TYPE I C*-ALGEBRAS OF REAL RANK ZERO

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Abstract. We show that a separable C*-algebra A of type I has real rank zero if and only if $d(\hat{A}) = 0$, where $d$ is a modified dimension. We also show that a separable C*-algebra of type I has real rank zero if and only if it is an AF-algebra.

1. Recently, there have been remarkable developments in the theory of classification of separable nuclear C*-algebras (see [E3] for a survey). For example, direct limits of (sub)homogeneous C*-algebras of real rank zero have been classified ([EG] and [DG]). All (sub)homogeneous C*-algebras are of type I. Some efforts are made to study direct limits of type I C*-algebras of real rank zero ([LS]). This leads to the question when a separable C*-algebra of type I has real rank zero and how to classify them. It turns out that the question can be fairly easily answered. We show in this short note that a separable C*-algebra of type I has real rank zero if and only if it is an AF-algebra. So they can be classified by their dimension groups ([E1]).

2. It is shown ([BP]) that an abelian C*-algebra $A = C_0(X)$, where $X$ is a locally compact Hausdorff space, has real rank zero if and only if $\dim(X) = 0$. A natural question is whether it can be generalized to type I C*-algebras, i.e., whether a separable C*-algebra of type I has real rank zero if and only if $\dim(\hat{A}) = 0$. It turns out that the usual definition for zero dimension does not work very well when the space is not Hausdorff. Let $A$ be a unital C*-algebra generated by $\{K, 1, p\}$, where $K$ is the C*-algebra of compact operators on an infinite dimensional separable Hilbert space, $1$ is the identity operator and both $1 - p$ and $p$ are infinite dimensional projections. Clearly, $A$ is an extension of $\mathbb{C} \oplus \mathbb{C}$ by $K$ and $A$ is a type I C*-algebra with real rank zero. Its spectrum $\hat{A} = \{x_1, x_2, x_3\}$ consists of only three points and its open subsets $T = \emptyset, \{x_3\}, \{x_1, x_3\}, \{x_2, x_3\}, \hat{A}$, where $x_1$ is the primitive ideal generated by $K$ and $p, x_2$ is the primitive ideal generated by $K$ and $1 - p$ and $x_3$ is the zero ideal. However, by the usual definition, $(X, T)$ is not of dimension zero. It does not have clopen base. The problem is that closure of $\{x_3\}$ is the whole $\hat{A}$. As far as we are concerned, at least in this note, we do not believe that a space which contains only finitely many points should have “dimension” other than zero.

We now introduce a new concept of “dimension” $d$. For any topological space $Y$, let

$$Y_0 = \{y \in Y : y \notin \{z\} \text{ for any } z \in Y, z \neq y\},$$

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where \( \overline{\{z\}} \) is the closure of \( \{z\} \). For any ordinal \( \beta \), suppose \( Y_\alpha \) is defined for any \( \alpha < \beta \); we define

\[
Y_\beta = \{ y \in Y \setminus \cup_{\alpha < \beta} Y_\alpha : y \notin \overline{\{z\}} \text{ for any } z \in Y \setminus \cup_{\alpha < \beta} Y_\alpha, z \neq y \}.
\]

We say \( d(Y) = n \) if \( \dim(Y_\alpha) \leq n \) for all \( \alpha \) and for some \( \alpha \), \( \dim(Y_\alpha) = n \), where \( \dim(X) \) is the covering dimension of \( X \). Here \( Y_\alpha \) is equipped with the relative topology. It is clear that \( d(Y) = \dim(Y) \) if \( Y \) is Hausdorff. In this note we are only interested in the case that \( d(Y) = 0 \). We write \( d(Y) = 0 \), if \( \dim(Y_\alpha) = 0 \) for all ordinals \( \alpha \). Here \( \dim(Y_\alpha) = 0 \) means that \( Y_\alpha \) has a clopen base. We will show that a separable \( C^* \)-algebra \( A \) of type I has real rank zero if and only if \( d(A) = 0 \).

3. For the reader’s convenience, before we go any further, we would like to remind the reader of several definitions. A \( C^* \)-algebra \( A \) is of type I, if every irreducible representation \( (H, \pi) \) of \( A \) contains \( K(H) \), the compact operators on \( H \). For type I \( C^* \)-algebra \( A \), its primitive ideals space is the same as its equivalence classes of irreducible representations which will be denoted by \( \hat{A} \). We will refer the reader to section 4.1 of \([P]\) for the notation \( hull \) and basic facts about the hull-kernel topology on \( \hat{A} \).

