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TYPE I C^*-ALGEBRAS OF REAL RANK ZERO

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ABSTRACT. We show that a separable C^*-algebra A of type I has real rank zero if and only if $d(\hat{A}) = 0$, where d is a modified dimension. We also show that a separable C^*-algebra of type I has real rank zero if and only if it is an AF-algebra.

1. Recently, there have been remarkable developments in the theory of classification of separable nuclear C^*-algebras (see [E3] for a survey). For example, direct limits of (sub)homogeneous C^*-algebras of real rank zero have been classified ([EG] and [DG]). All (sub)homogeneous C^*-algebras are of type I. Some efforts are made to study direct limits of type I C^*-algebras of real rank zero ([LS]). This leads to the question when a separable C^*-algebra of type I has real rank zero and how to classify them. It turns out the question can be fairly easily answered. We show in this short note that a separable C^*-algebra of type I has real rank zero if and only if it is an AF-algebra. So they can be classified by their dimension groups ([E1]).

2. It is shown ([BP]) that an abelian C^*-algebra $A = C_0(X)$, where X is a locally compact Hausdorff space, has real rank zero if and only if $dim(X) = 0$. A natural question is whether it can be generalized to type I C^*-algebras, i.e., whether a separable C^*-algebra of type I has real rank zero if and only if $dim(\hat{A}) = 0$. It turns out that the usual definition for zero dimension does not work very well when the space is not Hausdorff. Let A be a unital C^*-algebra generated by $\{\mathcal{K}, 1, p\}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on an infinite dimensional separable Hilbert space, 1 is the identity operator and, both $1-p$ and p are infinite dimensional projections. Clearly, A is an extension of $\mathbb{C} \oplus \mathbb{C}$ by $\mathcal{K}$ and A is a type I C^*-algebra with real rank zero. Its spectrum $\hat{A} = \{x_1, x_2, x_3\}$ consists of only three points and its open subsets $\mathcal{T} = \{\emptyset, \{x_3\}, \{x_1, x_3\}, \{x_2, x_3\}, \hat{A}\}$, where x_1 is the primitive ideal generated by $\mathcal{K}$ and p, x_2 is the primitive ideal generated by $\mathcal{K}$ and $1-p$ and x_3 is the zero ideal. However, by the usual definition, $(X, \mathcal{T})$ is not of dimension zero. It does not have clopen base. The problem is that closure of $\{x_3\}$ is the whole $\hat{A}$. As far as we are concerned, at least in this note, we do not believe that a space which contains only finitely many points should have "dimension" other than zero.

We now introduce a new concept of "dimension" d. For any topological space Y, let

$$Y_0 = \{y \in Y : y \notin \overline{\{z\}} \text{ for } \text{ any } z \in Y, z \neq y\},$$

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where $\overline{\{z\}}$ is the closure of $\{z\}$. For any ordinal β, suppose Y_α is defined for any $\alpha < \beta$; we define

$$Y_\beta = \{y \in Y \setminus \cup_{\alpha<\beta} Y_\alpha : y \notin \overline{\{z\}} \quad \text{for any} \quad z \in Y \setminus \cup_{\alpha<\beta} Y_\alpha, z \neq y\}.$$

We say $d(Y) = n$ if $dim(Y_\alpha) \leq n$ for all α and for some α, $dim(Y_\alpha) = n$, where $dim(X)$ is the covering dimension of X. Here Y_α is equipped with the relative topology. It is clear that $d(Y) = dim(Y)$ if Y is Hausdorff. In this note we are only interested in the case that $d(Y) = 0$. We write $d(Y) = 0$, if $dim(Y_\alpha) = 0$ for all ordinals α. Here $dim(Y_\alpha) = 0$ means that Y_α has a clopen base. We will show that a separable C^*-algebra A of type I has real rank zero if and only if $d(\hat{A}) = 0$.

3. For the reader's convenience, before we go any further, we would like to remind the reader of several definitions. A C^*-algebra A is of type I, if every irreducible representation (H, π) of A contains $\mathcal{K}(H)$, the compact operators on H. For type I C^*-algebra A, its primitive ideals space is the same as its equivalence classes of irreducible representations which will be denoted by $\hat{A}$. We will refer the reader to section 4.1 of [P] for the notation *hull* and basic facts about the hull-kernel topology on $\hat{A}$.

A C^*-algebra has real rank zero if invertible selfadjoint elements are dense in the set of selfadjoint elements. In particular, every AF-algebra has real rank zero. A positive element x in a type I C^*-algebra A has a continuous trace, if $Tr(\pi(x))$ $(\pi \in \hat{A})$ is (finite and) continuous on $\hat{A}$. A type I C^*-algebra is said to have continuous trace if the set of elements with continuous trace is dense in A_+. The proof of the main result uses the following facts:

(i) Given an extension $0 \to J \to A \to A/J \to 0$, then A is AF, if both J and A/J are AF ([B]).

