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COMMUTATIVE GROUP ALGEBRAS OF σ-SUMMABLE ABELIAN GROUPS

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Abstract. In this note we study the commutative modular and semisimple group rings of σ-summable abelian p-groups, which group class was introduced by R. Linton and Ch. Megibben. It is proved that $S(RG)$ is σ-summable if and only if G_p is σ-summable, provided G is an abelian group and R is a commutative ring with 1 of prime characteristic p, having a trivial nilradical. If G_p is a σ-summable p-group and the group algebras RG and RH over a field R of characteristic p are R-isomorphic, then H_p is a σ-summable p-group, too. In particular $G_p \cong H_p$ provided G_p is totally projective of a countable length.

Moreover, when K is a first kind field with respect to p and G is p-torsion, $S(KG)$ is σ-summable if and only if G is a direct sum of cyclic groups.

Let p be a fixed prime. Throughout this paper G is an abelian group, $G^{p^\omega} = \bigcap_{n<\omega} G^{p^n}$ is its first p-Ulm subgroup (a first Ulm subgroup with respect to p) and $G[p]$ is the socle of the p-primary component $G_p = \bigcup_{n<\omega} G[p^n]$.

In this note we investigate the commutative group algebras of σ-summable abelian p-groups. We obtain some characterizations of σ-summability for normed p-torsion components of abelian group rings. For this we need certain definitions (cf. [5]) and observations.

If λ is a limit ordinal, we call a p-group G a C_λ-group provided G/G^{p^α} is totally projective for all $\alpha < \lambda$. Thus every abelian p-group is a C_ω-group. Recall that the length of a reduced p-group G is just the smallest ordinal λ such that $G^{p^\lambda} = 1$. If $\lambda = \omega$, we call such an abelian p-group G separable, i.e. the p-primary G is said to be separable if $G^{p^\omega} = 1$. We shall say that a reduced abelian p-group G is σ-summable if its socle $G[p]$ is the ascending union of a sequence of subgroups $\{S_n\}_{n<\omega}$, where for each n there is an ordinal α_n less than the length of G such that $S_n \cap G^{p^{\alpha_n}} = 1$. Thus the reduced abelian p-group G is σ-summable (by Linton-Megibben [5]) if and only if $G[p] = \bigcup_{n<\omega} S_n$, where $S_n \subseteq S_{n+1}$ and $S_n \cap G^{p^{\alpha_n}} = 1$ for every natural n and some ordinal $\alpha_n < \operatorname{length} G$. This definition for σ-summability, as a generalization of the classical Kulikov criterion for direct sums of cyclic p-groups, possesses the following properties: Some subgroups and direct sums of σ-summable groups are σ-summable; the separable abelian p-group G is σ-summable if and only

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if it is a direct sum of cyclic groups; all summable groups with countable limit lengths are σ-summable; a totally projective p-group G of a limit length cofinal with ω is σ-summable.

More generally, the next statement is valid.

Theorem (Linton-Megibben [5])**.** *Let λ be a limit ordinal cofinal with ω. Then a p-group G of length λ is totally projective if and only if G is a σ-summable C_λ-group.*

Thus, if G is a σ-summable p-group, then G is not totally projective when G is not a C_λ-group. Furthermore, if G and H are σ-summable, then $G \ncong H$ is possible, even assuming that the Ulm-Kaplansky functions of G and H are equal (it is well-known that $G \cong H$ when both G and H are totally projective). The major open question, when is $G \cong H$, provided G and H are σ-summable, is interesting. Is it true if and only if $G[p]$ and $H[p]$ are isomorphic as valuated vector spaces (i.e. if and only if there is an isometry (a height-preserving isomorphism) of $G[p]$ onto $H[p]$)?

Now, because each countable limit ordinal is cofinal with ω, owing to the above theorem and following step-by-step the idea for proof in [7], we close the introduction with the following

Criterion. Suppose G is an abelian p-group of countable length λ. Then G is totally projective if and only if G/G^{p^α} is σ-summable for all limit $\alpha \leq \lambda$.

