A CHARACTERIZATION OF RINGS IN WHICH EACH PARTIAL ORDER IS CONTAINED IN A TOTAL ORDER

STUART A. STEINBERG

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Abstract. Rings in which each partial order can be extended to a total order are called $O^*$-rings by Fuchs. We characterize $O^*$-rings as subrings of algebras over the rationals that arise by freely adjoining an identity or one-sided identity to a rational vector space $N$ or by taking the direct sum of $N$ with an $O^*$-field. Each real quadratic extension of the rationals is an $O^*$-field.

A ring $R$ is called an $O^*$-ring if each of its ring partial orders can be extended to a total order of $R$. Two of the problems in the list at the back of Fuch's book [4] concern $O^*$-rings.

(A) Establish ring theoretical properties of $O^*$-rings.

(B) Does there exist a polynomial identity which forces each totally ordered ring that satisfies it to be an $O^*$-ring?

These problems were perhaps motivated by the well-known fact that each torsion-free abelian group is an $O^*$-group. Recently, Kreinovich [7] has shown that (B) has a negative answer in the sense that if $f(x_1,\ldots,x_n)=0$ is such an identity, then each $O^*$-ring that satisfies it must be trivial; that is, $R^2=0$. In the process of showing this he noted that an $O^*$-ring has two very restrictive properties: it is algebraic over $\mathbb{Z}$ and each nilpotent element has index at most two. To see this first recall that the partial order in a partially ordered ring $R$ is determined by its positive cone $R^+=\{x\in R: x\geq 0\}$; we will refer to such a positive cone as a partial order of $R$. Now if $a$ is an element of an $O^*$-ring $R$ that is not algebraic over $\mathbb{Z}$, then $\mathbb{Z}^+[-a^2]$ is a partial order of $R$ which is not contained in any total order of $R$. Also, if $a\in R$ is nilpotent of index $n>2$ let $b=-a^{n-2}$ if $n$ is even and let $b=-a^{n-1}$ if $n$ is odd. Then $\mathbb{Z}^+b$ is a partial order of $R$ that is not contained in any total order of $R$.

Clearly, each subring of an $O^*$-ring $R$ is an $O^*$-ring, and its divisible hull $d(R)$ is also an $O^*$-ring. For if $P$ is a partial order of $d(R)$ and $T$ is a total order of $R$ which contains $P\cap R$, then $d(T) = \{x \in d(R) : \exists n > 0 \text{ with } nx \in T\}$ is a total order of $d(R)$ which contains $P$. Consequently, in this paper we will deal exclusively with algebras over the rationals $\mathbb{Q}$. All such $O^*$-algebras are determined in the

Theorem. If $R$ is an $O^*$-algebra, then there is a $\mathbb{Q}$-vector space $N$ such that $R$ is (isomorphic to) one of the following algebras.

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(i) $R = F \oplus N$ (algebra direct sum) where $F$ is a subfield of the reals that is algebraic over $\mathbb{Q}$ and $N^2 = 0$.

(ii) $R = \left( \begin{array}{c} \mathbb{Q} \\ N \\ 0 \\ 0 \end{array} \right)$ or the dual $\left( \begin{array}{c} \mathbb{Q} \\ 0 \\ N \\ 0 \end{array} \right)$.

(iii) $R = \left\{ \left( \begin{array}{cc} a & b \\ 0 & a \end{array} \right) : a \in \mathbb{Q}, b \in N \right\}$.

Moreover, each of these algebras is an $O^*$-algebra where in (i) $F$ is an $O^*$-field.

Proof. If $G$ and $H$ are po-groups with positive cones $P_G$ and $P_H$ respectively, then $G \oplus H$ will denote the po-group whose underlying group is the direct sum $G \oplus H$ and whose positive cone is $\{ g + h : 0 \neq h \in P_H, \text{ or } h = 0 \text{ and } g \in P_G \}$; and $G \oplus H$ will denote the same group ordered similarly but with $G$ dominating. The same arrow notation will be used for other lexicographic orderings even if there are more than two summands.

In a totally ordered ring the set $N$ of nilpotent elements is an ideal and the quotient modulo $N$ is a totally ordered domain [4, p.130]. Assume that $R^2 \neq 0$. Then $R$ has a nonzero idempotent $e$, and by Albert’s theorem [1] $R/N$ is a field. Since $R/N$ can be embedded in the real closure of $\mathbb{Q}$ [6, p.285] we may assume that it is a subfield of the reals.

