

THE INFINITESIMAL CONE OF A TOTALLY POSITIVE SEMIGROUP

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ABSTRACT. Given a complex reductive linear algebraic group split over \mathbb{R} with a fixed pinning, it is shown that all elements of the Lie algebra \mathfrak{g} infinitesimal to the totally positive subsemigroup $G_{\geq 0}$ of G lie in the totally positive cone $\mathfrak{g}_{\geq 0} \subset \mathfrak{g}$.

1. INTRODUCTION

Classically, the totally positive subsemigroup of $GL_n\mathbb{R}$ is the semigroup consisting of the matrices in $GL_n\mathbb{R}$ all of whose minors are nonnegative. Corresponding to it, there is a cone in $\mathfrak{gl}_n\mathbb{R}$, the “infinitesimal semigroup”, studied by Loewner in [Loe]. It consists of all matrices with arbitrary entries along the diagonal, nonnegative entries next to the diagonal, and zeros everywhere else. Loewner showed that X lies in this cone if and only if $\exp(tX)$ lies in the totally positive semigroup for all $t \geq 0$. Loewner also pointed out that, as a consequence of a theorem of Whitney’s, the exponential image of the infinitesimal semigroup generates the totally positive semigroup.

Let G be an arbitrary complex reductive linear algebraic group split over \mathbb{R} . We will start by recalling Lusztig’s definitions of the totally ≥ 0 subsemigroup $G_{\geq 0}$ of G , and of a cone $\mathfrak{g}_{\geq 0}$ in $\mathfrak{g} = \text{Lie}(G)$. These definitions give both $G_{\geq 0}$ and $\mathfrak{g}_{\geq 0}$ in terms of a set of generators. The aim of this paper is to extend Loewner’s result relating $G_{\geq 0}$ and $\mathfrak{g}_{\geq 0}$ to this more general setting. This is partially done in [L1] where it is shown that if $X \in \mathfrak{g}_{\geq 0}$ then $\exp X$ lies in $G_{\geq 0}$. To complete the picture we show (Proposition 2.1) that if $\exp(tX)$ lies in $G_{\geq 0}$ for all $t \geq 0$, then X must lie in $\mathfrak{g}_{\geq 0}$.

The proof of Proposition 2.1 in the classical case ([Loe]) uses the definition of $G_{\geq 0}$ in terms of inequalities, rather than by generators. For arbitrary semisimple, simply laced G there is also a characterization of $G_{\geq 0}$ by inequalities, arising from the positivity property of Lusztig’s canonical basis (see Theorem 3.1, shown by Lusztig in [L1]; for a stronger version, see [L2]). We make use of these inequalities to prove Proposition 2.1 in the simple, simply laced case. The general case then follows by standard arguments.

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2. DEFINITIONS AND PROPOSITION

Let G be a complex reductive linear algebraic group split over \mathbb{R} . We will identify any algebraic group over \mathbb{R} with its \mathbb{R} -rational points. Fix a maximal torus $T \subset G$ that is split over \mathbb{R} , and let B^+ be a Borel subgroup containing T . Denote the set of simple roots corresponding to B^+ by $\Pi = \{\alpha_i | i \in I\}$, and the set of (positive) roots by R (resp. R^+). Let $X^\vee(T)$ denote the group of cocharacters $\chi : \mathbb{R}^* \rightarrow T$. For any root α , denote its coroot by α^\vee . Let $(T, B^+, B^-, x_i, y_i; i \in I)$ be a *pinning* of G (see [L1]). Here B^- is a Borel subgroup containing T opposed to B^+ . Let U^+, U^- be the unipotent radicals of $B^+, \text{ resp. } B^-$. Then the $x_i, \text{ resp. } y_i$, are embeddings $x_i : \mathbb{R} \rightarrow U^+$ and $y_i : \mathbb{R} \rightarrow U^-$ such that

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mapsto x_i(a), \quad \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mapsto y_i(b), \quad \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \mapsto \alpha_i^\vee(c)$$

defines a homomorphism $SL_2\mathbb{R} \rightarrow G_i = \text{Centralizer}(\text{Ker}(\alpha_i)^\circ)$.

The totally ≥ 0 submonoid $G_{\geq 0}$ of G is then defined (in [L1, p.535]) to be the semigroup with identity element generated by the elements $\{x_i(a), y_i(a), \chi(b) | a \in \mathbb{R}_{\geq 0}, b \in \mathbb{R}_{> 0}, i \in I, \chi \in X^\vee(T)\}$.