Modular group algebras of σ-summable abelian groups

First and foremost we denote by R a commutative ring with identity of prime characteristic p and by $S(RG)$ the normed p-component in a group ring RG. Define $M = M[N(R); \Pi(G/H)] = \{\sum_{g\in\Pi} r_g(1-g) \mid r_g \in N(R)\}$, where $N(R) = \bigcup_{n<\omega} R(p^n)$ is the nilradical (the Baer radical) of R with p^n-socles $R(p^n) = \{r \in R \mid r^{p^n} = 0\}$ and $\Pi = \Pi(G/H)$ is a complete system of representatives of G with respect to the subgroup $H \subseteq G$, containing the identity of G. Let $I = I(RG; H)$ be the relative augmentation ideal of RG with respect to H.

The following preliminary conclusions are well-known (see [3] and [4]), but for completeness and for the convenience of the reader we give proofs. We start with the following

Main Lemma.

$$(*) \qquad S(RG)[p] = 1 + I(RG; G[p]) \oplus M[R(p); \Pi(G/G[p])],$$

$$(**) \qquad S(RG) = 1 + I(RG; G_p) \oplus M[N(R); \Pi(G/G_p)].$$

Proof. Certainly, $S(RG) = 1 + I_p(RG; G)$, where $I_p(RG;G) \stackrel{def}{=} \{x \in I(RG;G) \mid x^{p^n} = 0\}$, hence it remains only to show that $I_p(RG;G) = I(RG;G_p) \oplus M[N(R); \Pi(G/G_p)]$.

Actually, for this purpose, given $x \in I_p(RG;G)$, clearly $x = \sum_{g\in\Pi}\sum_{h\in G_p} x_{gh} \cdot gh$, where $x_{gh} \in R$, $\sum_{h\in G_p} x_{gh} = r_g \in N(R)$, and $\sum_{g\in\Pi}\sum_{h\in G_p} x_{gh} = \sum_{g\in\Pi} r_g = 0$. Thus $x = \sum_{g\in\Pi}\sum_{h\in G_p\setminus\{1\}} x_{gh}\cdot g(h-1) + \sum_{g\in\Pi\setminus\{1\}} r_g(g-1) \in I + M$, since $x_{gh}\cdot g \in RG$. Besides, if $y \in I \cap M$, then $y = \sum_{g\in\Pi}\sum_{h\in G_p} x_{gh}\cdot gh$, along with the conditions $r_g = 0$ for each $g \in \Pi$ and $h = 1$. Therefore $x_{gh} = 0$ and $y = 0$, i.e. $I \cap M = 0$ implies $x \in I \oplus M$. The proof is complete.

Lemma 1. *For all ordinals* α

$$(***) \qquad S^{p^\alpha}(RG) = S(R^{p^\alpha}G^{p^\alpha})$$

is valid.

Proof. By a standard transfinite induction, we only need consider $\alpha = 1$. Now, in this case clearly $S^p(RG) \subseteq S(R^pG^p)$.

Conversely, take $x \in S(R^pG^p)$, hence $x = \sum_i r_i^p g_i^p$, where $r_i \in R$, $g_i \in G$ and besides $\sum_i r_i^p = 1$. Thus $x = \sum_i r_i^p g_i^p - \sum_i r_i^p + 1 = (1 - \sum_i r_i(1-g_i))^p = y^p \in S^p(RG)$, since $y = 1 - \sum_i r_i(1-g_i) \in S(RG)$. So, $S(R^pG^p) \subseteq S^p(RG)$, as required. The proof is finished.

Lemma 2. $S(RG) = 1$ *if and only if* $N(R) \neq 0$ *and* $G = 1$*, or* $N(R) = 0$ *and* $G_p = 1$.

Proof. If $N(R) = 0$ and $G_p = 1$, then $M = 0$ and $I = 0$. Consequently $S(RG) = 1$ by the Main Lemma. Conversely, if $S(RG) = 1$, then apparently $G_p = 1$ and $\Pi = 1$, when $N(R) \neq 0$. Therefore $G = G_p = 1$, as desired.

