpha} D.$$

It is known (see [BBEK], for example) that BD is an inductive limit of matrix algebras over $C(S^1)$, and BD is also of real rank zero (i.e., the set of invertible selfadjoint elements is dense in the set of selfadjoint elements). Furthermore,

$$K_0(BD) = \mathbb{Z}(\frac{1}{2}), \quad K_1(BD) = \mathbb{Z},$$

where

$$\mathbb{Z}(\frac{1}{2}) = \{\frac{k}{2^n}; k, n \text{ are integers}\} \subset \mathbb{R}.$$

For our purpose, we need to introduce the positive cone $K_*(A)_+$ of $K_*(A)$ (see [E]) and the scale $\mathbf{1}_A \in K_*(A)$ for a C^* -algebra A . First, $\mathbf{1}_A$ is the element in $K_0(A) \subset K_*(A)$ corresponding to the unit of A . And $K_*(A)_+$ is defined by

$$K_*(A)_+ = \{([p], [u + (1 - p)]) \in K_0(A) \oplus K_1(A); p, u \in M_k(A), uu^* = u^*u = p\}.$$

It is also known that $K_*(BD)_+ \setminus \{0\} = (\mathbb{Z}(\frac{1}{2})_+ \setminus \{0\}) \oplus \mathbb{Z}$ and $\mathbf{1}_{BD} = (1, 0) \in K_0(A) \oplus K_1(A)$.

The following two theorems prove that BD is a counterexample of Conjecture 2.

Theorem 1. $F_3K_1(BD) = \mathbb{Z}$.

Theorem 2. *If Conjecture 2 is true for the C^* -algebra BD , then $F_3K_1(BD) = 0$.*

In the proof of Theorem 1, we need the following lemma.

Lemma 1. *Suppose that $\alpha : K_*(A) \otimes K_*(B) \rightarrow K_*(A \otimes B)$ is the map defined in the Kunneth formula. If $x \in F_kK_*(A)$ and $y \in F_lK_*(B)$, then $\alpha(x \otimes y) \in F_{k+l}K_*(A \otimes B)$.*

Proof. By Definition 1, there are maps $\phi : C(S^k) \rightarrow M_a(A)$ and $\psi : C(S^l) \rightarrow M_b(B)$ such that

$$\phi_*b_k = x \quad \text{and} \quad \psi_*b_l = y .$$

We can identify $K_*(C(S^k)) \otimes K_*(C(S^l))$ with $K_*(C(S^k) \otimes C(S^l))$ via the Kunneth formula. Hence

$$b_k \otimes b_l \in K_*(C(S^k) \otimes C(S^l)) = K_*(C(S^k \times S^l)) .$$

Also, it is well known, from the definition of tensor product (or cup product) of cohomologies of topological spaces, that $b_k \otimes b_l$ is in $H^{k+l}(S^k \times S^l)$, via the Chern map. That is,

$$b_k \otimes b_l \in F_{k+l}K_*(C(S^k) \otimes C(S^l)) .$$

Consider the map $\phi \otimes \psi : C(S^k \times S^l) \rightarrow M_{ab}(A \otimes B)$. Since the exact sequence of the Kunneth formula is natural,

$$(\phi \otimes \psi)_*(b_k \otimes b_l) = \alpha(x \otimes y) .$$

This proves

$$\alpha(x \otimes y) \in F_{k+l}K_*(A \otimes B) .$$

□

Proof of Theorem 1. Let A be the AF algebra with $K_0(A) = \mathbb{Z}(\frac{1}{2}) \oplus \mathbb{Z}$, $K_0(A)_+ \setminus \{0\} = (\mathbb{Z}(\frac{1}{2})_+ \setminus \{0\}) \oplus \mathbb{Z}$, and $\mathbf{1}_A = (1, 0) \in K_0(A)$. Then by [EL], there is a homomorphism $\psi : C(T^2) \rightarrow A$ such that $\psi_*(x) = (0, 1) \in \mathbb{Z}(\frac{1}{2}) \oplus \mathbb{Z} = K_0(A)$, where $x \in K_0(C(T^2))$ is the generator of $H^2(T^2)$. This proves that $F_2(K_0(A)) = \mathbb{Z}$. Considering $A \otimes BD$, we know that, by the Kunneth formula,

$$K_0(A \otimes BD) = \mathbb{Z}(\frac{1}{2}) \oplus \mathbb{Z}(\frac{1}{2}), \quad K_1(A \otimes BD) = \mathbb{Z}(\frac{1}{2}) \oplus \mathbb{Z} .$$

