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ON THE COBORDISM INVARIANCE OF THE INDEX OF DIRAC OPERATORS

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ABSTRACT. We describe a "tunneling" proof of the cobordism invariance of the index of Dirac operators.

The goal of this note is to present a very short proof of the cobordism invariance of the index. More precisely, if $\hat{D}$ is a Dirac operator on an odd dimensional manifold $\hat{M}$ with boundary $\partial\hat{M} = M$ then we show that the index of its restriction D to M is zero. The novelty of this proof consists in the fact that we provide an *explicit* isomorphism between the kernel and the cokernel of D. This map can be viewed as a sort of "propagator" (see Sect. 4).

1. THE SETTING

Consider the following collection of data.

(a) A compact, oriented, $(2n+1)$-dimensional Riemann manifold $(\hat{M}^{2n+1}, \hat{g})$ with boundary $\partial\hat{M} = M^{2n}$ such that $\hat{g}$ is a product metric near the boundary. We denote by s the longitudinal coordinate on a collar neighborhood of M. The various orientations are defined as in Figure 1.

(b) A bundle of complex self-adjoint Clifford modules $\hat{\mathcal{E}} \to \hat{M}$ (in the sense of [BGV]). The Clifford multiplication is denoted by

$$\hat{\mathbf{c}} : T^*\hat{M} \to \operatorname{End}(\hat{\mathcal{E}}).$$

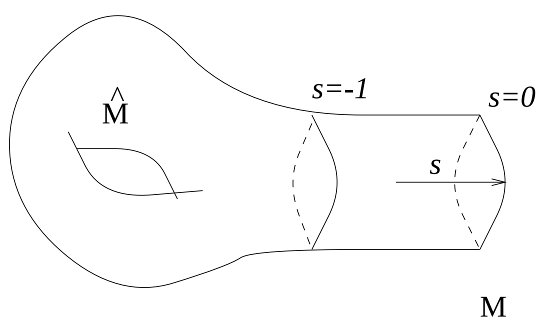

FIGURE 1. A $spin^c$ bordism

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Set $\mathcal{E} = \hat{\mathcal{E}}|_M$. We assume that

$$\text{End}\,(\mathcal{E}) \cong Cl(T^*M) \otimes \mathbb{C}, \tag{1}$$

i.e. M has a *spin*^{c} structure and $\mathcal{E}$ is in fact a bundle of complex spinors associated to this *spin*^{c} structure.

(c) A formally selfadjoint Dirac operator $\hat{D} : C^\infty(\hat{\mathcal{E}}) \to C^\infty(\hat{\mathcal{E}})$ such that along the neck it has the form

$$\hat{D} = \hat{\mathbf{c}}(ds)\,(\nabla_s + D) \tag{2}$$

where the operator

$$D : C^\infty(\hat{\mathcal{E}}|_{\{s\}\times M}) \to C^\infty(\hat{\mathcal{E}}|_{\{s\}\times M})$$

is formally selfadjoint and independent of s. We set $J = \hat{\mathbf{c}}(ds)$. Note that since both $\hat{D}$ and D are symmetric we have

$$\{J, D\} = 0 \tag{3}$$

where $\{\cdot,\cdot\}$ denotes the anticommutator of two operators.

Fix a local, oriented, orthonormal frame $(e^1, \cdots, e^{2n})$ of T^*M so that $(ds, e^1, \cdots, e^{2n})$ is an oriented orthonormal frame of $T^*\hat{M}$. If we denote by $\mathbf{c}$ the Clifford multiplication along the boundary then we have the equality

$$J\mathbf{c}(e^i) = \hat{\mathbf{c}}(ds)\mathbf{c}(e^i) = \hat{\mathbf{c}}(e^i)$$

and we can conclude from (2) that D is a Dirac operator on M with symbol $\mathbf{c}$.