Proposition 3. *If* $N(R) = 0$*, then*

($\circ$) $S(RG)$ *is reduced if and only if* G_p *is reduced,*

($\circ\circ$) $S(RG)$ *is separable if and only if* G_p *is separable.*

Proof. ($\circ$) Lemma 1 implies that the maximal divisible subgroup of $S(RG)$ is precisely $S(PG^*)$, where P is the maximal perfect subring of R and G^* is the maximal p-divisible subgroup of G. But, it is evident that the maximal divisible subgroup of G_p is equal to $(G^*)_p$. So, $(G^*)_p = 1$ and Lemma 2 yields $S(PG^*) = 1$, since $N(P) = 0$.

($\circ\circ$) $(G^{p^\omega})_p = (G_p)^{p^\omega} = 1$, therefore $S^{p^\omega}(RG) = S(R^{p^\omega}G^{p^\omega}) = 1$ by Lemmas 1 and 2. The proposition is true.

Lemma 4. *If* A *and* H *are subgroups of* G *and* L *is a subring of* R *with the same identity, then*

$$(\circ\circ\circ) \qquad \begin{aligned} &[1 + I(RG; H)] \cap S(LA) \subseteq 1 + I(LA; A \cap H), \\ &[1 + I(RG; H)] \cap G_pS(LA) \subseteq G_p[1 + I(LA; A \cap H)]. \end{aligned}$$

Proof. Let $x \in [1 + I(RG; H)] \cap S(LA)$, hence $x = \sum_{a \in A} x_a a$, $x_a \in L$, and

$$\sum_{a \in bH} x_a = \begin{cases} 1, & b \in H, \\ 0, & b \notin H, \end{cases} \quad \text{for each } b \in A.$$

But $b(H \cap A) = bH \cap A$ because $b \in A$. Thus

$$\sum_{a \in b(H \cap A)} x_a = \begin{cases} 1, & b \in H \cap A, \\ 0, & b \notin H \cap A, \end{cases} \quad \text{for every } b \in A,$$

and obviously $x \in 1 + I(LA; A \cap H)$.

To prove the second inequality, choose x in the left-hand side. Hence $x = g_p \sum_{a \in A} x_a a$, where $x_a \in L$, $g_p \in G_p$ and

$$\sum_{a \in bH} x_a = \begin{cases} 1, & b \in H, \\ 0, & b \notin H, \end{cases} \quad \text{for each } b \in A.$$

Therefore as above we see that x lies in $G_p[1 + I(LA; A \cap H)]$. The statement is shown.

The summability for $S(RG)$ in a modular aspect is discussed in [3]. Now we are in position to prove

Theorem. *Let R be a ring without nilpotent elements. Then $S(RG)$ is σ-summable if and only if G_p is σ-summable. Besides, if G_p is σ-summable of a limit length, then $S(RG)/G_p$ is σ-summable.*

Proof. If $S(RG)$ is σ-summable, then G_p is the same since it is a subgroup with equal length.

Now we treat the more difficult converse question. For this, suppose that G_p is σ-summable. Hence G_p and $S(RG)$ are both reduced by Proposition 3 and besides $\operatorname{length} S(RG) = \operatorname{length} G_p$ applying Lemmas 1 and 2. Moreover $G[p] = \bigcup_{n<\omega} G_n$, where $G_n \subseteq G_{n+1}$ and $G_n \cap G^{p^{\alpha_n}} = 1$ for ordinals α_n strictly less than the length of G_p. From the Main Lemma, $S(RG)[p] = 1 + I(RG; G[p])$, hence $S(RG)[p] = \bigcup_{n<\omega}[1 + I(RG; G_n)]$. Moreover, we compute $[1 + I(RG; G_n)] \cap S^{p^{\alpha_n}}(RG) = [1 + I(RG; G_n)] \cap S(R^{p^{\alpha_n}} G^{p^{\alpha_n}}) = 1 + I(R^{p^{\alpha_n}} G^{p^{\alpha_n}}; G^{p^{\alpha_n}} \cap G_n) = 1$ for all α_n, using Lemmas 1 and 4. Finally $S(RG)$ is also σ-summable.