(ii) Every hereditary C^*-subalgebra of a C^*-algebra of real rank zero has real rank zero ([BP]).

(iii) A result from [P] (Theorem 6.2.11) which yields an essential composition series $\{I_\alpha\}_{0 \leq \alpha \leq \beta}$ for any type I C^*-algebra A such that $I_{\alpha+1}/I_\alpha$ has continuous trace for each ordinal $\alpha < \beta$.

We then reduce the general case to the case that A has continuous trace.

4. Lemma. *Let A be a C^*-algebra of type I. Then*

$$\hat{A}_0 = \hat{A} \setminus hull(I_0),$$

where I_0 is the maximal essential ideal of A with continuous trace.

Proof. Let I_0 be the maximal essential ideal of A which has continuous trace. Note $I_0 \neq \{0\}$ (see [P, 6.2.11]). Since $\hat{A} \setminus hull(I_0)$ is homeomorphic to $\hat{I}_0$ and $\hat{I}_0$ is Hausdorff, we have

$$\hat{A} \setminus hull(I_0) \subset \hat{A}_0.$$

Fix $\zeta \in \hat{A}_0$. Let (π, H) be an irreducible representation of I_0 and let (ρ, K) be the irreducible representation of A induced by π. Since I_0 is an ideal, $H = K$. Suppose that $a \in ker(\rho)$. Then for any $i \in I_0$, ai, $ia \in Ker(\pi)$. Let $\phi : A \to A/Ker(\pi)$ be the quotient map. Then $\phi(ai) = \phi(ia) = 0$. Since $\phi(I_0)$ is an essential ideal of $\phi(A)$, $\phi(a) = 0$. Therefore $Ker(\rho) = Ker(\pi)$. Thus ζ does not contain $Ker(\pi)$. In particular, $I_0 \not\subset \zeta$. This implies that $\zeta \in \hat{A} \setminus hull(I_0)$. Therefore

$$\hat{A} \setminus hull(I_0) = \hat{A}_0. \quad \square$$

Let A be a type I C^*-algebra and let $J_0 = I_0$ and $K_0 = 0$. We define I_α, J_α and K_α as follows. Suppose that I_α, J_α and K_α have been defined for all ordinals $\alpha < \beta$. Set $K_\beta = cl(\cup_{\alpha<\beta} I_\alpha)$. Note that A/K_β is type I. Let J_β be the maximal essential ideal of A/K_β with continuous trace and let I_β the preimage of J_β in A. One of the purposes of the following lemma is to produce a certain composition series which is going to be used in the proof of Theorem 8.

5. Lemma. *Let A be a C^*-algebra of type I. Then*

$$\hat{A}_\alpha \cong \hat{J}_\alpha \cong (A/K_\alpha)\hat{} \setminus hull(J_\alpha) = ((A/K_\alpha)\hat{})_0$$

and

$$\hat{A} \setminus \hat{A}_\alpha = hull(I_\alpha),$$

where "$\cong$" means homeomorphic.

Proof. From the homeomorphism between $\hat{A} \setminus hull(I_0)$ and $\hat{I}_0$, we see that, by Lemma 4, the lemma holds for $\alpha = 0$. Let β be an ordinal number. Suppose that the lemma holds for any ordinal $\alpha < \beta$. Set $K_\beta = cl(\cup_{\alpha<\beta} I_\alpha)$. Let J_β be the maximal essential ideal of A/K_β with continuous trace and let I_β be the preimage of J_β.

Suppose that $\zeta \in \hat{A}_\beta$. Then

$$\begin{aligned}\zeta \in \hat{A} \setminus \cup_{\alpha<\beta} \hat{A}_\alpha &= \cap_{\alpha<\beta} (\hat{A} \setminus \hat{A}_\alpha) \\ &= \cap_{\alpha<\beta} hull(I_\alpha) = hull(K_\beta).\end{aligned}$$

The last equality follows from the definition of K_β. Note that $(A/K_\beta)\hat{} \cong hull(K_\beta)$. From the definition of $\hat{A}_\beta$ and by applying the same argument in Lemma 4, we see that $\zeta \in (A/K_\beta)\hat{} \setminus hull(J_\beta) = ((A/K_\beta)\hat{})_0$.

