COMMUTATIVE GROUP ALGEBRAS
OF \( \sigma \)-SUMMABLE ABELIAN GROUPS

PETER DANCHEV

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Abstract. In this note we study the commutative modular and semisimple group rings of \( \sigma \)-summable abelian \( p \)-groups, which group class was introduced by R. Linton and Ch. Megibben. It is proved that \( S(RG) \) is \( \sigma \)-summable if and only if \( G_p \) is \( \sigma \)-summable, provided \( G \) is an abelian group and \( R \) is a commutative ring with 1 of prime characteristic \( p \), having a trivial nilradical. If \( G_p \) is a \( \sigma \)-summable \( p \)-group and the group algebras \( RG \) and \( RH \) over a field \( R \) of characteristic \( p \) are \( R \)-isomorphic, then \( H_p \) is a \( \sigma \)-summable \( p \)-group, too.

In particular \( G_p \cong H_p \) provided \( G_p \) is totally projective of a countable length.

Moreover, when \( K \) is a first kind field with respect to \( p \) and \( G \) is \( p \)-torsion, \( S(RG) \) is \( \sigma \)-summable if and only if \( G \) is a direct sum of cyclic groups.

Let \( p \) be a fixed prime. Throughout this paper \( G \) is an abelian group, \( G^{p\omega} = \bigcap_{n<\omega} G^{p^n} \) is its first \( p \)-Ulm subgroup (a first Ulm subgroup with respect to \( p \)) and \( G[p] \) is the socle of the \( p \)-primary component \( G_p = \bigcup_{n<\omega} G[p^n] \).

In this note we investigate the commutative group algebras of \( \sigma \)-summable abelian \( p \)-groups. We obtain some characterizations of \( \sigma \)-summability for normed \( p \)-torsion components of abelian group rings. For this we need certain definitions (cf. [5]) and observations.

If \( \lambda \) is a limit ordinal, we call a \( p \)-group \( G \) a \( C_\lambda \)-group provided \( G/G^{p\omega} \) is totally projective for all \( \alpha < \lambda \). Thus every abelian \( p \)-group is a \( C_\omega \)-group. Recall that the length of a reduced \( p \)-group \( G \) is just the smallest ordinal \( \lambda \) such that \( G^{p\lambda} = 1 \). If \( \lambda = \omega \), we call such an abelian \( p \)-group \( G \) separable, i.e. the \( p \)-primary \( G \) is said to be separable if \( G^{p\omega} = 1 \). We shall say that a reduced abelian \( p \)-group \( G \) is \( \sigma \)-summable if its socle \( G[p] \) is the ascending union of a sequence of subgroups \( \{S_n \}_{n<\omega} \), where for each \( n \) there is an ordinal \( \alpha_n \) less than the length of \( G \) such that \( S_n \cap G^{p^{\alpha_n}} = 1 \).

Thus the reduced abelian \( p \)-group \( G \) is \( \sigma \)-summable (by Linton-Megibben [5]) if and only if \( G[p] = \bigcup_{n<\omega} S_n \), where \( S_n \subseteq S_{n+1} \) and \( S_n \cap G^{p^{\alpha_n}} = 1 \) for every natural \( n \) and some ordinal \( \alpha_n < \text{length } G \). This definition for \( \sigma \)-summability, as a generalization of the classical Kulikov criterion for direct sums of cyclic \( p \)-groups, possesses the following properties: Some subgroups and direct sums of \( \sigma \)-summable groups are \( \sigma \)-summable; the separable abelian \( p \)-group \( G \) is \( \sigma \)-summable if and only

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if it is a direct sum of cyclic groups; all summmable groups with countable limit lengths are \(\sigma\)-summmable; a totally projective \(p\)-group \(G\) of a limit length cofinal with \(\omega\) is \(\sigma\)-summmable.

More generally, the next statement is valid.

**Theorem** (Linton-Megibbon [5]). Let \(\lambda\) be a limit ordinal cofinal with \(\omega\). Then a \(p\)-group \(G\) of length \(\lambda\) is totally projective if and only if \(G\) is a \(\sigma\)-summmable \(C_\lambda\)-group.

