

COLLAPSING SUCCESSORS OF SINGULARS

JAMES CUMMINGS

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ABSTRACT. Let κ be a singular cardinal in V , and let $W \supseteq V$ be a model such that $\kappa_V^+ = \lambda_W^+$ for some W -cardinal λ with $W \models \text{cf}(\kappa) \neq \text{cf}(\lambda)$. We apply Shelah's pcf theory to study this situation, and prove the following results.

- 1) W is not a κ^+ -c.c generic extension of V .
- 2) There is no "good scale for κ " in V , so in particular weak forms of square must fail at κ .
- 3) If $V \models \text{cf}(\kappa) = \aleph_0$ then $V \models \text{"}\kappa \text{ is strong limit} \implies 2^\kappa = \kappa^+\text{"}$, and also ${}^\omega \kappa \cap W \supseteq {}^\omega \kappa \cap V$.
- 4) If $\kappa = \aleph_\omega^V$ then $\lambda \leq (2^{\aleph_0})_V$.

1. INTRODUCTION

The results in this paper were motivated by the following natural question.

Question 1. Can there exist a pair of inner models of ZFC, V and W say, such that $V \subseteq W$ and $\aleph_{\omega+1}^V = \aleph_2^W$?

This question was raised by Bukovský and Copláková-Hartová [1] who observed (Theorem 1.7 of [1]) that if we have V and W as above then $W \models 2^{\aleph_0} \geq \aleph_2$, because $V \models \aleph_\omega^{\aleph_0} \geq \aleph_{\omega+1}$. They also note that if Question 1 has a positive answer then it is possible to violate CH by adding a real.

We can ask more general questions about the possibilities for collapsing successors of singulars; for example

Question 2. Let $V \models \text{"}\kappa \text{ is singular"}$. Can there be $W \supseteq V$ such that

$$W \models \text{"}\kappa_V^+ \text{ is a cardinal and } \text{cf}(\kappa) \neq \text{cf}(|\kappa|)\text{"}?$$

The restriction " $\text{cf}(\kappa) \neq \text{cf}(|\kappa|)$ " is not unreasonable, because for example it is easy to collapse $\aleph_{\omega,2+1}^V$ to $\aleph_{\omega+1}^{V[G]}$ by forcing with $\text{Coll}(\omega, \aleph_\omega)$.

Work of Shelah [10] shows that a positive solution to Question 1 or Question 2 would require the use of large cardinals.

Definition 1.1. Let κ be a cardinal. Then AD_κ holds iff there exists a sequence $\langle A_\alpha : \alpha < \kappa^+ \rangle$ such that

1. For each $\alpha < \kappa^+$, A_α is an unbounded subset of κ .
2. For each $\beta < \kappa^+$ there exists $g : \beta \rightarrow \kappa$ such that $\langle A_\alpha \setminus g(\alpha) : \alpha < \beta \rangle$ is a sequence of mutually disjoint sets.

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Shelah proved ([10], page 440) that

1. If κ is regular, or if κ is singular and \square_κ holds, then AD_κ holds.
2. If $V \models \text{AD}_\kappa$, $W \supseteq V$ and $W \models \text{“}\kappa_V^+ \text{ is a cardinal”}$, then $W \models \text{cf}(\kappa) = \text{cf}(|\kappa|)$.

In particular, if $V \models \text{AD}_\kappa + \text{cf}(\kappa) = \omega$ then $W \models \text{cf}(|\kappa|) = \omega$, so that in W the cardinal κ_V^+ must be \aleph_1^W or the successor of a cardinal of cofinality ω .

Remark 1.2. For κ singular the weak square principle \square_κ^* is enough to derive AD_κ . See [3] for a proof.

The connection with large cardinals arises because for κ singular the principle \square_κ^* holds unless there exist inner models of some very strong large cardinal axioms (at the level of Woodin cardinals). See [6], [7] and [8].

The results in this paper are more general than Shelah's in the sense that they can be applied in some situations where $V \models \neg \text{AD}_\kappa$. See [4] and [3] for a discussion of the implications between AD_κ , squares, scales and reflection principles.

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2. PCF THEORY, GOOD POINTS AND APPROACHABILITY

In this section we review some facts about pcf theory that will be used later. For a detailed development of pcf theory see [2] or [11]. All the facts quoted here are due to Shelah unless otherwise stated.

