TYPE I C*-ALGEBRAS OF REAL RANK ZERO

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Abstract. We show that a separable C*-algebra A of type I has real rank zero if and only if \( d(\hat{A}) = 0 \), where \( d \) is a modified dimension. We also show that a separable C*-algebra of type I has real rank zero if and only if it is an AF-algebra.

1. Recently, there have been remarkable developments in the theory of classification of separable nuclear C*-algebras (see [E3] for a survey). For example, direct limits of (sub)homogeneous C*-algebras of real rank zero have been classified ([EG] and [DG]). All (sub)homogeneous C*-algebras are of type I. Some efforts are made to study direct limits of type I C*-algebras of real rank zero ([LS]). This leads to the question when a separable C*-algebra of type I has real rank zero and how to classify them. It turns out the question can be fairly easily answered. We show in this short note that a separable C*-algebra of type I has real rank zero if and only if it is an AF-algebra. So they can be classified by their dimension groups ([E1]).

2. It is shown ([BP]) that an abelian C*-algebra \( A = C_0(X) \), where \( X \) is a locally compact Hausdorff space, has real rank zero if and only if \( \dim(X) = 0 \). A natural question is whether it can be generalized to type I C*-algebras, i.e., whether a separable C*-algebra of type I has real rank zero if and only if \( \dim(\hat{A}) = 0 \). It turns out that the usual definition for zero dimension does not work very well when the space is not Hausdorff. Let \( A \) be a unital C*-algebra generated by \( \{K, 1, p\} \), where \( K \) is the C*-algebra of compact operators on an infinite dimensional separable Hilbert space, 1 is the identity operator and, both \( 1 - p \) and \( p \) are infinite dimensional projections. Clearly, \( A \) is an extension of \( \mathbb{C} \oplus \mathbb{C} \) by \( K \) and \( A \) is a type I C*-algebra with real rank zero. Its spectrum \( \hat{A} = \{x_1, x_2, x_3\} \) consists of only three points and its open subsets \( T = \{\emptyset, \{x_3\}, \{x_1, x_3\}, \{x_2, x_3\}, \hat{A}\} \), where \( x_1 \) is the primitive ideal generated by \( K \) and \( p \), \( x_2 \) is the primitive ideal generated by \( K \) and \( 1 - p \) and \( x_3 \) is the zero ideal. However, by the usual definition, \( (X, T) \) is not of dimension zero. It does not have clopen base. The problem is that closure of \( \{x_3\} \) is the whole \( \hat{A} \). As far as we are concerned, at least in this note, we do not believe that a space which contains only finitely many points should have “dimension” other than zero.

We now introduce a new concept of “dimension” \( d \). For any topological space \( Y \), let

\[
Y_0 = \{y \in Y : y \notin \{z\} \text{ for any } z \in Y, z \neq y\},
\]

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where \( \overline{\{z\}} \) is the closure of \( \{z\} \). For any ordinal \( \beta \), suppose \( Y_\alpha \) is defined for any \( \alpha < \beta \); we define
\[
Y_\beta = \{ y \in Y \setminus \cup_{\alpha < \beta} Y_\alpha : y \not\in \overline{\{z\}} \text{ for any } z \in Y \setminus \cup_{\alpha < \beta} Y_\alpha, z \neq y \}.
\]
We say \( d(Y) = n \) if \( \dim(Y_\alpha) \leq n \) for all \( \alpha \) and for some \( \alpha \), \( \dim(Y_\alpha) = n \), where \( \dim(X) \) is the covering dimension of \( X \). Here \( Y_\alpha \) is equipped with the relative topology. It is clear that \( d(Y) = \dim(Y) \) if \( Y \) is Hausdorff. In this note we are only interested in the case that \( d(Y) = 0 \). We write \( d(Y) = 0 \), if \( \dim(Y_\alpha) = 0 \) for all ordinals \( \alpha \). Here \( \dim(Y_\alpha) = 0 \) means that \( Y_\alpha \) has a clopen base. We will show that a separable \( C^* \)-algebra \( A \) of type I has real rank zero if and only if \( d(\hat{A}) = 0 \).

3. For the reader’s convenience, before we go any further, we would like to remind the reader of several definitions. A \( C^* \)-algebra \( A \) is of type I, if every irreducible representation \((\pi, H)\) of \( A \) contains \( K(H) \), the compact operators on \( H \). For type I \( C^* \)-algebra \( A \), its primitive ideals space is the same as its equivalence classes of irreducible representations which will be denoted by \( \hat{A} \). We will refer the reader to section 4.1 of [P] for the notation \( hull \) and basic facts about the hull-kernel topology on \( \hat{A} \).