Suppose first that $R$ is unital and $N \neq 0$. If $F = R/N$ is a proper extension of $\mathbb{Q}$ let $\{a_i : i \geq 1\}$ be a basis of $\mathbb{Q}F$ with $a_1 = 1$. Then $F = \oplus_{i \geq 1} \mathbb{Q}a_i$, is a totally ordered group. Since $N^2 = 0$ $N$ is a vector space over $F$. Let $0 \neq x \in N$. Then $F^+x$ is a partial order of $R$ and hence is contained in a total order $T$ of $R$. This total order induces a total order $T_F$ of the field $F$. Since $(F, T_F)$ is archimedean, $T_F \not\subseteq F^+$. Let $a \in T$ with $a + N \not\in F^+$. Then $(a + N)x = ax \in T \cap Fx = F^+x$ yields the contradiction that $a + N \not\in F^+$. Thus $F = Q$ and $R = \mathbb{Q}1 + N$ is isomorphic to a ring of type (iii).

Suppose now that $R$ is not unital. Since the left and right annihilator ideals of $R$ are convex ideals one of them is contained in the other. Suppose that the right annihilator $r(R)$ is contained in the left annihilator $l(R)$. According to [2, Theorem 9.4.15] (also see [5, 2.4]) the Pierce decomposition of $R$ is $R = B \oplus C \oplus D$ where $B = eRe$, $D = r(R) = (1 - e)R$, $C = eR(1 - e)$ and $C \oplus D = l(R) = R(1 - e)$. Also, any total order of $R$ is of the form $(B \oplus C \oplus D)^+$. If $C \neq 0$ and $D \neq 0$, then a total order $(C \oplus D)^+$ of $C \oplus D$ could be extended to a total order of $R$. Thus one of $C$ or $D$ is zero but the other is nonzero. If $C = 0$ then $B$ and $D$ are ideals of $R$. If $B$ is not a field and $0 \neq b \in B$ with $b^2 = 0$ and $0 \neq d \in D$, then the partial order $(\mathbb{Z}b \oplus \mathbb{Z}d)^+$ could be extended to a total order of $R$. Thus $B$ is a field and $R$ is of type (i). Suppose then that $D = 0$. If $0 \neq b \in B$ with $b^2 = 0$, then, since in any total order of $R$ either $b \geq C$ or $-b \geq C$, we must have $bc = 0$. But then for $0 \neq c \in C$ there is a total order of $R$ containing $(\mathbb{Z}b \oplus \mathbb{Z}c)^+$. So $B$ is a field. By an argument similar to the one given when $R$ is unital we see that $B = \mathbb{Q}$. Thus $R$ is of type (ii).

We next show that each of these algebras is an $O^*$-algebra. If $R = F \oplus N$ is of type (i) and $P$ is a partial order of $R$, then $P_F = \{ \alpha \in F : \alpha + x \in P \text{ for some } x \in N \}$ is a partial order of $F$. For $P_F$ is closed under addition and multiplication; and if $\alpha + x$ and $-\alpha + y$ are in $P$ then $-\alpha^2 \in P \cap F$. Thus $\alpha = 0$ since $F$ is an $O^*$-field. Now, if $T_F$ is a total order of the field $F$ with $T_F \supseteq P_F$ and $T_N$ is a total order of the group $N$ with $T_N \supseteq P \cap N$, then $R = [(F, T_F) \oplus (N, T_N)]^+$ is a total order of $R$ which contains $P$. 

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Suppose that \( R \) is of type (iii) and that \( P \) is a partial order of \( R \). If \( x = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in P \) with \( a < 0 \) then we may assume that \( a = -1 \). But then \(-1 = x^2 + 2x \in P \) and this is impossible. So if \( T_N \) is a total order of the group \( \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} \)
which contains \( \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} \cap P \), then \( \mathbb{Q} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \left( \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} \right) = T_N \) gives a total order of \( R \) which contains \( P \). Similarly, each ring of type (ii) is an \( O^* \) ring.

It is interesting to note that the unique totally ordered (right or left) self-injective rings that are not unital are \( O^* \) rings \([8, \text{ Theorem 5.4}]\).

A well-known result of Serre’s \([4, \text{ p.117}]\) implies that a real algebraic extension of \( \mathbb{Q} \) is an \( O^* \) field if and only if each of the algebraic number fields that it contains is an \( O^* \) field.

**Example.** Each real quadratic extension of \( \mathbb{Q} \) is an \( O^* \) field.

To see this we may assume that \( F = \mathbb{Q}(\sqrt{e}) \) where \( e \in \mathbb{Z}^+ \) is square-free. Let \( P \) be a partial order of \( F \). By replacing \( P \) by \( \mathbb{Q}^+ P + \mathbb{Q}^+ \) we may assume that \( \mathbb{Q}^+ P \subseteq P \) and \( 1 \in P \). Now, \( F \) has exactly two total orders \([6, \text{ p. 287}]\): \( T_1 = F \cap \mathbb{R}^+ \) and \( T_2 = \{ a + b \sqrt{e} : a - b \sqrt{e} \in \mathbb{R}^+, a, b \in \mathbb{Q} \} \) where \( \mathbb{R}^+ \) is the total order of \( \mathbb{R} \).