We can regard $J|_M$ as an endomorphism of the bundle $\mathcal{E}$ and as such it satisfies the anticommutation relations

$$\{J, \mathbf{c}(e^i)\} = 0.$$

Using (1) we conclude that $\mathbf{i}J$ ($\mathbf{i} = \sqrt{-1}$) is a multiple of the chiral operator

$$\Gamma_{\mathcal{E}} = \mathbf{i}^n \mathbf{c}(e^1)\cdots\mathbf{c}(e^{2n}).$$

We fix the orientations such that $\mathbf{i}J = \Gamma$. Thus $\mathbf{i}J$ defines a $\mathbb{Z}_2$ grading on $\mathcal{E}$

$$\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-, \quad \mathcal{E}_\pm = \ker(\pm 1 - \mathbf{i}J).$$

The anticommutation equality (3) implies that D has a block decomposition

$$D = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}$$

where

$$D_\pm : C^\infty(\mathcal{E}_\pm) \to C^\infty(\mathcal{E}_\mp) \quad \text{and} \quad D_- = D_+^*.$$

Define

$$\text{ind}\ D = \dim\ker D_+ - \dim\ker D_.$$

We will show that ind $D = 0$ by explicitly producing an isometry $\ker D_+ \to \ker D_-$.

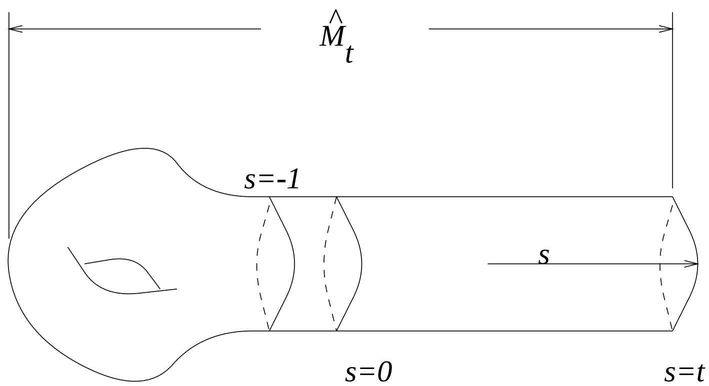

FIGURE 2. Stretching the neck

2. Cauchy data spaces and their adiabatic limits

Denote by $\hat{M}_t$ $(t \gg 0)$ the manifold obtained from $\hat{M}$ by attaching the long cylinder $[0,t] \times M$ (see Figure 2).

The bundle $\hat{\mathcal{E}}$ and the operator $\hat{D}$ have natural extensions $\hat{\mathcal{E}}_t$ and $\hat{D}_t$ to $\hat{M}_t$. For every $r \geq 0$ denote by $L^{r,2}$ the Sobolev space of distributions in L^2 with L^2-derivatives up to order r and set

$$\mathcal{K}_t = \{u \in L^{1/2,2}(\hat{\mathcal{E}}_t)\ ;\ \hat{D}_t u = 0\}.$$

In [BW] it is shown there exists a well defined continuous restriction map

$$r_t : \mathcal{K}_t \to L^2(\hat{\mathcal{E}}_t|_{\partial \hat{M}_t}).$$

The Cauchy data space of $\hat{D}_t$ is defined as

$$\Lambda_t = r_t(\mathcal{K}_t).$$

Λ_t is a closed subspace of $L^2(\mathcal{E})$ and satisfies a crucial condition (also established in [BW]), namely

$$\Lambda_t^{\perp} = J\Lambda_t.$$

The family $(\Lambda_t)_{t>0}$ has an especially nice behavior as $t \to \infty$. To describe it we need a bit more terminology.

For every interval $I \subset \mathbb{R}$ we denote by $\mathcal{H}_I$ the closed subspace of $L^2(\mathcal{E})$ spanned by the eigenvectors of D corresponding to eigenvalues in I. In [N1] we proved that there exist $E \geq 0$ and a D invariant subspace $L_\infty \subset \mathcal{H}_{[-E,E]}$ such that

$$JL_\infty = L_\infty^{\perp} \tag{4}$$

and

$$\Lambda_t \overset{t\to\infty}{\to} \Lambda_\infty = L_\infty \oplus \mathcal{H}_{(-\infty,-E]} \quad \text{in the} \quad \text{gap topology of [K].} \tag{5}$$