A \( C^* \)-algebra has real rank zero if invertible selfadjoint elements are dense in the set of selfadjoint elements. In particular, every AF-algebra has real rank zero. A positive element \( x \) in a type I \( C^* \)-algebra \( A \) has a continuous trace, if \( \text{Tr}(\pi(x)) \) (\( \pi \in A \)) is (finite and) continuous on \( \hat{A} \). A type I \( C^* \)-algebra is said to have continuous trace if the set of elements with continuous trace is dense in \( A_+ \). The proof of the main result uses the following facts:

(i) Given an extension \( 0 \to J \to A \to A/J \to 0 \), then \( A \) is AF, if both \( J \) and \( A/J \) are AF ([B]).

(ii) Every hereditary \( C^* \)-subalgebra of a \( C^* \)-algebra of real rank zero has real rank zero ([BP]).

(iii) A result from [P] (Theorem 6.2.11) which yields an essential composition series \( \{I_\alpha\}_{0 \leq \alpha \leq \beta} \) for any type I \( C^* \)-algebra \( A \) such that \( I_{\alpha+1}/I_\alpha \) has continuous trace for each ordinal \( \alpha < \beta \).

We then reduce the general case to the case that \( A \) has continuous trace.

4. Lemma. Let \( A \) be a \( C^* \)-algebra of type I. Then
\[
\hat{A}_0 = \hat{A} \setminus \text{hull}(I_0),
\]
where \( I_0 \) is the maximal essential ideal of \( A \) with continuous trace.

Proof. Let \( I_0 \) be the maximal essential ideal of \( A \) which has continuous trace. Note \( I_0 \neq \{0\} \) (see [P, 6.2.11]). Since \( \hat{A} \setminus \text{hull}(I_0) \) is homeomorphic to \( \hat{I}_0 \) and \( \hat{I}_0 \) is Hausdorff, we have
\[
\hat{A} \setminus \text{hull}(I_0) \subset \hat{A}_0.
\]
Fix \( \zeta \in \hat{A}_0 \). Let \((\pi, H)\) be an irreducible representation of \( I_0 \) and let \((\rho, K)\) be the irreducible representation of \( A \) induced by \( \pi \). Since \( I_0 \) is an ideal, \( H = K \). Suppose that \( a \in \text{ker}(\rho) \). Then for any \( i \in I_0 \), \( ai, ia \in \text{ker}(\pi) \). Let \( \phi : A \to A/\text{ker}(\pi) \) be the quotient map. Then \( \phi(ai) = \phi(ia) = 0 \). Since \( \phi(I_0) \) is an essential ideal of \( \phi(A) \), \( \phi(a) = 0 \). Therefore \( \text{ker}(\rho) = \text{ker}(\pi) \). Thus \( \zeta \) does not contain \( \text{ker}(\pi) \). In particular, \( I_0 \not\subseteq \zeta \). This implies that \( \zeta \in \hat{A} \setminus \text{hull}(I_0) \). Therefore
\[
\hat{A} \setminus \text{hull}(I_0) = \hat{A}_0.
\]
Let $A$ be a type I $C^*$-algebra and let $J_0 = I_0$ and $K_0 = 0$. We define $I_\alpha$, $J_\alpha$ and $K_\alpha$ as follows. Suppose that $I_\alpha$, $J_\alpha$ and $K_\alpha$ have been defined for all ordinals $\alpha < \beta$. Set $K_\beta = \text{cl}(\cup_{\alpha < \beta} I_\alpha)$. Note that $A/K_\beta$ is type I. Let $J_\beta$ be the maximal essential ideal of $A/K_\beta$ with continuous trace and let $I_\beta$ be the preimage of $J_\beta$ in $A$.

One of the purposes of the following lemma is to produce a certain composition series which is going to be used in the proof of Theorem 8.

5. **Lemma.** Let $A$ be a $C^*$-algebra of type I. Then

$$\hat{A}_\alpha \cong \hat{I}_\alpha \cong (A/K_\alpha) \setminus \text{hull}(J_\alpha) = ((A/K_\alpha)_0$$

and

$$\hat{A} \setminus \hat{A}_\alpha = \text{hull}(I_\alpha),$$

where “$\cong$” means homeomorphic.

**Proof.** From the homeomorphism between $\hat{A} \setminus \text{hull}(I_0)$ and $\hat{I}_0$, we see that, by Lemma 4, the lemma holds for $\alpha = 0$. Let $\beta$ be an ordinal number. Suppose that the lemma holds for any ordinal $\alpha < \beta$. Set $K_\beta = \text{cl}(\cup_{\alpha < \beta} I_\alpha)$. Let $J_\beta$ be the maximal essential ideal of $A/K_\beta$ with continuous trace and let $I_\beta$ be the preimage of $J_\beta$.