A \( C^* \)-algebra has real rank zero if invertible selfadjoint elements are dense in the set of selfadjoint elements. In particular, every AF-algebra has real rank zero. A positive element \( x \) in a type I \( C^* \)-algebra \( A \) has a continuous trace, if \( Tr(\pi(x)) \) \( (\pi \in \hat{A}) \) is (finite and) continuous on \( \hat{A} \). A type I \( C^* \)-algebra is said to have continuous trace if the set of elements with continuous trace is dense in \( A_+ \). The proof of the main result uses the following facts:

(i) Given an extension \( 0 \to J \to A \to A/J \to 0 \), then \( A \) is AF, if both \( J \) and \( A/J \) are AF ([B]).

(ii) Every hereditary \( C^* \)-subalgebra of a \( C^* \)-algebra of real rank zero has real rank zero ([BP]).

(iii) A result from [P] (Theorem 6.2.11) which yields an essential composition series \( \{I_\alpha\}_{0 \leq \alpha \leq \beta} \) for any type I \( C^* \)-algebra \( A \) such that \( I_{\alpha+1}/I_\alpha \) has continuous trace for each ordinal \( \alpha < \beta \).

We then reduce the general case to the case that \( A \) has continuous trace.

4. Lemma. Let \( A \) be a \( C^* \)-algebra of type I. Then

\[
\hat{A}_0 = \hat{A} \setminus hull(I_0),
\]

where \( I_0 \) is the maximal essential ideal of \( A \) with continuous trace.

Proof. Let \( I_0 \) be the maximal essential ideal of \( A \) which has continuous trace. Note \( I_0 \neq \{0\} \) (see [P, 6.2.11]). Since \( \hat{A} \setminus hull(I_0) \) is homeomorphic to \( \hat{I}_0 \) and \( \hat{I}_0 \) is Hausdorff, we have

\[
\hat{A} \setminus hull(I_0) \subset \hat{A}_0.
\]

Fix \( \zeta \in \hat{A}_0 \). Let \( (\pi, H) \) be an irreducible representation of \( I_0 \) and let \( (\rho, K) \) be the irreducible representation of \( A \) induced by \( \pi \). Since \( I_0 \) is an ideal, \( H = K \). Suppose that \( a \in \ker(\rho) \). Then for any \( i \in I_0, ai, ia \in Ker(\pi) \). Let \( \phi : A \to A/\ker(\pi) \) be the quotient map. Then \( \phi(ai) = \phi(ia) = 0 \). Since \( \phi(I_0) \) is an essential ideal of \( \phi(A) \), \( \phi(a) = 0 \). Therefore \( Ker(\rho) = Ker(\pi) \). Thus \( \zeta \) does not contain \( Ker(\pi) \). In particular, \( I_0 \notin \zeta \). This implies that \( \zeta \in \hat{A} \setminus hull(I_0) \). Therefore

\[
\hat{A} \setminus hull(I_0) = \hat{A}_0.
\]
Let $A$ be a type I $C^*$-algebra and let $J_0 = I_0$ and $K_0 = 0$. We define $I_\alpha$, $J_\alpha$ and $K_\alpha$ as follows. Suppose that $I_\alpha$, $J_\alpha$ and $K_\alpha$ have been defined for all ordinals $\alpha < \beta$. Set $K_\beta = cl(\cup_{\alpha < \beta} I_\alpha)$. Note that $A/K_\beta$ is type I. Let $J_\beta$ be the maximal essential ideal of $A/K_\beta$ with continuous trace and let $I_\beta$ be the preimage of $J_\beta$ in $A$. One of the purposes of the following lemma is to produce a certain composition series which is going to be used in the proof of Theorem 8.

5. Lemma. Let $A$ be a $C^*$-algebra of type I. Then

$$\hat{A}_\alpha \cong \hat{I}_\alpha \cong (A/K_\alpha) \setminus hull(J_\alpha) = ((A/K_\alpha)_0)$$

and

$$\hat{A} \setminus \hat{A}_\alpha = hull(I_\alpha),$$

where “$\cong$” means homeomorphic.

Proof. From the homeomorphism between $\hat{A} \setminus hull(I_0)$ and $\hat{I}_0$, we see that, by Lemma 4, the lemma holds for $\alpha = 0$. Let $\beta$ be an ordinal number. Suppose that the lemma holds for any ordinal $\alpha < \beta$. Set $K_\beta = cl(\cup_{\alpha < \beta} I_\alpha)$. Let $J_\beta$ be the maximal essential ideal of $A/K_\beta$ with continuous trace and let $I_\beta$ be the preimage of $J_\beta$.