Thus, if \(G\) is a \(\sigma\)-summmable \(p\)-group, then \(G\) is not totally projective when \(G\) is not a \(C_\lambda\)-group. Furthermore, if \(G\) and \(H\) are \(\sigma\)-summmable, then \(G \not\cong H\) is possible, even assuming that the Ulm-Kaplansky functions of \(G\) and \(H\) are equal (it is well-known that \(G \cong H\) when both \(G\) and \(H\) are totally projective). The major open question, when is \(G \cong H\), provided \(G\) and \(H\) are \(\sigma\)-summmable, is interesting. Is it true if and only if \(G[p]\) and \(H[p]\) are isomorphic as valued vector spaces (i.e. if and only if there is an isometry (a height-preserving isomorphism) of \(G[p]\) onto \(H[p]\))?  

Now, because each countable limit ordinal is cofinal with \(\omega\), owing to the above theorem and following step-by-step the idea for proof in [7], we close the introduction with the following

**Criterion.** Suppose \(G\) is an abelian \(p\)-group of countable length \(\lambda\). Then \(G\) is totally projective if and only if \(G/G^{\omega^\alpha}\) is \(\sigma\)-summmable for all limit \(\alpha \leq \lambda\).

**Modular group algebras of \(\sigma\)-summmable abelian groups**

First and foremost we denote by \(R\) a commutative ring with identity of prime characteristic \(p\) and by \(S(RG)\) the normed \(p\)-component in a group ring \(RG\). Define \(M = M[N(R); \Pi(G/H)] = \{\sum_{g \in \Pi} r_g (1 - g) \mid r_g \in N(R)\}\), where \(N(R) = \bigcup_{n < \omega} R(p^n)\) is the nilradical (the Baer radical) of \(R\) with \(p^\alpha\)-socles \(R(p^n) = \{r \in R \mid r^{p^n} = 0\}\) and \(\Pi = \Pi(G/H)\) is a complete system of representatives of \(G\) with respect to the subgroup \(H \subseteq G\), containing the identity of \(G\). Let \(I = I(RG; H)\) be the relative augmentation ideal of \(RG\) with respect to \(H\).

The following preliminary conclusions are well-known (see [3] and [4]), but for completeness and for the convenience of the reader we give proofs. We start with the following

**Main Lemma.**

\[
\begin{align*}
(*) & \quad S(RG)[p] = 1 + I(RG; G[p]) \oplus M[N(R); \Pi(G/G[p])], \\
(**) & \quad S(RG) = 1 + I(RG; G_p) \oplus M[N(R); \Pi(G/G_p)].
\end{align*}
\]

**Proof.** Certainly, \(S(RG) = 1 + I_p(RG; G)\), where \(I_p(RG; G) \overset{\text{def}}{=} \{x \in I(RG; G) \mid x^{p_\alpha} = 0\}\), hence it remains only to show that \(I_p(RG; G) = I(RG; G_p) \oplus M[N(R); \Pi(G/G_p)]\).

Actually, for this purpose, given \(x \in I_p(RG; G)\), clearly \(x = \sum_{g \in \Pi} \sum_{h \in G_p} x_{gh} \cdot gh\), where \(x_{gh} \in R, \sum_{h \in G_p} x_{gh} = r_g \in N(R)\), and \(\sum_{g \in \Pi} \sum_{h \in G_p} x_{gh} = \sum_{g \in \Pi} r_g = 0\). Thus \(x = \sum_{g \in \Pi} \sum_{h \in G_p \setminus \{1\}} x_{gh} \cdot gh \cdot (g(h - 1) + \sum_{g \in \Pi \setminus \{1\}} r_g (g - 1) \in I + M\), since \(x_{gh} \cdot g \in RG\). Besides, if \(y \in I \cap M\), then \(y = \sum_{g \in \Pi} \sum_{h \in G_p} x_{gh} \cdot gh\), along with the conditions \(r_g = 0\) for each \(g \in \Pi\) and \(h = 1\). Therefore \(x_{gh} = 0\) and \(y = 0\), i.e. \(I \cap M = 0\) implies \(x \in I \oplus M\). The proof is complete.
Let $G$.

By a standard transfinite induction, we only need consider $\alpha = 1$. Now, in this case clearly $S^p(RG) \subseteq S(R^pG^p)$.