Definition 2.1. Let κ be a singular cardinal. A *scale for κ* (of length κ^+) is $(\vec{\kappa}, \vec{f})$ where

1. $\vec{\kappa} = \langle \kappa_i : i < \text{cf}(\kappa) \rangle$ is a strictly increasing sequence of regular cardinals with $\sup_i \kappa_i = \kappa$.
2. $\vec{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle$ is a sequence such that
 - (a) $f_\alpha \in \prod_i \kappa_i$.
 - (b) \vec{f} is strictly increasing and cofinal in $(\prod_i \kappa_i, <^*)$ where $<^*$ is the eventual domination relation on $\prod \kappa_i$.

Fact 2.2. If κ is singular then there exists a scale of length κ^+ for κ .

Definition 2.3. Let $(\vec{\kappa}, \vec{f})$ be a scale of length κ^+ for κ . $\alpha < \kappa^+$ is *good* (for \vec{f}) iff there exist $A \subseteq \alpha$ unbounded in α and $i < \text{cf}(\kappa)$ such that whenever $\beta, \gamma \in A$ with $\beta < \gamma$ and $j > i$ then $f_\beta(j) < f_\gamma(j)$.

Remark 2.4. If $\text{cf}(\alpha) < \text{cf}(\kappa)$ then it is easy to see that α is good.

Remark 2.5. If $\text{cf}(\alpha) > \text{cf}(\kappa)$, α being good is equivalent to $\langle f_\beta : \beta < \alpha \rangle$ having a least upper bound g such that $\text{cf}(g(i)) = \text{cf}(\alpha)$ for all large i .

Remark 2.6. The definition of *good point* actually makes sense for any $\vec{\kappa}$ and any increasing sequence from $(\prod_i \kappa_i, <^*)$.

Fact 2.7. If $\rho = \text{cf}(\rho) < \kappa$ and $(\vec{\kappa}, \vec{f})$ is a scale of length κ^+ for κ then

$$\{ \alpha < \kappa^+ : \text{cf}(\alpha) = \rho \text{ and } \alpha \text{ is good} \}$$

is stationary in κ^+ .

Since we will use some of the ideas later, we sketch a proof of this fact.

Proof. We prove it first when $\text{cf}(\kappa) < \rho$. Let $C \subseteq \kappa^+$ be club. Fix θ some large regular cardinal. Build an increasing and continuous sequence $\langle N_i : i < \rho \rangle$ such that

1. $N_i \prec (H_\theta, \in, <_\theta)$ where $<_\theta$ is some wellordering of H_θ .
2. $C, \vec{\kappa}, \vec{f} \in N_0$ and $\text{cf}(\kappa) \subseteq N_0$.
3. $|N_i| < \rho$.
4. $\langle N_i : i \leq j \rangle \in N_{j+1}$ for all $j < \rho$ (it will follow that $i \subseteq N_i$ and hence that $i < j \implies N_i \in N_j$).

Notice that if $i < j$ then $\text{sup}(N_i \cap \kappa^+) \in N_j$, so that $\langle \text{sup}(N_i \cap \kappa^+) : i < \rho \rangle$ is increasing and continuous. The limit of this sequence is $\text{sup}(\bigcup_i N_i \cap \kappa^+)$. It follows from elementarity that C is unbounded in $N_i \cap \kappa^+$, so that $\text{sup}(N_i \cap \kappa^+) \in C$.

Define $\chi_i \in \prod_n \kappa_n$ by setting $\chi_i(n) = 0$ for $\kappa_n < \rho$, $\chi_i(n) = \text{sup}(N_i \cap \kappa_n)$ for $\kappa_n \geq \rho$. Since $N_i \in N_{i+1}$ we see that $\chi_i \in N_{i+1}$, so that there exists $\alpha_i \in N_{i+1} \cap \kappa^+$ such that $\alpha_i > \text{sup}(N_i \cap \kappa^+)$ and $\chi_i <^* f_{\alpha_i}$. Since $\text{cf}(\kappa) \subseteq N_{i+1}$ we see that for all $n < \text{cf}(\kappa)$ we have $f_{\alpha_i}(\kappa_n) \in N_{i+1} \cap \kappa_n$, in particular we have $f_{\alpha_i} <^* \chi_{i+1}$.