Suppose that $\zeta \in \hat{A}_\beta$. Then

$$\zeta \in \hat{A} \setminus \cup_{\alpha < \beta} \hat{A}_\alpha = \cap_{\alpha < \beta} (\hat{A} \setminus \hat{A}_\alpha)$$

$$= \cap_{\alpha < \beta} hull(I_\alpha) = hull(K_\beta).$$

The last equality follows from the definition of $K_\beta$. Note that $(A/K_\beta) \cong hull(K_\beta)$. From the definition of $\hat{A}_\beta$ and by applying the same argument in Lemma 4, we see that $\zeta \in (A/K_\beta) \setminus hull(J_\beta) = ((A/K_\beta)_0)$.

Conversely,

$$((A/K_\beta)_0) = (A/K_\beta) \setminus hull(J_\beta)$$

$$\cong hull(K_\beta) \setminus hull(I_\beta) = (\hat{A} \setminus \cup_{\alpha < \beta} \hat{A}_\alpha) \setminus hull(I_\beta).$$

The last subset is open in $\hat{A} \setminus \cup_{\alpha < \beta} \hat{A}_\alpha$ and is Hausdorff, since the first one is (see Lemma 4). Therefore, it must be a subset of $\hat{A}_\beta$ which is a subset of $\hat{A} \setminus \cup_{\alpha < \beta} \hat{A}_\alpha$. This implies that

$$\hat{A}_\beta = ((A/K_\beta)_0).$$

So, by Lemma 4,

$$\hat{A} \setminus \hat{A}_\beta \cong hull(J_\beta).$$

This ends the proof.

6. Lemma. Let $A$ be a separable $C^*$-algebra which has continuous trace. Suppose that $dim(A) = 0$. Then $A$ is an AF-algebra.

Proof. We first recall that $\hat{A}$ is Hausdorff and $A$ is a continuous field of elementary $C^*$-algebras.

Consider $A \otimes K$, where $K$ is the $C^*$-algebra of compact operators on $l^2$. Clearly, $A \otimes M_n$ has continuous trace, where $M_n$ is the $C^*$-algebra of $n \times n$ matrices (over $\mathbb{C}$). Since $A \otimes M_n$ is dense in $A \otimes K$, $A \otimes K$ has continuous trace. We also have $(A \otimes K) = \hat{A}$ and every irreducible representation has dimension $\aleph_0$. So $A \otimes K$ is a separable $C^*$-algebra with continuous trace, homogeneous of rank $\aleph_0$. Note also,
since \( \hat{A} \) has dimension zero, \( H^3(\hat{A}, \mathbb{Z}) = \{0\} \). It follows from Corollary 10.9.6 in [Dix] that \( A \otimes \mathcal{K} \cong C_0(\hat{A}) \otimes \mathcal{K} \). Since \( \dim(\hat{A}) = 0 \), \( A \otimes \mathcal{K} \) is an AF-algebra. We conclude that \( A \) is itself AF. \( \square \)

7. Remark. One can prove Lemma 8 without introducing \( H^3(T, \mathbb{Z}) \). But the above proof is shorter.

8. Theorem. Let \( A \) be a separable \( C^* \)-algebra of type I. Then \( A \) has real rank zero if and only if \( d(\hat{A}) = 0 \).

Proof. We first show the theorem holds for the case that \( A \) has continuous trace. In this special case, \( A \) is Hausdorff.

Let \( \mathcal{A} \) be the continuous field of nonzero elementary \( C^* \)-algebras defined by \( A \). Then \( A \) satisfies the Fell’s condition (see [Dix, 10.5.8], i.e., for any \( t_0 \in \hat{A} \), there are a neighborhood \( U(t_0) \) and a vector field \( p \) of \( \mathcal{A} \), defined and continuous in \( U(t_0) \) such that \( p(t) \) is a projection of rank 1, for every \( t \in U(t_0) \). Since \( \hat{A} \) is locally compact and Hausdorff, we may assume that there is a neighborhood \( \hat{V}(t_0) \) such that \( \hat{V}(t_0) \subset \overline{V(t_0)} \subset U(t_0) \), where \( \overline{V(t_0)} \) is compact. There is a non-negative function \( f \in C_0(\hat{A}) \) such that \( f(t) = 1 \) when \( t \in \overline{V(t_0)} \) and \( f(t) = 0 \) if \( t \not\in U(t_0) \). So there is \( x \in A \) such that \( x = f \cdot p \). Suppose that \( I \) is the ideal of \( A \) defined by elements in \( \mathcal{A} \) which vanish in \( \overline{V(t_0)} \). If \( A \) has real rank zero, then so is \( A/I \) ( [BP] )

Furthermore, \( \pi(x)(A/I)\pi(x) \) has real rank zero. Clearly \( \pi(x)A/I\pi(x) \cong C(\overline{V(t_0)}) \). Therefore, the Hausdorff space \( \overline{V(t_0)} \) has dimension zero. In particular, \( V(t_0) \) has a clopen base. Since this is true for every \( t \in \hat{A} \), we conclude that \( \dim(\hat{A}) = 0 \).