3. The cobordism invariance of the index

Denote by P_∞ the orthogonal projection onto Λ_∞ and denote by $R_\infty = 2P_\infty - 1$ the orthogonal reflection in Λ_∞. The condition (4) is equivalent to

$$\{R_\infty, J\} = 0. \tag{6}$$

Since $\mathcal{H}_{[-E,E]}$ is J-invariant (by (3)) we obtain a splitting

$$\mathcal{H}_{[-E,E]} = \mathcal{H}_E^+ \oplus \mathcal{H}_E^-$$

where $\mathcal{H}_E^\pm$ is the ± 1-eigenspace of $\mathbf{i}J$ on $\mathcal{H}_{[-E,E]}$. The equality (6) implies that R_∞ switches the components $\mathcal{H}_E^\pm$, i.e.

$$R_\infty(\mathcal{H}_E^\pm) = \mathcal{H}_E^\mp.$$

This "switch" is obviously an isomorphism. Note that

$$\ker D_\pm \subset \mathcal{H}_E^\pm.$$

Since L_∞ is also $D-invariant$ we deduce that R_∞ maps $\ker D_+$ isometrically onto $\ker D_-$. Geometrically this switch is the reflection in L_∞. This shows $\operatorname{ind} D = 0$.

Remark. In [N2] we use this adiabatic limit technique to establish the cobordism invariance of the index of *arbitrary families* of Dirac operators. The extreme generality of that situation may obscure some nice phenomena in special cases such as the one discussed below.

4. An example

Consider a cobordism as in Figure 3 and $\hat{D}$ a Dirac operator on $\hat{M}$ with all the properties listed in Section 1. The boundary operator D consists of two pieces $D_{\pm\infty}$ corresponding to the two components of the boundary and each is equipped with the induced chiral grading

$$D_{\pm\infty} = \begin{bmatrix} 0 & D_{\pm\infty}^- \\ D_{\pm\infty}^+ & 0 \end{bmatrix}.$$

We assume that $\ker D_{\pm\infty}^- = \{0\}$. Set

$$\mathcal{H} = \mathcal{H}_{-\infty} \oplus \mathcal{H}_\infty \stackrel{def}{=} \ker D_{-\infty}^+ \oplus \ker D_\infty^+.$$

Denote by $\mathcal{L}$ the space of extended L^2-solutions of $\hat{D}$ on the manifold $\hat{M}_\infty$ obtained by attaching infinite half-cylinders at its ends (we refer to [APS] for the exact definition). Then the space $L_\infty \cap \mathcal{H}$ described in Section 3 consists of the asymptotic values of these extended solutions and we can set $L_\infty \cap \mathcal{H} = \mathcal{L}|_{\partial\hat{M}}$. According to the considerations in Section 3 any $u \in \mathcal{L}|_{\partial\hat{M}}$ has an unique decomposition

$$u = u_+ + u_+, \quad u_\pm \in \mathcal{H}_{\pm\infty}. \tag{7}$$

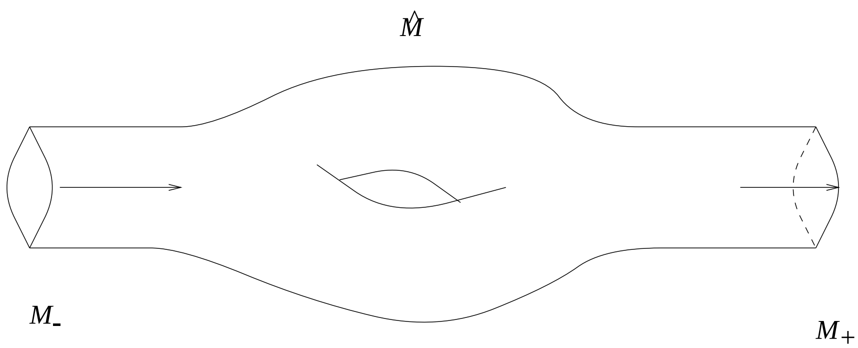

Figure 3. Cobordism

In other words $\mathcal{L}|_{\partial\hat{M}}$ is the graph of a linear operator (the "propagator")

$$\mathcal{P} : u_- \mapsto u_+.$$

The uniqueness of the decomposition (7) implies that the "propagator" $\mathcal{P}$ is an isomorphism $\mathcal{P} : \mathcal{H}_{-\infty} \to \mathcal{H}_{\infty}$.

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