All of the inequalities that subsequently appear will refer to this total order. If \( a + b \sqrt{e} \in P \), then \( \overline{a} = a - b \sqrt{e} \). We first note that

\[
\begin{align*}
(\ast) & & b \geq 0 & \iff \sqrt{e} \in P & \iff \overline{x} \in P & \iff x \in P \\
(\ast\ast) & & b < 0 & \iff -\sqrt{e} \in P & \iff x \in P \\

\end{align*}
\]

For \( b \sqrt{e} = x - a \in P \); so \( b \geq 0 \) (respectively, \( b < 0 \)) \( \iff \sqrt{e} \) (respectively, \( -\sqrt{e} \)) \( \in P \). Also, \( x^2 - ax = eb^2 + ab \sqrt{e} \in P \); so \( 1 + \frac{a}{eb} \sqrt{e}, -\left( 1 + \frac{b}{a} \sqrt{e} \right) \in P \), and consequently \( \frac{a^2 - b^2 e}{ab} \sqrt{e} = \left( \frac{a}{eb} - \frac{b}{a} \right) \sqrt{e} \in P \). Thus, \( a^2 - b^2 e < 0 \) in both cases. If \( x < 0 \) and also \( b > 0 \), then \( b \sqrt{e} < a^2 \); so \( b < 0 \). Trivially, \( b < 0 \) gives \( x < 0 \). The other case is similar.

Suppose that \( P \not\subseteq T_1, T_2 \). Then there are \( x \in P \setminus T_1 \) and \( y \in P \setminus T_2 \). So \( x = a + b \sqrt{e} < 0 \) and \( y = c + d \sqrt{e} \) with \( \overline{y} < 0 \); hence \( a < 0 \) or \( b < 0 \), and \( c < 0 \) or \( d > 0 \). We consider each of the four cases separately.

(I) \( a < 0 \) and \( c < 0 \). This case is impossible by \((\ast\ast)\) and \((\ast)\).

(II) \( a < 0 \) and \( d > 0 \). By \((\ast\ast)\) \( -\sqrt{e} \in P \) and hence \( c > 0 \). But then \( y_1 = -\sqrt{e} y = -de - c \sqrt{e} \in P \) and \( \overline{y}_1 = -de + c \sqrt{e} = \sqrt{e} \overline{y} < 0 \). This is case I.

(III) \( b < 0 \) and \( c < 0 \). After passing to \( P \) this is case II.

(IV) \( b < 0 \) and \( d > 0 \). To avoid the other cases \( a \geq 0 \) and \( c \geq 0 \). If \( a = 0 \) then \( -\sqrt{e} \in P \), and hence \( -\sqrt{e} y = -de - c \sqrt{e} \in P \); so \( c \sqrt{e} > d^2 e^2 \) by \((\ast)\). But \( c < d \sqrt{e} \) since \( \overline{y} < 0 \). Thus \( a > 0 \). If \( c = 0 \), then \( \sqrt{e} \in P \) and \( \sqrt{e} x = be + a \sqrt{e} \in P \); and hence \( y \in P \) gives case II. Thus \( c > 0 \) and \( xy = (ac + bde) + (ad + bd) \sqrt{e} \in P \) with \( ac + bde < 0 \), since \( a < -b \sqrt{e} \) and \( c < d \sqrt{e} \). By \((\ast)\) and \((\ast\ast)\) \( \sqrt{e} \in P \) or \( -\sqrt{e} \in P \). If the former holds then \( \sqrt{e} x = be + a \sqrt{e} \in P \); this is case II. If the latter holds, \( y_1 = -\sqrt{e} y = -de - c \sqrt{e} \in P \) and \( \overline{y}_1 < 0 \). This contradicts \((\ast)\).
This calculation actually gives the

Corollary. The following statements are equivalent for the quadratic extension $F = K(\sqrt{e})$ of the $O^*$-field $K$.

1. $F$ is an $O^*$-field.
2. $e$ is totally positive in $K$ (that is, $e$ is positive in each total order of $K$), and for each partial order $P$ of $F$ there is a total order $T$ of $K$ such that $PT$ is a partial order of $F$.
3. Each maximal partial order of $F$ contains $e$ and a total order of $K$.

Note that $\mathbb{Q}(\sqrt{e})$ is not an $O^*$-field if $0 < e \in \mathbb{Z}$ is square-free.

References

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Department of Mathematics, The University of Toledo, Toledo, Ohio 43606-3390
E-mail address: ssteinb@uoft02.utoledo.edu