Now for every $i < \rho$ we may choose $j < \text{cf}(\kappa)$ such that

$$k > j \implies \chi_i(k) < f_{\alpha_i}(k) < \chi_{i+1}(k).$$

Since $\rho > \text{cf}(\kappa)$ we may fix $A \subseteq \rho$ unbounded and $j < \text{cf}(\kappa)$ such that

$$\forall i \in A \forall k > j \chi_i(k) < f_{\alpha_i}(k) < \chi_{i+1}(k).$$

It follows that if $i_0, i_1 \in A$ with $i_0 < i_1$ then for $k > j$ we have

$$f_{\alpha_{i_0}}(k) < \chi_{i_0+1}(k) \leq \chi_{i_1}(k) < f_{\alpha_{i_1}}(k).$$

Now we know that $\text{sup}(N_i \cap \kappa^+) < \alpha_i < \text{sup}(N_{i+1} \cap \kappa^+)$, so this shows that $\text{sup}_i \alpha_i = \text{sup}_i \text{sup}(N_i \cap \kappa^+)$ is a good point. This point is also a point of C with cofinality ρ , so we have shown the set of good points of cofinality ρ to be stationary. This concludes the proof when $\rho > \text{cf}(\kappa)$.

To finish we observe that if A witnesses α to be a good point for \vec{f} , and β is a limit point of A , then $A \cap \beta$ witnesses that β is good for \vec{f} . Given $\rho = \text{cf}(\rho) \leq \text{cf}(\kappa)$ and a club set C , we first use the argument above to find $\alpha \in \lim(C)$ with $\text{cf}(\alpha) = \text{cf}(\kappa)^+$ and α good and choose A witnessing this; then we choose $\beta \in C$ a limit point of A with cofinality ρ , so that $A \cap \beta$ witnesses β is good. \square

Remark 2.8. It is easy to see that if $(\vec{\kappa}, \vec{f})$ and $(\vec{\kappa}, \vec{g})$ are two scales of length κ^+ for κ in the same product $\prod_i \kappa_i$, then the set of good points for \vec{f} is equal modulo clubs to the set of good points for \vec{g} .

Definition 2.9. A *good scale* for κ is a scale for κ in which, modulo the club filter, almost every point of cofinality greater than $\text{cf}(\kappa)$ is good.

Now we look at a form of weak square (the approachability square AP_κ) introduced by Shelah in [9]. AP_κ is substantially weaker than \square_κ^* .

Definition 2.10. AP_κ holds iff there exists $\langle C_\alpha : \alpha < \kappa^+ \rangle$ such that

1. C_α is a club subset of α of order type $\text{cf}(\alpha)$ for each limit α .
2. $C_{\gamma+1}$ is a subset of κ^+ for each successor ordinal $\gamma + 1$.
3. For a club of $\alpha < \kappa^+$, $\forall \beta < \alpha \exists \gamma < \alpha C_\alpha \cap \beta = C_{\gamma+1}$.

We are interested in this principle mainly because of the following fact, which is Claim 4.4 in [4].

Fact 2.11. *If AP_κ holds and $(\vec{\kappa}, \vec{f})$ is a scale then almost every point α with $\text{cf}(\alpha) > \text{cf}(\kappa)$ is good for \vec{f} . That is, every scale is good.*

We also collect here a few more facts about good points.

Fact 2.12. *Let $\text{cf}(\kappa) = \omega$ and κ be strong limit with $2^\kappa > \kappa^+$. Then there exists a scale $(\vec{\kappa}, \vec{f})$ for κ such that every point of uncountable cofinality is good for \vec{f} .*

Fact 2.13. *If $(\vec{\kappa}, \vec{f})$ is a scale for \aleph_ω and $\alpha < \aleph_{\omega+1}$, $\text{cf}(\alpha) > 2^{\aleph_0}$ then α is good for \vec{f} .*

3. THE MAIN LEMMA

In this section we will prove the key technical lemma of the paper, which connects the notion of “good point” with Questions 1 and 2 from the Introduction. We originally proved the lemma for points of cofinality λ , with the additional assumption that λ is regular in W ; Burke pointed out that it holds in the more general form given here.

