NOTE ON FAITHFUL REPRESENTATIONS
AND A LOCAL PROPERTY OF LIE GROUPS

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Abstract. Let $G$ be any analytic group, let $T$ be a maximal toroid of the radical of $G$, and let $S$ be a maximal semisimple analytic subgroup of $G$.

If $L = \mathcal{L}(G)$ is the Lie algebra of $G$, $\text{rad}[L, L]$ is the radical of $[L, L]$, and $\mathcal{Z}(L)$ is the center of $L$, we show that $G$ has a faithful representation if and only if

(i) $\text{rad}[L, L] \cap \mathcal{Z}(L) \cap \mathcal{L}(T) = (0)$, and

(ii) $S$ has a faithful representation.

A theorem of M. Moskowitz [4, Thm. 2], shows that if $L$ is a finite-dimensional (real) Lie algebra, then all analytic groups with Lie algebra $L$ have faithful representations if and only if (i) $\text{rad}[L, L] \cap \mathcal{Z}(L) = (0)$, and (ii) for some maximal semisimple subalgebra $S$ of $L$, the simply connected analytic group with Lie algebra $S$ has a faithful representation. So it would be of interest to find a similar criterion for a single analytic group $G$ to have a faithful representation. Such a criterion is given in Theorem 2 below. As a consequence, we obtain Moskowitz’ Theorem in Corollary 3. So our criterion in the solvable case says that $G$ has a faithful representation if and only if $[L, L] \cap \mathcal{Z}(L) \cap \mathcal{L}(T) = (0)$ for some maximal toroid $T$ of $G$ where $L = \mathcal{L}(G)$; whereas the well-known criterion in the solvable case is that $G$ has a faithful representation if and only if $[G, G]$ is closed in $G$ and simply connected [2, p. 220]. For the case of semisimple analytic groups, we refer the reader to [2, pp. 199–201].

Our proof uses the notion of nuclei of analytic groups introduced by Hochschild and Mostow. A nucleus $K$ of an analytic group $G$ is a closed normal simply connected solvable analytic subgroup of $G$ such that $G/K$ is reductive. An analytic group $G$ has a faithful representation if and only if $G$ has a nucleus; if $G$ has a nucleus $K$, then $G = K \cdot P$ (semi-direct) for every maximal reductive analytic subgroup $P$ of $G$ [3, Section 2]. Recall that an analytic group is reductive if it has a faithful representation and all its representations are semisimple.

If $G$ is an analytic group, $\mathcal{L}(G)$ is its Lie algebra, $\text{rad} G$ is its radical, and $[G, G]$ is its commutator (derived) subgroup. Similarly, if $L$ is a Lie algebra, $\text{rad} L$ is its radical, and $[L, L]$ is its commutator (derived) subalgebra. All representations of analytic groups are assumed to be analytic and finite dimensional.

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Lemma 1. Let $G$ be any analytic group, and let $A$ and $B$ be analytic subgroups of $G$ such that $A$ is normal in $G$. If $G = AB$ and $A \cap B = (1)$, then $A$ and $B$ are closed in $G$.

Proof. Let $G^+ = A \times B$ be the cartesian product of the analytic manifolds of $A$ and $B$ underlying the analytic groups $A$ and $B$. Then, in addition to its manifold structure, $G^+ = A \times B$ is also an abstract group via the conjugation action of $B$ on $A$ in $G$. We now show that these structures turn $G^+$ into an analytic group. Let $f : A \times B \to A$ be the mapping given by $f(a, b) = bab^{-1}$. Then one can easily check that $f$ is analytic on a neighborhood of $(1, 1_B)$ by using the exponential maps in the analytic groups $A, B$ and $G$. Since $A$ is connected, it follows that $f$ is analytic on $A \times B$ [1, Lemma 3, p. 362].

Hence $G^+ = A \times B$ is an analytic group with the above group and manifold structures. Now let $p : G^+ = A \times B \to AB = G$ be the mapping given by $p(a, b) = ab$. Then $p$ is a surjective continuous homomorphism between locally compact connected topological groups, so $p$ must be an open map [2, Thm. 2.5, p. 7] or [2, Exercise 1, p. 13]. But $p$ is also bijective since $A \cap B = (1)$. Consequently, $p : G^+ \to G$ is an isomorphism of analytic groups. Hence $A$ and $B$ are closed in $G$ since they are closed in $G^+$.

We shall need the fact that if $G$ has a faithful representation, then its representation radical $N = \text{rad}[G, G]$ is closed in $G$ and simply connected. This is true, for example, because $N$ is contained in every nucleus $K$ of $G$ [3, Section 2] and each $K$ is a closed simply connected solvable analytic subgroup of $G$. For a direct proof, see the proof of Theorem 1 in [4].