Let dx_i, dy_i be the derivatives of x_i, y_i at zero. Lusztig defines in [L1, p. 549] a cone in $\mathfrak{g} := \text{Lie}(G)$ as follows:

$$\mathfrak{g}_{\geq 0} := \{t + \sum_{i \in I} (a_i dx_i + b_i dy_i) | t \in \text{Lie}(T), a_i, b_i \geq 0\}.$$

We propose to show

Proposition 2.1. *If $X \in \mathfrak{g}$ satisfies*

$$(*) \quad \exp(tX) \in G_{\geq 0} \quad \text{for } t \geq 0$$

then X lies in $\mathfrak{g}_{\geq 0}$.

The converse, $X \in \mathfrak{g}_{\geq 0}$ implies $(*)$, was shown by Lusztig in [L1].

3. CONNECTIONS WITH LUSZTIG'S CANONICAL BASIS

In the following assume G semisimple and of simply laced type. Let \mathbb{U} denote the universal enveloping algebra of \mathfrak{g} . Corresponding to the chosen pinning there is a set of Chevalley generators of \mathbb{U} given by $E_i := dx_i$ and $F_i := dy_i$ (here \mathfrak{g} is considered a subset of \mathbb{U}). Let \mathbb{U}^- be the subalgebra of \mathbb{U} generated by the elements $F_i (i \in I)$. Denote by \mathbb{B}^- the basis of \mathbb{U}^- obtained from Lusztig's canonical basis [L0] of the corresponding quantized universal enveloping algebra by specializing at $v = 1$ and extending scalars from \mathbb{Q} to \mathbb{C} .

Consider a finite dimensional, irreducible, algebraic representation V of G over \mathbb{C} and the corresponding representation of \mathbb{U} on V . Let $B(\eta)$ denote Lusztig's canonical basis of V arising from applying the elements of \mathbb{B}^- to a chosen highest weight vector η . In relation to $G_{\geq 0}$, the basis $B(\eta)$ has the following positivity property (see [L1, Prop 3.2]).

Theorem 3.1(Lusztig). *Let $g \in G_{\geq 0}$. Then the matrix entries of $g : V \rightarrow V$ with respect to $B(\eta)$ are nonnegative real numbers.*

This property will be central to our proof of Proposition 2.1. It has the following consequence.

Lemma 3.2. *Suppose X satisfies property (*) of Proposition 2.1 and B is a basis of V such that $G_{\geq 0}$ acts on V by matrices with nonnegative real entries with respect to B . Then for any $b, b' \in B$ with $b \neq b'$, the coefficient of b' in $X.b$ is nonnegative.*

Proof. Let \langle, \rangle be an inner product making B orthonormal. By our assumptions, we have

$$\langle \exp(tX).b, b' \rangle \geq 0 \quad \forall t \geq 0.$$

At $t = 0$ the above becomes an equality (note that $b \neq b'$). Therefore taking the derivative at zero gives

$$\langle X.b, b' \rangle \geq 0.$$

□

3.3. Canonical basis of the adjoint representation. Suppose G is simple. We consider the adjoint representation $V = \mathfrak{g}$. Let $B = \{X_\alpha, t_i | \alpha \in R, i \in I\}$ be a basis of V such that, as in [L3, p 259],

$$\begin{aligned}
 E_i.X_\alpha &= \begin{cases} X_{\alpha+\alpha_i}, & \alpha + \alpha_i \in R, \\ t_i, & \alpha + \alpha_i = 0, \\ 0, & \text{otherwise,} \end{cases} \\
 F_i.X_\alpha &= \begin{cases} X_{\alpha-\alpha_i}, & \alpha - \alpha_i \in R, \\ t_i, & \alpha - \alpha_i = 0, \\ 0, & \text{otherwise,} \end{cases} \\
 E_i.t_j &= \begin{cases} -\langle \alpha_i, \alpha_j^\vee \rangle X_{\alpha_i}, & i \neq j, \\ 2X_{\alpha_i}, & i = j, \end{cases} \\
 F_i.t_j &= \begin{cases} -\langle \alpha_i, \alpha_j^\vee \rangle X_{-\alpha_i}, & i \neq j, \\ 2X_{-\alpha_i}, & i = j. \end{cases}
 \end{aligned}$$