Conversely,

$$\begin{aligned}((A/K_\beta)\hat{})_0 &= (A/K_\beta)\hat{} \setminus hull(J_\beta) \\ &\cong hull(K_\beta) \setminus hull(I_\beta) = (\hat{A} \setminus \cup_{\alpha<\beta} \hat{A}_\alpha) \setminus hull(I_\beta).\end{aligned}$$

The last subset is open in $\hat{A} \setminus \cup_{\alpha<\beta} \hat{A}_\alpha$ and is Hausdorff, since the first one is (see Lemma 4). Therefore, it must be a subset of $\hat{A}_\beta$ which is a subset of $\hat{A} \setminus \cup_{\alpha<\beta} \hat{A}_\alpha$. This implies that

$$\hat{A}_\beta = ((A/K_\beta)\hat{})_0.$$

So, by Lemma 4,

$$\hat{A} \setminus \hat{A}_\beta \cong hull(J_\beta).$$

This ends the proof. □

6. Lemma. *Let A be a separable C^*-algebra which has continuous trace. Suppose that $dim(\hat{A}) = 0$. Then A is an AF-algebra.*

Proof. We first recall that $\hat{A}$ is Hausdorff and A is a continuous field of elementary C^*-algebras.

Consider $A \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on l^2. Clearly, $A \otimes M_n$ has continuous trace, where M_n is the C^*-algebra of $n \times n$ matrices (over $\mathbb{C}$). Since $A \otimes M_n$ is dense in $A \otimes \mathcal{K}$, $A \otimes \mathcal{K}$ has continuous trace. We also have $(A \otimes \mathcal{K})\hat{} = \hat{A}$ and every irreducible reperesentation has dimension $\aleph_0$. So $A \otimes \mathcal{K}$ is a separable C^*-algebra with continuous trace, homogeneous of rank $\aleph_0$. Note also, since $\hat{A}$ has dimension zero, $H^3(\hat{A}, \mathbf{Z}) = \{0\}$. It follows from Corollary 10.9.6 in [Dix] that $A \otimes \mathcal{K} \cong C_0(\hat{A}) \otimes \mathcal{K}$. Since $dim(\hat{A}) = 0$, $A \otimes \mathcal{K}$ is an AF-algebra. We conclude that A is itself AF. $\square$

7. *Remark.* One can prove Lemma 8 without introducing $H^3(T, \mathbf{Z})$. But the above proof is shorter.

8. Theorem. *Let A be a separable C^*-algebra of type I. Then A has real rank zero if and only if $d(\hat{A}) = 0$.*

Proof. We first show the theorem holds for the case that A has continuous trace. In this special case, $\hat{A}$ is Hausdorff.

Let $\mathcal{A}$ be the continuous field of nonzero elementary C^*-algebras defined by A. Then A satisfies the Fell's condition (see [Dix, 10.5.8], i.e., for any $t_0 \in \hat{A}$, there are a neighborhood $U(t_0)$ and a vector field p of $\mathcal{A}$, defined and continuous in $U(t_0)$ such that $p(t)$ is a projection of rank 1, for every $t \in U(t_0)$. Since $\hat{A}$ is locally compact and Hausdorff, we may assume that there is a neighborhood $V(t_0)$ such that $V(t_0) \subset \overline{V(t_0)} \subset U(t_0)$, where $\overline{V(t_0)}$ is compact. There is a non-negative function $f \in C_0(\hat{A})$ such that $f(t) = 1$ when $t \in \overline{V(t_0)}$ and $f(t) = 0$ if $t \notin U(t_0)$. So there is $x \in A$ such that $x = f \cdot p$. Suppose that I is the ideal of A defined by elements in $\mathcal{A}$ which vanish in $\overline{V(t_0)}$. If A has real rank zero, then so is A/I ([BP]). Furthermore, $\pi(x)(A/I)\pi(x)$ has real rank zero. Clearly $\pi(x)A/I\pi(x) \cong C(\overline{V(t_0)})$. Therefore, the Hausdorff space $\overline{V(t_0)}$ has dimension zero. In particular, $V(t_0)$ has a clopen base. Since this is true for every $t \in \hat{A}$, we conclude that $dim(\hat{A}) = 0$.

The converse follows from Lemma 6. Moreover, in this case, A is an AF-algebra.

Now for the general case, by Lemma 5, A has an essential composition series $\{I_\alpha | 0 \le \alpha \le \beta\}$ such that $I_{\alpha+1}/I_\alpha$ has continuous trace for each $\alpha < \beta$ and $(I_{\alpha+1}/I_\alpha)\hat{} = \hat{A}_{\alpha+1}$.

If A has real rank zero, then by [BP] , each J_α has real rank zero. Thus, by Lemma 5 and the above, $dim(\hat{A}_\alpha) = dim(\hat{J}_\alpha) = 0$. Therefore $d(\hat{A}) = 0$.