Theorem 2. Let $G$ be any analytic group with Lie algebra $L$, let $T$ be a maximal toroid of $\text{rad}(G)$, and let $S$ be a maximal semisimple analytic subgroup of $G$. Then $G$ has a faithful representation if and only if

(i) $\text{rad}[L, L] \cap Z(L) \cap L(T) = (0)$, and

(ii) $S$ has a faithful representation.

Proof. Let $\text{Ad}$ and $\text{ad}$ be the adjoint representations of $G$ and $L(G)$ respectively on the Lie algebra $L$ of $G$. Since $\text{rad}[L, L]$ acts nilpotently on any representation space of $L$ [2, Thm. 3.2, p. 128], $\text{ad}(\text{rad}[L, L])$ consists of nilpotent elements. Since $\text{ad}(L(T)) = L(\text{Ad}(T))$ and $T$ is a toroid, it follows that $\text{ad}(L(T))$ consists of semisimple elements. Hence $\text{ad}(\text{rad}[L, L] \cap L(T)) = (0)$. Thus $(\text{rad}[L, L] \cap L(T)) \subseteq Z(L)$. Since $\text{rad}[L, L] = L(N)$ where $N = \text{rad}[G, G]$ [2, Thm. 3.1, p. 138], it follows that $\text{rad}[L, L] \cap Z(L) \cap L(T) = (0)$ if and only if $L(N) \cap L(T) = (0)$.

So first suppose that $G$ has a faithful representation. Then $S$ has a faithful representation. Moreover, as remarked above, $N$ is a closed simply connected solvable analytic subgroup of $G$, so $N$ has no non-trivial compact subgroups [2, Thm. 2.3, p. 138]. Hence $L(N) \cap L(T) = (0)$.

Conversely, suppose $L(N) \cap L(T) = (0)$. Then there exists a subspace $K$ of $\text{rad} L(G)$ containing $L(N)$ such that $\text{rad} L(G) = K \oplus L(T)$. Since $K$ contains $L(N) = \text{rad}[L, L]$ and $\text{rad}[L, L] = [L, \text{rad} L]$ [2, Thm. 3.2, p. 128], it follows that $K$ is an ideal of $L(G)$. Hence $\text{rad} L(G) = K + L(T)$ (semi-direct). Thus if $K$ is the analytic subgroup of $G$ corresponding to $K$, then $K$ is normal in $G$, $\text{rad}(G) = K \cdot T$, and the subgroup $K \cap T$ is discrete in the analytic group $K$. Thus the projection morphism $K \to K/(K \cap T)$ is a covering. Moreover, $K/(K \cap T)$ is homeomorphic to $\text{rad}(G)/T$ which is known to be a simply connected homogeneous space since $T$
is a maximal toroid of $\text{rad}(G)$ [2, Exercise 1, p. 187]. Hence the covering morphism $K \rightarrow K/(K \cap T)$ is a homeomorphism of simply connected homogeneous spaces. Thus $K \cap T = (1)$ and $K$ is simply connected. Since $\text{rad}(G) = K \cdot T$, it follows by Lemma 1 that $K$ is closed in $\text{rad}(G)$. Hence $K$ is a nucleus of $\text{rad}(G)$. Consequently, $\text{rad}(G)$ has a faithful representation [3, Section 2]. Since $S$ has also a faithful representation, it follows that $G$ has a faithful representation [2, Thm. 4.2, p. 221]. This proves Theorem 2.

\begin{corollary}[Thm. 2 of \cite{4}]
Let $L$ be a finite-dimensional (real) Lie algebra. Let $S$ be a maximal semisimple subalgebra of $L$, and let $S^*$ be the simply connected analytic group with Lie algebra $S$. Then all analytic groups with Lie algebra $L$ have faithful representations if and only if

(i) $\text{rad}[L, L] \cap Z(L) = (0)$, and

(ii) $S^*$ has a faithful representation.

\end{corollary}

\begin{proof}
Suppose $\text{rad}[L, L] \cap Z(L) = (0)$, and $S^*$ has a faithful representation. Let $G$ be any analytic group with Lie algebra $L$, and let $S_g$ be the (maximal) semisimple analytic subgroup of $G$ corresponding to the Lie algebra $S$. Since $S^*$ has a faithful representation, it follows that $S_g$ also has a faithful representation [4, Cor. 1a]. Hence $G$ has a faithful representation by Theorem 2.

For the converse, we may use the proof in [4, top of p. 197] since it refers only to the fact that $N = \text{rad}[G, G]$ is simply connected whenever $G$ has a faithful representation. \end{proof}

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