Lemma 3.1. *Let V, W be inner models of set theory with $V \subseteq W$. Let κ be singular in V , and fix $(\vec{\kappa}, \vec{f})$ a scale of length κ^+ for κ . Suppose that $\kappa_V^+ = \lambda_W^+$ where $\lambda > \aleph_0$ and $W \models \text{cf}(\lambda) \neq \text{cf}(\kappa)$. Then there exists $\eta < \lambda$ such that for every $\delta \in \text{REG}^W \cap (\eta, \lambda]$*

$$\{ \gamma < \kappa_V^+ : W \models \text{“cf}(\gamma) = \delta \text{ and } \gamma \text{ is good for } \vec{f}\text{”} \}$$

is non-stationary in W .

Proof. We work in W . Since $\kappa_V^+ = \kappa_W^+$ we will just denote this cardinal by “ κ^+ ”. Observe that $\mu =_{\text{def}} \text{cf}_W(\kappa) < \lambda$ because λ is the largest cardinal less than κ^+ . $\text{cf}_W(\text{cf}_V(\kappa)) = \mu$, so we choose $C \subseteq \text{cf}_V(\kappa)$ a cofinal set of order type μ . $|\kappa| = \lambda$ so we may write $\kappa = \bigcup_{i < \lambda} X_i$ where \vec{X} is an increasing sequence of sets with $|X_i| < \lambda$ for all $i < \lambda$.

Now for all $\alpha < \kappa^+$, $\text{rge}(f_\alpha \upharpoonright C) \subseteq \kappa = \bigcup_i X_i$. Since $\mu \neq \text{cf}(\lambda)$, for every $\alpha < \kappa^+$ there is $i < \lambda$ such that $\{ n \in C : f_\alpha(n) \in X_i \}$ is unbounded in μ . Hence we can fix $i < \lambda$ such that the set

$$B =_{\text{def}} \{ \alpha < \kappa^+ : \{ n \in C : f_\alpha(n) \in X_i \} \text{ is unbounded in } \mu \}$$

is unbounded in κ^+ .

Now let $\eta = \max\{|X_i|, \mu\}$, and fix $\delta \in \text{REG}^W$ with $\eta < \delta \leq \lambda$. Since B is unbounded it has a club set of limit points; we claim that every limit point of B with cofinality δ is not good for \vec{f} . To see this, suppose for a contradiction that γ is a limit point of B of cofinality δ and that γ is good. This means that there is $A \subseteq \gamma$ unbounded and $m < \text{cf}_V(\kappa)$ such that if $\alpha, \beta \in A$, $\alpha < \beta$ and $m < n < \text{cf}_V(\kappa)$ then $f_\alpha(n) < f_\beta(n)$.

Now we build an increasing sequence of ordinals $\langle \gamma_j : j < \delta \rangle$ such that $\gamma_{2j} \in A$ and $\gamma_{2j+1} \in B$. Notice that $f_{\gamma_{2j}} <^* f_{\gamma_{2j+1}} <^* f_{\gamma_{2j+2}}$ for each $j < \delta$. For each j let us choose $n_j > m$ such that $n_j \in C$,

$$f_{\gamma_{2j}}(n_j) < f_{\gamma_{2j+1}}(n_j) < f_{\gamma_{2j+2}}(n_j),$$

and $f_{\gamma_{2j+1}}(n_j) \in X_i$.

Since $\delta > \mu = |C|$ we can find $Z \subseteq \delta$ unbounded and a fixed $n \in C$ such that for all $j \in Z$

$$f_{\gamma_{2j}}(n) < f_{\gamma_{2j+1}}(n) < f_{\gamma_{2j+2}}(n)$$

and $f_{\gamma_{2j+1}}(n) \in X_i$. Now if j, k are two elements of Z with $j < k$ then

$$f_{\gamma_{2j+1}}(n) < f_{\gamma_{2j+2}}(n) \leq f_{\gamma_{2k}}(n) < f_{\gamma_{2k+1}}(n).$$

This implies that $|\{ f_{\gamma_{2j+1}}(n) : j \in Z \}| = |Z| = \delta$. However for each $j \in Z$ we have $f_{\gamma_{2j+1}}(n) \in X_i$, and $|X_i| < \delta$, which is a contradiction. So almost every point of cofinality δ is not good and we have proved the claim of the lemma. \square

Remark 3.2. $REG^W \cap (\eta, \lambda]$ is always non-empty. If λ is regular in W then λ will do, and if λ is singular then λ is limit and we are guaranteed many suitable δ .