Suppose that $\zeta \in \hat{A}_\beta$. Then

$$\zeta \in \hat{A} \setminus \cup_{\alpha < \beta} \hat{A}_\alpha = \cap_{\alpha < \beta} (\hat{A} \setminus \hat{A}_\alpha) = \cap_{\alpha < \beta} \text{hull}(I_\alpha) = \text{hull}(K_\beta).$$

The last equality follows from the definition of $K_\beta$. Note that $(A/K_\beta) \cong \text{hull}(K_\beta)$.

From the definition of $\hat{A}_\beta$ and by applying the same argument in Lemma 4, we see that $\zeta \in (A/K_\beta) \setminus \text{hull}(J_\beta) = ((A/K_\beta)_0$.

Conversely,

$$((A/K_\beta)_0 = (A/K_\beta) \setminus \text{hull}(J_\beta) \cong \text{hull}(K_\beta) \setminus \text{hull}(I_\beta) = (\hat{A} \setminus \cup_{\alpha < \beta} \hat{A}_\alpha) \setminus \text{hull}(I_\beta).$$

The last subset is open in $\hat{A} \setminus \cup_{\alpha < \beta} \hat{A}_\alpha$ and is Hausdorff, since the first one is (see Lemma 4). Therefore, it must be a subset of $\hat{A}_\beta$ which is a subset of $\hat{A} \setminus \cup_{\alpha < \beta} \hat{A}_\alpha$. This implies that

$$\hat{A}_\beta = ((A/K_\beta)_0.$$

So, by Lemma 4,

$$\hat{A} \setminus \hat{A}_\beta \cong \text{hull}(J_\beta).$$

This ends the proof. \qed

6. **Lemma.** Let $A$ be a separable $C^*$-algebra which has continuous trace. Suppose that $\dim(A) = 0$. Then $A$ is an AF-algebra.

**Proof.** We first recall that $\hat{A}$ is Hausdorff and $A$ is a continuous field of elementary $C^*$-algebras.

Consider $A \otimes K$, where $K$ is the $C^*$-algebra of compact operators on $l^2$. Clearly, $A \otimes M_n$ has continuous trace, where $M_n$ is the $C^*$-algebra of $n \times n$ matrices (over $\mathbb{C}$). Since $A \otimes M_n$ is dense in $A \otimes K$, $A \otimes K$ has continuous trace. We also have $(A \otimes K) = \hat{A}$ and every irreducible representation has dimension $\aleph_0$. So $A \otimes K$ is a separable $C^*$-algebra with continuous trace, homogeneous of rank $\aleph_0$. Note also,
since $\hat{A}$ has dimension zero, $H^3(\hat{A}, \mathbb{Z}) = \{0\}$. It follows from Corollary 10.9.6 in [Dix] that $A \otimes \mathcal{K} \cong C_0(\hat{A}) \otimes \mathcal{K}$. Since $\dim(A) = 0$, $A \otimes \mathcal{K}$ is an AF-algebra. We conclude that $A$ is itself AF. 

7. Remark. One can prove Lemma 8 without introducing $H^3(T, \mathbb{Z})$. But the above proof is shorter.

8. Theorem. Let $A$ be a separable $C^*$-algebra of type I. Then $A$ has real rank zero if and only if $d(\hat{A}) = 0$.

Proof. We first show the theorem holds for the case that $A$ has continuous trace. In this special case, $A$ is Hausdorff.

Let $\mathcal{A}$ be the continuous field of nonzero elementary $C^*$-algebras defined by $A$. Then $A$ satisfies the Fell’s condition (see [Dix, 10.5.8], i.e., for any $t_0 \in \hat{A}$, there are a neighborhood $U(t_0)$ and a vector field $p$ of $\mathcal{A}$, defined and continuous in $U(t_0)$ such that $p(t)$ is a projection of rank 1, for every $t \in U(t_0)$. Since $\hat{A}$ is locally compact and Hausdorff, we may assume that there is a neighborhood $V(t_0)$ such that $V(t_0) \subset \overline{V(t_0)} \subset U(t_0)$, where $\overline{V(t_0)}$ is compact. There is a non-negative function $f \in C_0(\hat{A})$ such that $f(t) = 1$ when $t \in \overline{V(t_0)}$ and $f(t) = 0$ if $t \notin U(t_0)$. So there is $x \in A$ such that $x = f \cdot p$. Suppose that $I$ is the ideal of $A$ defined by elements in $\mathcal{A}$ which vanish in $V(t_0)$. If $A$ has real rank zero, then so is $A/I$ ([BP]). Furthermore, $\pi(x)(A/I)\pi(x)$ has real rank zero. Clearly $\pi(x)A/I\pi(x) \cong C(\overline{V(t_0)})$. Therefore, the Hausdorff space $\overline{V(t_0)}$ has dimension zero. In particular, $V(t_0)$ has a clopen base. Since this is true for every $t \in \hat{A}$, we conclude that $\dim(\hat{A}) = 0$.