Furthermore, by Lemma 1, the \mathbb{Z} summand of $K_1(A \otimes BD)$ is in $F_3K_1(A \otimes BD)$ (notice that it comes from the tensor product of $\mathbb{Z} = F_2K_0(A)$ and $\mathbb{Z} = K_1(BD)$). We also have

$$K_*(A \otimes BD)_+ \setminus \{0\} = ((\mathbb{Z}(\frac{1}{2})_+ \setminus \{0\}) \oplus \mathbb{Z}(\frac{1}{2})) \oplus (\mathbb{Z}(\frac{1}{2}) \oplus \mathbb{Z})$$

and

$$\mathbf{1}_{A \otimes BD} = (1, 0, 0, 0) \in K_0(A \otimes BD) \oplus K_1(A \otimes BD) .$$

Let $\beta : K_*(A \otimes BD) \rightarrow K_*(BD)$ be defined by

$$\beta(a, b, c, d) = (a, d) \in K_0(BD) \oplus K_1(BD) = K_*(BD) .$$

Then β is a graded homomorphism which preserves the order and the scale. By the Elliott classification theory (see 7.3 of [El]), there is a homomorphism $\phi : A \otimes BD \rightarrow BD$ such that $\phi_* = \beta$. Therefore, we have proved that

$$\mathbb{Z} \subset \phi_* F_3 K_1(A \otimes BD) \subset F_3 K_1(BD) = \mathbb{Z} .$$

This ends the proof of Theorem 1. □

Proof of Theorem 2. The proof is similar to that of Proposition 7.8 in [ExL]. Let u be the canonical unitary generator of $C(S^1) \subset C(S^1) \rtimes_{\alpha} D$, and γ be the action on BD defined by

$$\gamma(u) = e^{2\pi i \sqrt{2}} u; \quad \gamma(g) = g \quad \text{for } g \in D.$$

Since γ is homotopic to the identity homomorphism, one has that $\gamma_* = \text{id}$ on $K_1(BD)$. Let τ be the canonical trace on BD , i.e.,

$$\tau\left(\sum_{d \in D} f_d[d]\right) = \int_{S^1} f_0 d\theta,$$

for any $\sum_{d \in D} f_d[d] \in C(S^1) \rtimes_{\alpha} D$. Define the rotation number map

$$\rho_{\gamma}^{\tau} : K_1(BD) \longrightarrow \mathbb{R}/\tau_*(K_0(BD))$$

of γ by

$$\rho_{\gamma}^{\tau}([v]) = \det(\gamma(v^*)v)$$

(see [ExL] and [Ex] for details about ρ_{γ}^{τ} .) It is easy to see that

$$\rho_{\gamma}^{\tau}([u]) = \det(\gamma(u^*)u) = \sqrt{2} \notin \mathbb{Z}\left(\frac{1}{2}\right) = \tau_*(K_0(BD))$$

for the generator $[u] \in K_1(BD)$. Hence $\rho_{\gamma}^{\tau} : K_1(BD) \rightarrow \mathbb{R}/\tau_*(K_0(BD))$ is injective.

Suppose that we are given $\phi : C(S^3) \rightarrow M_k \otimes BD$. Then ϕ and $\gamma \circ \phi$ are homotopic. So by the conjecture,

$$\rho_{\gamma}^{\tau}([\phi(b)]) = -\det(\gamma \circ \phi(b)) + \det(\phi(b)) = 0.$$

Hence $\phi_*(b) = 0$ in $K_1(BD)$. □

Combining Theorem 1 and Theorem 2, we disproved Conjecture 2.

Inspired by this paper, the second counterexample of the conjecture was discovered by Elliott-Gong [EG]. Their example involves a deep structure theory of the irrational rotation algebras, and the reduction of inductive limits of matrix algebras over $C(T^2)$ to that over $C(S^1)$.

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