Lusztig has pointed out that B coincides with the canonical basis $B(X_{\alpha_0})$ of \mathfrak{g} corresponding to highest weight vector X_{α_0} (if α_0 is the highest root). This can be proved as follows. To check that X_α lies in $B(X_{\alpha_0})$ note that α is conjugate to α_0 by some element of the Weyl group. Then [L4, 28.1.4] gives an explicit formula for the canonical basis element of weight α and it is seen to coincide with X_α . To show that the t_i lie in $B(X_{\alpha_0})$ pick an element b of the canonical basis of \mathbb{U}^- such that $X_{\alpha_i} = b.X_{\alpha_0}$. It follows that $b \notin F_i.\mathbb{U}^-$. Now applying [L4, 14.3.2(c)] and the positivity property of the canonical basis of \mathbb{U}^- we find that

$$F_i.b = b' + \text{elements in } F_i^2.\mathbb{U}^-,$$

where $b' \in \mathbb{U}^-$ is again an element of the canonical basis. Then we see that

$$t_i = F_i.X_{\alpha_i} = F_i.b.X_{\alpha_0} = b'.X_{\alpha_0} + 0.$$

Therefore t_i lies in $B(X_{\alpha_0})$ and the proof is complete.

We note that, following [L1, Proof of Proposition 3.2], one can also show directly that B has the property of Theorem 3.1, i.e. that for all $g \in G_{\geq 0}$ the matrix entries of $g : V \rightarrow V$ with respect to B lie in $\mathbb{R}_{\geq 0}$. In fact, this property need only be checked for the generators $\exp(aE_i), \exp(aF_i)$, and $\chi(b)$ ($a \in \mathbb{R}_{\geq 0}, b \in \mathbb{R}_{> 0}$) of $G_{\geq 0}$, where it can be deduced from the observation that the E_i and F_i act by matrices with nonnegative entries with respect to B .

4. PROOF OF PROPOSITION 2.1

Assume first that G is simple, as above, and consider the basis of the adjoint representation from 3.3. We choose an element $i_0 \in I$ and rescale the basis such that $X_{\alpha_{i_0}} = dx_{i_0}$. The resulting basis is the canonical basis for the rescaled highest weight vector and is again denoted by B .

Definition and Lemma 4.1. For $i \in I$ define $\sigma_i \in \mathbb{R}$ by $X_{\alpha_i} = \sigma_i dx_i$. Then

- (1) $\sigma_i = \pm 1$,
- (2) σ_i alternates along edges of the Dynkin diagram.

Proof. If i and $j \in I$ are connected by an edge, then

$$[X_{\alpha_i}, X_{\alpha_j}] = [\sigma_i dx_i, X_{\alpha_j}] = \sigma_i X_{\alpha_i + \alpha_j},$$

wherefore

$$[X_{\alpha_j}, X_{\alpha_i}] = -\sigma_i X_{\alpha_i + \alpha_j} = -\sigma_i [dx_j, X_{\alpha_i}].$$

This implies $X_{\alpha_j} = -\sigma_i dx_j$. So $\sigma_j = -\sigma_i$, and (2) follows. Since in every connected component of the Dynkin diagram there is an i_0 with $\sigma_{i_0} = 1$, (1) also follows. \square

Remark 4.2. We recall the following two facts about finite type, simply laced root systems:

- (1) Let $\alpha \in R^+$. Either α is simple or there are a positive root α' and a simple reflection S_{α_i} such that $S_{\alpha_i} \alpha' = \alpha' + \alpha_i = \alpha$.
- (2) For any $\lambda = \sum_{i \in I} \lambda_i \alpha_i \in R$, the set of vertices $\text{Supp}(\lambda) := \{i \in I | \lambda_i \neq 0\}$ is connected in the Dynkin diagram.

4.3 Lemma. Let $\alpha = \sum_{j \in I} n_j \alpha_j$ be a positive root of height m . Then

$$(**) \quad [X_\alpha, X_{-\alpha}] = (-1)^{m+1} \sum_{j \in I} \sigma_j n_j t_j.$$

Proof. If $ht(\alpha) = 1$, say $\alpha = \alpha_i$, then

$$[X_\alpha, X_{-\alpha}] = \sigma_i E_i \cdot X_{-\alpha_i} = \sigma_i t_i.$$

We prove the general case by induction. Let $ht(\alpha) = m \geq 2$. Using (4.2.1), pick $\alpha' \in R^+$ such that $\alpha = \alpha' + \alpha_i = S_{\alpha_i} \alpha'$ for some $i \in I$. Then

$$\begin{aligned} [X_\alpha, X_{-\alpha}] &= [E_i \cdot X_{\alpha'}, X_{-\alpha}] = E_i \cdot [X_{\alpha'}, X_{-\alpha}] - [X_{\alpha'}, X_{-\alpha'}] \\ &= E_i \cdot [X_{\alpha'}, F_i \cdot X_{-\alpha'}] - [X_{\alpha'}, X_{-\alpha'}] = (E_i F_i - 1) \cdot [X_{\alpha'}, X_{-\alpha'}]. \end{aligned}$$