Further, since G_p is balanced in $S(RG)$ [3] we derive

$$(S(RG)/G_p)[p] = S(RG)[p]G_p/G_p = \bigcup_{n<\omega} [(1 + I(RG; G_n))G_p/G_p].$$

Moreover using Lemmas 1 and 4 we calculate

$$\begin{aligned}
&[(1 + I(RG; G_n))G_p/G_p] \cap (S(RG)/G_p)^{p^{\alpha_n}} \\
&\quad = [(1 + I(RG; G_n))G_p/G_p] \cap [S(R^{p^{\alpha_n}} G^{p^{\alpha_n}})G_p/G_p] \\
&\quad = [[(1 + I(RG; G_n))G_p] \cap [S(R^{p^{\alpha_n}} G^{p^{\alpha_n}})G_p]]/G_p \\
&\quad = G_p[(1 + I(RG; G_n)) \cap [S(R^{p^{\alpha_n}} G^{p^{\alpha_n}})G_p]]/G_p \\
&\quad = G_p(1 + I(R^{p^{\alpha_n}} G^{p^{\alpha_n}}; G^{p^{\alpha_n}} \cap G_n))/G_p = 1.
\end{aligned}$$

Besides using the fact that G_p is nice in $S(RG)$ [3], it is elementary to verify that $S(RG)/G_p$ is reduced and $\operatorname{length}(S(RG)/G_p) = \operatorname{length} G_p$. Finally, $S(RG)/G_p$ must be σ-summable, as stated. This completes the proof.

Problem. Is $S(RG)/G_p$ totally projective assuming G_p is σ-summable of a limit length (in particular of a length cofinal with ω) and $N(R) = 0$? Moreover, is G_p a direct factor of $S(RG)$?

Corollary 5 ([2, 3, 4])**.** *If $N(R) = 0$, then $S(RG)$ is a direct sum of cyclic groups if and only if G_p is a direct sum of cyclic groups. Besides if G_p is a direct sum of cyclics, then G_p is a direct factor of $S(RG)$ with a complement which is a direct sum of cyclics.*

Proof. By the Theorem and Proposition 3, G_p is separable σ-summable if and only if $S(RG)$ is separable σ-summable. But this is a direct sum of cyclic groups. Further, if G_p is a direct sum of cyclics, then again the Theorem implies that $S(RG)/G_p$ is separable σ-summable, i.e. a direct sum of cyclics. But G_p is pure in $S(RG)$ and so from a well-known fact due to L. Kulikov in the abelian group theory, G_p is a direct factor of $S(RG)$. The proof is fulfilled.

More recently, W. May in [6] has proved that, if G is a totally projective p-group and R is a perfect field (eventually R is a perfect ring with $N(R) = 0$), then $S(RG)$ is totally projective. We can now state

Proposition 6. *If G_p is totally projective with a limit length cofinal with ω and $N(R) = 0$, then $S(RG)$ is σ-summable.*

Proof. By the Linton-Megibben theorem, G_p is σ-summable and we need only apply the central theorem. So, the assertion holds.

In the work [6], W. May also proves that, if G is totally projective p-primary and $RH \cong RG$ as R-algebras for any group H, then H is also totally projective p-primary, and therefore $H \cong G$, because the corresponding group cardinal invariants of Ulm-Kaplansky of H and G are equal.

Now we conclude an assertion analogous to the above.

Proposition 7. *If G_p is σ-summable and H is a group so that $RH \cong RG$ as R-algebras, then H_p is σ-summable.*

Proof. We can assume that R is a field, hence $S(RG) \cong S(RH)$, and the Theorem yields the result immediately.

In particular, as a corollary, if G is a σ-summable p-group and $RG \cong RH$ as R-algebras, then H is one also. But whether $G \cong H$ is unknown.