The converse follows from Lemma 6. Moreover, in this case, $A$ is an AF-algebra.

Now for the general case, by Lemma 5, $A$ has an essential composition series $\{I_\alpha|0 \leq \alpha \leq \beta\}$ such that $I_{\alpha+1}/I_\alpha$ has continuous trace for each $\alpha < \beta$ and $(I_{\alpha+1}/I_\alpha) \sim A_{\alpha+1}$.

If $A$ has real rank zero, then by [BP], each $J_\alpha$ has real rank zero. Thus, by Lemma 5 and the above, $\dim(\hat{A}_\alpha) = \dim(\hat{J}_\alpha) = 0$. Therefore $d(\hat{A}) = 0$.

If $d(A) = 0$, then $\dim(\hat{J}_\alpha) = 0$ for each $\alpha$. From the above, each $J_\alpha$ is an AF-algebra. We will show that $A$ itself is an AF-algebra. We prove this by induction on $\beta$. Suppose that we have shown this for all ordinals $\alpha < \beta$. If $\beta$ is not a limit ordinal, say $\beta = \alpha + 1$. By the assumption, $I_\alpha$ is an AF-algebra. We also have that $A/I_\alpha = J_\beta$ is an AF-algebra. By [B], $A$ is an AF-algebra. If $\beta$ is a limit ordinal, $A$ is the norm closure $\bigcup_{\alpha < \beta} I_\alpha$, where each $I_\alpha$ is an AF-algebra. Since we assume that each $I_\alpha$ is an AF-algebra, we conclude that $A$ is an AF-algebra. 

9. Corollary. A separable $C^*$-algebra of type I has real rank zero if and only if it is an AF-algebra.

10. There is also a way to “Hausdorffize” $\hat{A}$. For the convenience, we will assume that $A$ is unital. Note that if $A$ is type I, so is $\hat{A}$, the unitization of $A$. Furthermore, $A$ has real rank zero if and only if $\hat{A}$ is. Now suppose that $A$ is a unital type I $C^*$-algebra. Then there is a compact Hausdorff space $H(\hat{A})$ such that $C(\hat{A}) \cong C(H(\hat{A}))$ by Gelfand transform. The Hausdorff space is unique up to homeomorphism. Therefore one may use the dimension of $H(\hat{A})$ instead of the dimension of $\hat{A}$. In fact, a separable $C^*$-algebra $A$ of type I has real rank zero if and only if $\dim(H(\hat{A})) = 0$. 

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11. **Theorem.** Let $A$ be a unital separable $C^*$-algebra of type I. Then the following are equivalent.

(a) $A$ has real rank zero,
(b) $A$ is an AF-algebra,
(c) $d(\hat{A}) = 0$,
(d) $\dim(H(\hat{A})) = 0$,
(e) The center of $A$ has real rank zero,
(f) The center of $A$ is an AF-algebra.

**Proof.** We have established the equivalence of (a), (b) and (c).

Let $I_0 = \{f \in \mathcal{C}(\hat{A}) : f(t) = 0, \ t \notin \hat{A}_0\}$,

$I_\alpha = \{f \in \mathcal{C}(\hat{A}) : f(t) = 0, \ if \ t \notin \hat{A} \cup \gamma \leq_\alpha \hat{A}_\gamma\}$.

We have $I_{\alpha+1}/I_\alpha = C_0(\hat{A}_\alpha)$. So if (c) holds, $I_{\alpha+1}/I_\alpha$ is an AF-algebra. Using the same argument that we used earlier, by [B], we see that $\mathcal{C}(\hat{A})$ is an AF-algebra. Thus $\mathcal{C}(H(\hat{A}))$ is an AF-algebra. Therefore, $\dim(H(\hat{A})) = 0$. This proves (c) implies (d). Conversely, if $\dim(H(\hat{A})) = 0$, then $\mathcal{C}(H(\hat{A}))$ is an AF-algebra, whence $\mathcal{C}(\hat{A})$ is an AF-algebra. Therefore $I_{\alpha+1}/I_\alpha$ is AF, which implies that $\dim(\hat{A}_\alpha) = 0$. So $d(\hat{A}) = 0$. Therefore (c) and (d) are equivalent.

To include (e) and (f), we apply Dauns-Hofmann’s theorem which says that the center of $A$ is isomorphic to $\mathcal{C}(\hat{A})$.

\[ \square \]

**ADDED IN PROOF**

After this paper was submitted, George Elliott kindly pointed to us that there is an overlap between this paper and [BE] which showed, among other things, that a separable type I $C^*$-algebra $A$ is AF if and only if $\hat{A}$ has a base of compact open sets.

**References**

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