If $d(\hat{A}) = 0$, then $dim(\hat{J}_\alpha) = 0$ for each α. From the above, each J_α is an AF-algebra. We will show that A itself is an AF-algebra. We prove this by induction on β. Suppose that we have shown this for all ordinals $\alpha < \beta$. If β is not a limit ordinal, say $\beta = \alpha + 1$. By the assumption, I_α is an AF-algebra. We also have that $A/I_\alpha = J_\beta$ is an AF-algebra. By [B] , A is an AF-algebra. If β is a limit ordinal, A is the norm closure $\cup_{\alpha<\beta} I_\alpha$, where each I_α is an AF-algebra. Since we assume that each I_α is an AF-algebra, we conclude that A is an AF-algebra. $\square$

9. Corollary. *A separable C^*-algebra of type I has real rank zero if and only if it is an AF-algebra.*

10. There is also a way to "Hausdorffize" $\hat{A}$. For the convenience, we will assume that A is unital. Note that if A is type I, so is $\tilde{A}$, the unitization of A. Furthermore, A has real rank zero if and only if $\tilde{A}$ is. Now suppose that A is a unital type I C^*-algebra . Then there is a compact Hausdorff space $H(\hat{A})$ such that $C(\hat{A}) \cong C(H(\hat{A}))$ by Gelfand transform. The Hausdorff space is unique up to homeomorphism. Therefore one may use the dimension of $H(\hat{A})$ instead of the dimension of $\hat{A}$. In fact, a separable C^*-algebra A of type I has real rank zero if and only if $dim(H(\hat{A})) = 0$.

11. Theorem. *Let A be a unital separable C^*-algebra of type I. Then the following are equivalent.*

(a) *A has real rank zero,*
(b) *A is an AF-algebra,*
(c) $d(\hat{A}) = 0$,
(d) $dim(H(\hat{A})) = 0$,
(e) *The center of A has real rank zero,*
(f) *The center of A is an AF-algebra.*

Proof. We have established the equivalence of (a), (b) and (c).

Let

$$I_0 = \{f \in C(\hat{A}) : f(t) = 0, \quad t \notin \hat{A}_0\},$$

$$I_\alpha = \{f \in C(\hat{A}) : f(t) = 0, \text{ if } t \notin \hat{A} \setminus \cup_{\gamma \leq \alpha} \hat{A}_\gamma\}.$$

We have $I_{\alpha+1}/I_\alpha = C_0(\hat{A}_\alpha)$. So if (c) holds, $I_{\alpha+1}/I_\alpha$ is an AF-algebra. Using the same argument that we used earlier, by [B] , we see that $C(\hat{A})$ is an AF-algebra. Thus $C(H(\hat{A}))$ is an AF-algebra. Therefore, $dim(H(\hat{A})) = 0$. This proves (c) implies (d). Conversely, if $dim(H(\hat{A})) = 0$, then $C(H(\hat{A}))$ is an AF-algebra, whence $C(\hat{A})$ is an AF-algebra. Therefore $I_{\alpha+1}/I_\alpha$ is AF, which implies that $dim(\hat{A}_\alpha) = 0$. So $d(\hat{A}) = 0$. Therefore (c) and (d) are equivalent.

To include (e) and (f), we apply Dauns-Hofmann's theorem which says that the center of A is isomorphic to $C(\hat{A})$. □

Added in proof

After this paper was submitted, George Elliott kindly pointed to us that there is an overlap between this paper and [BE] which showed, among other things, that a separable type I C^*-algebra A is AF if and only if $\hat{A}$ has a base of compact open sets.

References

[BE] O. Bratteli and G. A. Elliott, *Structure spaces of approximately finite dimensional C^*-algebras, II*, J. Funct. Anal. **30** (1978), 74-82. MR **80d:**46111

[B] L. G. Brown, Extensions of AF algebras: The projection lifting problem, Proceedings Symposia in Pure Mathematics 38 (American Mathematical Society, Providence, 1982),

[BP] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99**(1991), 131-149. MR **92m:**46086

[Dix] J. Dixmier, *C^*-Algebras*, North-Holland, Amsterdam/New York/Oxford, 1977. MR **56:**16388

[DG] M. Dadarlat and G. Gong, *A classification result for approximately homogeneous C^*-algebras of real rank zero*, preprint.

[E1] G. A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1876), 29-44.

[E2] G. A. Elliott, *On the classification of C^*-algebras of real rank zero*, J. Reine Angew. Math. **443** (1993), 179-219. MR **94i:**46074

[E3] G. A. Elliott, *The classification problem for amenable C^*-algebras*, Proc. ICM '94, to appear. CMP 96:16

[EG] G. A. Elliott and G. Gong, *On the classification of C^*-algebras of real rank zero*, II, Ann. Math. **144** (1996), 497–610. CMP 97:06

[LS] H. Lin and H. Su, *Classification of direct limits of generalized Toeplitz algebras*, Pacific J. Math., to appear.
[P] G. K. Pedersen, *C^*-algebras and Their Automorphism Groups*, Academic Press, London/New York /San Francisco, 1979. MR **81e:**46037

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