The converse follows from Lemma 6. Moreover, in this case, \( A \) is an AF-algebra.

Now for the general case, by Lemma 5, \( A \) has an essential composition series \( \{I_\alpha|0 \leq \alpha \leq \beta \} \) such that \( I_{\alpha+1}/I_\alpha \) has continuous trace for each \( \alpha < \beta \) and \( (I_{\alpha+1}/I_\alpha) = \hat{A}_{\alpha+1} \).

If \( A \) has real rank zero, then by [BP] , each \( J_\alpha \) has real rank zero. Thus, by Lemma 5 and the above, \( \dim(\hat{A}_\alpha) = \dim(\hat{J}_\alpha) = 0 \). Therefore \( d(\hat{A}) = 0 \).

If \( d(\hat{A}) = 0 \), then \( \dim(\hat{I}_\alpha) = 0 \) for each \( \alpha \). From the above, each \( J_\alpha \) is a Banach algebra. We will show that \( A \) itself is an AF-algebra. We prove this by induction on \( \beta \). Suppose that we have shown this for all ordinals \( \alpha < \beta \). If \( \beta \) is not a limit ordinal, say \( \beta = \alpha + 1 \). By the assumption, \( I_\alpha \) is an AF-algebra. We also have that \( A/I_\alpha = J_\beta \) is an AF-algebra. By [BP] , \( A \) is an AF-algebra. If \( \beta \) is a limit ordinal, \( A \) is the norm closure \( \cup_{\alpha < \beta}I_\alpha \), where each \( I_\alpha \) is an AF-algebra. Since we assume that each \( I_\alpha \) is an AF-algebra, we conclude that \( A \) is an AF-algebra. \( \square \)

9. Corollary. A separable \( C^* \)-algebra of type I has real rank zero if and only if it is an AF-algebra.

10. There is also a way to “Hausdorffize” \( \hat{A} \). For the convenience, we will assume that \( A \) is unital. Note that if \( A \) is type I, so is \( \hat{A} \), the unitization of \( A \). Furthermore, \( A \) has real rank zero if and only if \( \hat{A} \) is. Now suppose that \( A \) is a unital type I \( C^* \)-algebra . Then there is a compact Hausdorff space \( H(\hat{A}) \) such that \( C(\hat{A}) \cong C(H(\hat{A})) \) by Gelfand transform. The Hausdorff space is unique up to homeomorphism. Therefore one may use the dimension of \( H(\hat{A}) \) instead of the dimension of \( \hat{A} \). In fact, a separable \( C^* \)-algebra \( A \) of type I has real rank zero if and only if \( \dim(H(\hat{A})) = 0 \).
11. Theorem. Let $A$ be a unital separable $C^*$-algebra of type I. Then the following are equivalent.

(a) $A$ has real rank zero,
(b) $A$ is an AF-algebra,
(c) $d(\hat{A}) = 0$,
(d) $\dim(H(\hat{A})) = 0$,
(e) The center of $A$ has real rank zero,
(f) The center of $A$ is an AF-algebra.

Proof. We have established the equivalence of (a), (b) and (c).

Let

$I_0 = \{ f \in C(\hat{A}) : f(t) = 0, \ t \notin \hat{A}_0 \}$

$I_\alpha = \{ f \in C(\hat{A}) : f(t) = 0, \text{ if } t \notin \hat{A} \cup \bigcup_{\gamma \leq \alpha} \hat{A}_\gamma \}$.

We have $I_{\alpha+1}/I_\alpha = C_0(\hat{A}_\alpha)$. So if (c) holds, $I_{\alpha+1}/I_\alpha$ is an AF-algebra. Using the same argument that we used earlier, by [B], we see that $C(\hat{A})$ is an AF-algebra. Thus $C(H(\hat{A}))$ is an AF-algebra. Therefore, $\dim(H(\hat{A})) = 0$. This proves (c) implies (d). Conversely, if $\dim(H(\hat{A})) = 0$, then $C(H(\hat{A}))$ is an AF-algebra, whence $C(\hat{A})$ is an AF-algebra. Therefore $I_{\alpha+1}/I_\alpha$ is AF, which implies that $\dim(\hat{A}_\alpha) = 0$. So $d(\hat{A}) = 0$. Therefore (c) and (d) are equivalent.

To include (e) and (f), we apply Dauns-Hofmann’s theorem which says that the center of $A$ is isomorphic to $C(A)$.

ADDED IN PROOF

After this paper was submitted, George Elliott kindly pointed to us that there is an overlap between this paper and [BE] which showed, among other things, that a separable type I $C^*$-algebra $A$ is AF if and only if $\hat{A}$ has a base of compact open sets.

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