Conversely, take $x \in S(R^pG^p)$, hence $x = \sum_i r_i^p g^p_i$, where $r_i \in R, g_i \in G$ and besides $\sum_i r_i^p = 1$. Thus $x = \sum_i r_i^p g^p_i - \sum_i r_i^p + 1 = (1 - \sum_i r_i(1 - g_i))^p = y^p \in S^p(RG)$, since $y = 1 - \sum_i r_i(1 - g_i) \in S(RG)$. So, $S(R^pG^p) \subseteq S^p(RG)$, as required. The proof is finished.

**Lemma 2.** $S(RG) = 1$ if and only if $N(R) \neq 0$ and $G = 1$, or $N(R) = 0$ and $G_p = 1$.

**Proof.** If $N(R) = 0$ and $G_p = 1$, then $M = 0$ and $I = 0$. Consequently $S(RG) = 1$ by the Main Lemma. Conversely, if $S(RG) = 1$, then apparently $G_p = 1$ and $\Pi = 1$, when $N(R) \neq 0$. Therefore $G = G_p = 1$, as desired.

**Proposition 3.** If $N(R) = 0$, then

(\circ) \quad S(RG)$ is reduced if and only if $G_p$ is reduced,

(\circ\circ) \quad S(RG)$ is separable if and only if $G_p$ is separable.

**Proof.** (\circ) Lemma 1 implies that the maximal divisible subgroup of $S(RG)$ is precisely $S(PG^*)$, where $P$ is the maximal perfect subring of $R$ and $G^*$ is the maximal $p$-divisible subgroup of $G$. But, it is evident that the maximal divisible subgroup of $G_p$ is equal to $(G^*)_p$. So, $(G^*)_p = 1$ and Lemma 2 yields $S(PG^*) = 1$, since $N(P) = 0$.

(\circ\circ) $(G^*)^p = (G_p)^p = 1$, therefore $S^p(G^*) = S(R^pG^p) = 1$ by Lemmas 1 and 2. The proposition is true.

**Lemma 4.** If $A$ and $H$ are subgroups of $G$ and $L$ is a subring of $R$ with the same identity, then

(\circ\circ\circ) \quad [1 + I(RG;H)] \cap S(LA) \subseteq 1 + I(LA;A \cap H),

\quad [1 + I(RG;H)] \cap G_pS(LA) \subseteq G_p[1 + I(LA;A \cap H)].

**Proof.** Let $x \in [1 + I(RG;H)] \cap S(LA)$, hence $x = \sum_{a \in A} x_a a, x_a \in L$, and

$$\sum_{a \in bH} x_a = \begin{cases} 1, & b \in H, \\ 0, & b \notin H, \end{cases} \text{ for each } b \in A.$$  

But $b(H \cap A) = bH \cap A$ because $b \in A$. Thus

$$\sum_{a \in b(b(H \cap A))} x_a = \begin{cases} 1, & b \in H \cap A, \\ 0, & b \notin H \cap A, \end{cases} \text{ for every } b \in A,$$

and obviously $x \in 1 + I(LA;A \cap H)$.

To prove the second inequality, choose $x$ in the left-hand side. Hence $x = g_p \sum_{a \in A} x_a a$, where $x_a \in L, g_p \in G_p$ and

$$\sum_{a \in bH} x_a = \begin{cases} 1, & b \in H, \\ 0, & b \notin H, \end{cases} \text{ for each } b \in A.$$
Therefore as above we see that $x$ lies in $G_p[1 + I(LA; A \cap H)]$. The statement is shown.

The summability for $S(RG)$ in a modular aspect is discussed in [3]. Now we are in position to prove

**Theorem.** Let $R$ be a ring without nilpotent elements. Then $S(RG)$ is $\sigma$-summable if and only if $G_p$ is $\sigma$-summable. Besides, if $G_p$ is $\sigma$-summable of a limit length, then $S(RG)/G_p$ is $\sigma$-summable.

**Proof.** If $S(RG)$ is $\sigma$-summable, then $G_p$ is the same since it is a subgroup with equal length.

Now we treat the more difficult converse question. For this, suppose that $G_p$ is $\sigma$-summable. Hence $G_p$ and $S(RG)$ are both reduced by Proposition 3 and besides length $S(RG) = \text{length } G_p$ applying Lemmas 1 and 2. Moreover $G[p] = \bigcup_{n<\omega} G_n$, where $G_n \subseteq G_{n+1}$ and $G_n \cap G^{\alpha n} = 1$ for ordinals $\alpha_n$ strictly less than the length of $G_p$. From the Main Lemma, $S(RG)[p] = 1 + I(G_p; G[p])$, hence $S(RG)[p] = \bigcup_{n<\omega} [1 + I(G_p; G_n)]$. Moreover, we compute $[1 + I(G_p; G_n)] \cap S^{\alpha n}(RG) = 1 + I(G_p; G_n) \cap (RG) = 1$ for all $\alpha_n$, using Lemmas 1 and 4. Finally $S(RG)$ is also $\sigma$-summable.