We shall begin in this paragraph with the following

Main Proposition. *Suppose $RG \cong RH$. Then for each ordinal α, the isomorphism $R(G/G_p^{p^\alpha}) \cong R(H/H_p^{p^\alpha})$ holds.*

Proof. Really, we may assume that R is a field and that $RG = RH$, where $H \subseteq V(RG)$ is a normalized group basis in the group $V(RG)$ of all normed units. Therefore in view of the Main Lemma, $S(RG) = 1 + I(RG; G_p) = 1 + I(RH; H_p) = S(RH)$. Hence $I(RG; G_p) = I(RH; H_p)$ and obviously

$$I^{p^\alpha}(RG; G_p) = I(R^{p^\alpha} G^{p^\alpha}; G_p^{p^\alpha}) = I(R^{p^\alpha} H^{p^\alpha}; H_p^{p^\alpha}) = I^{p^\alpha}(RH; H_p)$$

for every ordinal α. But then

$$I(RG; G_p^{p^\alpha}) = RG \cdot I(R^{p^\alpha} G^{p^\alpha}; G_p^{p^\alpha}) = RH \cdot I(R^{p^\alpha} H^{p^\alpha}; H_p^{p^\alpha}) = I(RH; H_p^{p^\alpha}).$$

So, finally we derive

$$R(G/G_p^{p^\alpha}) \cong RG/I(RG; G_p^{p^\alpha}) = RH/I(RH; H_p^{p^\alpha}) \cong R(H/H_p^{p^\alpha}),$$

as stated.

We are ready to attack

Proposition 8. *Suppose G_p is totally projective of a countable length. Then $RH \cong RG$ as R-algebras for any group H implies $H_p \cong G_p$.*

Proof. Indeed, we may harmlessly presume that length G_p is limit and thus according to the modified criterion of Linton-Megibben for total projectivity, given by us in the introduction, $G_p/G_p^{p^\alpha} = (G/G_p^{p^\alpha})_p$ is σ-summable for all limit $\alpha \leq \operatorname{length} G_p$. But the Main Proposition and Proposition 7 yield that $(H/H_p^{p^\alpha})_p = H_p/H_p^{p^\alpha}$ is σ-summable. By applications of Lemmas 1 and 2 when R is a field, it is a routine matter to see that $\operatorname{length} S(RG) = \operatorname{length} G_p$. As a consequence, $RG \cong RH$ does imply $\operatorname{length} G_p = \operatorname{length} H_p$ and so the criterion for total projectivity cited above is applicable to obtain that H_p is totally projective. But by the well-known and documented classical result of May, the R-isomorphism $RG \cong RH$ implies that G_p and H_p have the same functions of Ulm-Kaplansky. Thus, $G_p \cong H_p$ as claimed.

Remark. Proposition 8 partially solves a problem posed by W. May in [6].

Semisimple group algebras of σ-summable abelian groups

Denote by K the field of the first kind with respect to p with a p-torsion part $U_p(K)$, and by $S(KG)$ and $U_p(KG)$ the p-components in the group algebra KG. The following formula is well-known:

$$U_p(KG) = S(KG) \times U_p(K). \qquad (\diamond)$$

The summability for $S(KG)$ and $U_p(KG)$ in a semisimple aspect are discussed in [1] and [3], respectively. Now we can state

Theorem. *Let G be a p-group.*

($\diamond\diamond$) *$S(KG)$ is σ-summable if and only if G is a direct sum of cyclic groups.*

($\diamond\diamond\diamond$) *$U_p(KG)$ is σ-summable if and only if G is a direct sum of cyclic groups.*

Proof. In fact, if G is a direct sum of cyclic groups, then both $S(KG)$ and $U_p(KG)$ are direct sums of cyclics by formula ($\diamond$), since $U_p(K)$ is cyclic (cf. [1]). Thus they are σ-summable. Conversely, if $S(KG)$ or $U_p(KG)$ are σ-summable, then $S(KG)$ is reduced, i.e. its maximal divisible subgroup, equal to $S^{p^\omega}(KG)$ (cf. [1]), is trivial. Hence $S(KG)$ is separable σ-summable, i.e. in other words $S(KG)$ is a direct sum of cyclics, i.e. G is a direct sum of cyclics. The theorem is verified.

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