4. EXPLOITING THE MAIN LEMMA

Throughout this section our running assumptions are that $V \subseteq W$, $\kappa_V^+ = \lambda_W^+$, $V \models \text{“}\kappa \text{ is a singular cardinal”}$, and $W \models \text{“}\text{cf}(\kappa) \neq \text{cf}(\lambda)\text{”}$.

Theorem 1. *W is not a κ^+ -c.c. extension of V .*

Proof. Fix $(\vec{\kappa}, \vec{f})$ a scale for κ . Applying Lemma 3.1, let $\delta \in REG^W$ be such that in W the set of good points with cofinality δ is non-stationary.

Let us define

$$S = \{ \gamma < \kappa^+ : V \models \text{“}\text{cf}(\gamma) = \delta \text{ and } \gamma \text{ is good for } \vec{f}\text{”} \}.$$

By Lemma 2.7, $V \models \text{“}S \text{ is stationary”}$. Since δ is regular in W we know that

$$V \models \text{“}\text{cf}(\gamma) = \delta \text{ and } \gamma \text{ is good”} \implies W \models \text{“}\text{cf}(\gamma) = \delta \text{ and } \gamma \text{ is good”},$$

and so by the choice of δ we have $W \models \text{“}S \text{ is nonstationary”}$. Since any κ^+ -c.c. forcing preserves stationary subsets of κ^+ , W is not a κ^+ -c.c. generic extension of V . \square

We originally proved the following result under the additional assumptions that λ is regular and $V \models \text{cf}(\kappa) < \lambda$; Burke pointed out that it is true in the general form given here.

Theorem 2. *$V \models \text{“}There \text{ is no good scale of length } \kappa^+ \text{ for } \kappa\text{”}$.*

Proof. Let $(\vec{\kappa}, \vec{f})$ be a good scale for κ in V , and let C be a club in κ^+ such that

$$V \models \text{“every point of } C \text{ with cofinality greater than } \text{cf}(\kappa) \text{ is good”}.$$

Let B be the unbounded set constructed in the proof of Lemma 3.1, and let $\delta \in REG^W$ be such that every limit point of B with W -cofinality δ is not good and $\delta > \text{cf}_W(\kappa)$.

Choose $\gamma \in C \cap \text{acc}(B)$ such that $W \models \text{cf}(\gamma) = \delta$. By the choice of B and δ , γ cannot be good. On the other hand $W \models \text{cf}(\gamma) = \delta > \text{cf}(\kappa)$ so that $V \models \text{cf}(\gamma) \neq \text{cf}(\kappa)$; if $V \models \text{cf}(\gamma) < \text{cf}(\kappa)$ then γ is good by Remark 2.4 and if $V \models \text{cf}(\gamma) > \text{cf}(\kappa)$ then γ is good by the choice of C . Contradiction! \square

Theorem 3. *Let $V \models \text{cf}(\kappa) = \omega$. Then*

1. $V \models \text{“}\kappa \text{ is strong limit} \implies 2^\kappa = \kappa^+\text{”}$.
2. If $\lambda = \aleph_1^W$ then ${}^\omega \kappa \cap W \supseteq {}^\omega \kappa \cap V$.

Proof. The first claim follows immediately from Fact 2.12 and Theorem 2. For the second claim, suppose for a contradiction that ${}^\omega\kappa \cap W = {}^\omega\kappa \cap V$ and fix (in V) a scale $(\vec{\kappa}, \vec{f})$ of length κ^+ for κ .

Now $W \models \text{cf}(\kappa_n) > \omega$ for all $n < \omega$ and $W \models \text{“}\vec{f} \text{ is cofinal in } \prod_n \kappa_n\text{”}$. This is enough for us to imitate the proof of Lemma 2.7 and to prove (by building in W an \aleph_1 -chain of countable structures) that

$$W \models \text{“}\{ \gamma < \kappa^+ : \text{cf}(\gamma) = \aleph_1 \text{ and } \gamma \text{ is good } \} \text{ is stationary”}.$$

This contradicts Lemma 3.1. \square

Theorem 4. *If $\kappa = \aleph_\omega^V$ then $\lambda \leq (2^{\aleph_0})_V$.*

Proof. Fix $(\vec{\kappa}, \vec{f})$ a scale of length κ^+ for κ . Recall that by Fact 2.13 all points of cofinality greater than 2^{\aleph_0} are good for \vec{f} .