Further, since $G_p$ is balanced in $S(RG)$ [3] we derive

$$S(RG)/G_p[p] = S(RG)[p]G_p/G_p = \bigcup_{\alpha < \omega} [(1 + I(G_p; G_n))G_p/G_p].$$

Moreover using Lemmas 1 and 4 we calculate

$$[(1 + I(G_p; G_n))G_p/G_p] \cap (S(RG)/G_p)^{\alpha n} = [(1 + I(G_p; G_n))G_p/G_p] \cap [S(R^{\alpha n}G^{\alpha n})G_p/G_p] = [(1 + I(G_p; G_n))G_p/G_p] \cap [S(R^{\alpha n}G^{\alpha n})G_p/G_p]$$

$$= G_p[(1 + I(G_p; G_n)) \cap [S(R^{\alpha n}G^{\alpha n})G_p]/G_p]$$

$$= G_p(1 + I(R^{\alpha n}G^{\alpha n}; G^{\alpha n} \cap G_n)/G_p).$$

Besides using the fact that $G_p$ is nice in $S(RG)$ [3], it is elementary to verify that $S(RG)/G_p$ is reduced and length$(S(RG)/G_p) = \text{length } G_p$. Finally, $S(RG)/G_p$ must be $\sigma$-summable, as stated. This completes the proof.

**Problem.** Is $S(RG)/G_p$ totally projective assuming $G_p$ is $\sigma$-summable of a limit length (in particular of a length cofinal with $\omega$) and $N(R) = 0$? Moreover, is $G_p$ a direct factor of $S(RG)$?

**Corollary 5 ([2, 3, 4]).** If $N(R) = 0$, then $S(RG)$ is a direct sum of cyclic groups if and only if $G_p$ is a direct sum of cyclic groups. Besides if $G_p$ is a direct sum of cyclics, then $G_p$ is a direct factor of $S(RG)$ with a complement which is a direct sum of cyclics.

**Proof.** By the Theorem and Proposition 3, $G_p$ is separable $\sigma$-summable if and only if $S(RG)$ is separable $\sigma$-summable. But this is a direct sum of cyclic groups. Further, if $G_p$ is a direct sum of cyclics, then again the Theorem implies that $S(RG)/G_p$ is separable $\sigma$-summable, i.e. a direct sum of cyclics. But $G_p$ is pure in $S(RG)$ and so from a well-known fact due to L. Kulikov in the abelian group theory, $G_p$ is a direct factor of $S(RG)$. The proof is fulfilled.
More recently, W. May in [6] has proved that, if $G$ is a totally projective $p$-group and $R$ is a perfect field (eventually $R$ is a perfect ring with $N(R) = 0$), then $S(RG)$ is totally projective. We can now state

**Proposition 6.** If $G_p$ is totally projective with a limit length cofinal with $\omega$ and $N(R) = 0$, then $S(RG)$ is $\sigma$-summable.

**Proof.** By the Linton-Megibben theorem, $G_p$ is $\sigma$-summable and we need only apply the central theorem. So, the assertion holds.

In the work [6], W. May also proves that, if $G$ is totally projective $p$-primary and $RH \cong RG$ as $R$-algebras for any group $H$, then $H$ is also totally projective $p$-primary, and therefore $H \cong G$, because the corresponding group cardinal invariants of Ulm-Kaplansky of $H$ and $G$ are equal.

Now we conclude an assertion analogous to the above.

**Proposition 7.** If $G_p$ is $\sigma$-summable and $H$ is a group so that $RH \cong RG$ as $R$-algebras, then $H_p$ is $\sigma$-summable.

**Proof.** We can assume that $R$ is a field, hence $S(RG) \cong S(RH)$, and the Theorem yields the result immediately.

In particular, as a corollary, if $G$ is a $\sigma$-summable $p$-group and $RG \cong RH$ as $R$-algebras, then $H$ is one also. But whether $G \cong H$ is unknown.