Denote by I_i the set of vertices $j \in I \setminus \{i\}$ adjacent to i in the Dynkin diagram. If $\alpha' = \sum_j n'_j \alpha_j$, then

$$n_i = \left(\sum_{j \in I_i} n'_j \right) - n'_i, \quad \text{and} \quad n_j = n'_j \quad (j \neq i).$$

Now (**) holds for α' , by induction hypothesis. So, using the above identities and Lemma 4.1,

$$\begin{aligned}
 [X_\alpha, X_{-\alpha}] &= (E_i F_i - 1) \cdot (-1)^m \sum_{j \in I} \sigma_j n'_j t_j \\
 &= (-1)^m \sigma_i n'_i t_i + (-1)^m \sum_{j \in I_i} \sigma_j n'_j (t_i - t_j) + (-1)^m \sum_{j \notin I_i \cup \{i\}} \sigma_j n'_j (0 - t_j) \\
 &= \left((-1)^m \sigma_i n'_i + (-1)^m \sum_{j \in I_i} \sigma_j n'_j \right) t_i - (-1)^m \sum_{j \neq i} \sigma_j n'_j t_j \\
 &= (-1)^m \sigma_i \left(n'_i - \sum_{j \in I_i} n'_j \right) t_i + (-1)^{m+1} \sum_{j \neq i} \sigma_j n'_j t_j \\
 &= (-1)^{m+1} \sum_{j \in I} \sigma_j n_j t_j.
 \end{aligned}$$

□

4.4 Proof of Proposition 2.1. Let $X \in \mathfrak{g}$ satisfy (*). Write

$$X = \sum_{i \in I} \lambda_i dx_i + \sum_{i \in I} \lambda_{-i} dy_i + \sum_{\alpha \in R \setminus \pm \Pi} \mu_\alpha X_\alpha + t, \quad t \in \text{Lie}(T).$$

We must show that $\lambda_{\pm i} \geq 0$ and $\mu_\alpha = 0$. To do this we use the positivity properties of B established in 3.3.

- (1) $[X, X_{-\alpha_i}]$ has $\text{Lie}(T)$ -component $\lambda_i t_i$. Therefore, by Lemma 3.2, we have $\lambda_i \geq 0$. Similarly, considering the $\text{Lie}(T)$ -component of $[X, X_{\alpha_i}]$ gives that $\lambda_{-i} \geq 0$.
- (2) Write $\alpha = \sum_j n_j \alpha_j$. Consider $[X, X_{-\alpha}]$. Its $\text{Lie}(T)$ -component is

$$\mu_\alpha [X_\alpha, X_{-\alpha}] = \varepsilon \mu_\alpha \sum_j \sigma_j n_j t_j,$$

where $\varepsilon = \pm 1$ depending on the height of α (Lemma 4.3). Choose vertices $i, j \in \text{Supp}(\alpha)$ adjacent in the Dynkin diagram (4.2.2). By Lemma 3.2, $\varepsilon \mu_\alpha \sigma_j n_j \geq 0$ and $\varepsilon \mu_\alpha \sigma_i n_i \geq 0$. Since $\sigma_i = -\sigma_j$ (Lemma 4.1), this implies $\mu_\alpha = 0$.

This proves the proposition in the simple, simply laced case. The more general assertion of the proposition can be reduced to this special case by standard arguments.

Remark 4.5. Proposition 2.1 can also be proved using the theory of Lie semigroups. Let $U_{\geq 0}^+$ and $U_{\geq 0}^-$ be the subsemigroups of U^+ , resp. U^- , generated by $\{x_i(a) | a \in \mathbb{R}_{\geq 0}, i \in I\}$, resp. by $\{y_i(a) | a \in \mathbb{R}_{\geq 0}, i \in I\}$. Correspondingly define $\mathfrak{u}_{\geq 0}^+ = \sum_{i \in I} \mathbb{R}^+ dx_i$ in \mathfrak{u}^+ , and $\mathfrak{u}_{\geq 0}^- = \sum_{i \in I} \mathbb{R}^+ dy_i$ in \mathfrak{u}^- . Then one can check that the conditions for [HN, Prop. 1.43] with $G = U^\pm$, $W = \mathfrak{u}_{\geq 0}^\pm$ and $N = (U^\pm, U^\pm)$, are satisfied. From that proposition it follows that $\mathfrak{u}_{\geq 0}^\pm$ is the tangent cone to $U_{\geq 0}^\pm$. This result together with [L1, Lemma 2.3(b)] suffices to prove that $\mathfrak{g}_{\geq 0}$ is the tangent cone of $G_{\geq 0}$.

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