Suppose for a contradiction that $\lambda > (2^{\aleph_0})_V$. Using Lemma 3.1 find γ such that $W \models \text{cf}(\gamma) > (2^{\aleph_0})_V$, and γ is not good for \vec{f} . Now $V \models \text{cf}(\gamma) > (2^{\aleph_0})_V$, contradicting 2.13. \square

5. CONCLUDING REMARKS

The problems discussed in this paper are connected with some other open problems in set theory.

5.1. Chang’s conjecture. The consistency of “ $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_2, \aleph_1)$ ” is open. If this holds then $S =_{\text{def}} \{ X \subseteq \aleph_{\omega+1} : \text{ot}(X) = \aleph_2 \}$ is stationary, and supposing that there exists a Woodin cardinal δ we can force with Woodin’s stationary tower forcing [12] below the condition S . It is easy to see that $S \Vdash \aleph_{\omega+1}^V = \aleph_2^{V[G]}$.

5.2. Good points. Let $(\vec{\kappa}, \vec{f})$ be a scale of length $\aleph_{\omega+1}$ for \aleph_ω . It is open whether the set of points $\{ \alpha < \aleph_{\omega+1} : \text{cf}(\alpha) > \aleph_1 \text{ and } \alpha \text{ is not good } \}$ can ever be stationary. If this set cannot be stationary then Lemma 3.1 implies that $\aleph_{\omega+1}$ can only be collapsed to \aleph_1 or \aleph_2 .

This problem is also connected with “ $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_2, \aleph_1)$ ”, because if this holds then by Claim 4.3 in [4] there is a stationary set of cofinality \aleph_2 points which are not good. It is known [4] that if MM holds or $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$ then the set $\{ \alpha < \aleph_{\omega+1} : \text{cf}(\alpha) = \aleph_1 \text{ and } \alpha \text{ is not good } \}$ is stationary. The statement “ $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$ ” is known [5] to be consistent.

5.3. Saturated ideals. Burke and Matsubara have pointed out that the results in this paper can be used to show that the nonstationary ideal on $\mathcal{P}_\kappa\lambda$ is not saturated for certain values of κ and λ .

REFERENCES

- [1] L. Bukovský and E. Copláková-Hartová, Minimal collapsing extensions of models of ZFC. *Annals of Pure and Applied Logic*, 46:265–298, 1990. MR **92e**:03077
- [2] M. Burke and M. Magidor, Shelah’s pcf theory and its applications. *Annals of Pure and Applied Logic*, 50:207–254, 1990. MR **92f**:03053
- [3] J. Cummings, M. Foreman and M. Magidor. Scales, squares and stationary reflection. In preparation.
- [4] M. Foreman and M. Magidor, A very weak square principle. To appear.
- [5] J. P. Levinski, M. Magidor and S. Shelah, Chang’s conjecture for \aleph_ω . *Israel Journal of Mathematics*, 69:161–172, 1990. MR **91g**:03071

- [6] W. J. Mitchell and E. Schimmerling, Covering without countable closure. *Mathematical Research Letters*, 2:595–609, 1995. MR **96k**:03123
- [7] W. J. Mitchell, E. Schimmerling, and J. R. Steel, The Covering Lemma up to a Woodin cardinal. To appear in *Annals of Pure and Applied Logic*.
- [8] E. Schimmerling, Combinatorial principles in the core model for one Woodin cardinal. *Annals of Pure and Applied Logic*, 74:153–201, 1995. MR **96f**:03041
- [9] S. Shelah, On successors of singulars. In *Logic Colloquium '78*, pp 357–380, Amsterdam, 1979, North-Holland. MR **82d**:03079
- [10] S. Shelah, Proper Forcing. Berlin, 1982, Springer-Verlag. MR **84h**:03002
- [11] S. Shelah, Cardinal Arithmetic. Oxford, 1994, Oxford University Press. MR **96e**:03001
- [12] H. Woodin, Supercompact cardinals, sets of reals and weakly homogeneous trees. *Proceedings of the National Academy of Sciences of the USA*, 85:6587–6591. MR **89m**:03040

DEPARTMENT OF MATHEMATICS 2-390, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

E-mail address: `cummings@math.mit.edu`

Current address: Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213-3890

E-mail address: `jcumming@andrew.cmu.edu`