We shall begin in this paragraph with the following

**Main Proposition.** Suppose $RG \cong RH$. Then for each ordinal $\alpha$, the isomorphism $R(G/G_p^{\alpha}) \cong R(H/H_p^{\alpha})$ holds.

**Proof.** Really, we may assume that $R$ is a field and that $RG = RH$, where $H \subseteq V(RG)$ is a normalized group basis in the group $V(RG)$ of all normed units. Therefore in view of the Main Lemma, $S(RG) = 1 + I(RG; G_p) = 1 + I(RH; H_p) = S(RH)$. Hence $I(RG; G_p) = I(RH; H_p)$ and obviously

$$I_p^\alpha(RG; G_p) = I(R^p\sigma G_p^{\sigma}; G_p^{\sigma}) = I(R^\sigma H_p^{\sigma}; H_p^{\sigma}) = I_p^\sigma(RH; H_p)$$

for every ordinal $\alpha$. But then

$$I(RG; G_p^{\sigma}) = RG \cdot I(R^p\sigma G_p^{\sigma}; G_p^{\sigma}) = RH \cdot I(R^\sigma H_p^{\sigma}; H_p^{\sigma}) = I(RH; H_p^{\sigma}).$$

So, finally we derive

$$R(G/G_p^{\sigma}) \cong RG/I(RG; G_p^{\sigma}) = RH/I(RH; H_p^{\sigma}) \cong R(H/H_p^{\sigma}),$$

as stated.

We are ready to attack

**Proposition 8.** Suppose $G_p$ is totally projective of a countable length. Then $RH \cong RG$ as $R$-algebras for any group $H$ implies $H_p \cong G_p$.

**Proof.** Indeed, we may harmlessly presume that length $G_p$ is limit and thus according to the modified criterion of Linton-Megibben for total projectivity, given by us in the introduction, $G_p/G_p^{\sigma} = (G/G_p^{\sigma})_p$ is $\sigma$-summable for all limit $\alpha \leq \text{length } G_p$.

But the Main Proposition and Proposition 7 yield that $(H/H_p^{\sigma})_p = H_p/H_p^{\sigma}$ is $\sigma$-summable. By applications of Lemmas 1 and 2 when $R$ is a field, it is a routine matter to see that length $S(RG) = \text{length } G_p$. As a consequence, $RG \cong RH$ does imply length $G_p = \text{length } H_p$ and so the criterion for total projectivity cited above
is applicable to obtain that $H_p$ is totally projective. But by the well-known and documented classical result of May, the $R$-isomorphism $RG \cong RH$ implies that $G_p$ and $H_p$ have the same functions of Ulm-Kaplansky. Thus, $G_p \cong H_p$ as claimed.

**Remark.** Proposition 8 partially solves a problem posed by W. May in [6].

**Semisimple group algebras of $\sigma$-summable abelian groups**

Denote by $K$ the field of the first kind with respect to $p$ with a $p$-torsion part $U_p(K)$, and by $S(KG)$ and $U_p(KG)$ the $p$-components in the group algebra $KG$. The following formula is well-known:

$$U_p(KG) = S(KG) \times U_p(K).$$

The summability for $S(KG)$ and $U_p(KG)$ in a semisimple aspect are discussed in [1] and [3], respectively. Now we can state

**Theorem.** Let $G$ be a $p$-group.

$$S(KG)$$ is $\sigma$-summable if and only if $G$ is a direct sum of cyclic groups.

$$U_p(KG)$$ is $\sigma$-summable if and only if $G$ is a direct sum of cyclic groups.

**Proof.** In fact, if $G$ is a direct sum of cyclic groups, then both $S(KG)$ and $U_p(KG)$ are direct sums of cyclics by formula $(\circ)$, since $U_p(K)$ is cyclic (cf. [1]). Thus they are $\sigma$-summable. Conversely, if $S(KG)$ or $U_p(KG)$ are $\sigma$-summable, then $S(KG)$ is reduced, i.e. its maximal divisible subgroup, equal to $S\omega^c(KG)$ (cf. [1]), is trivial. Hence $S(KG)$ is separable $\sigma$-summable, i.e. in other words $S(KG)$ is a direct sum of cyclics, i.e. $G$ is a direct sum of cyclics. The theorem is verified.

**References**


Department of Algebra, Plovdiv University, Plovdiv 4000, Bulgaria