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AN EXTENSION OF THE RABINOWITZ BIFURCATION THEOREM TO LIPSCHITZ POTENTIAL OPERATORS IN HILBERT SPACES

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Abstract. The main result of the paper is an extension of the bifurcation theorem of Rabinowitz to equations $Ax + \varphi_\lambda(x) = \lambda x$ with φ continuous jointly in (λ, x) and $\varphi_\lambda(\cdot)$ of class $C^{1,1}$. We also prove a bifurcation theorem for critical points of the function $g_\lambda(x)$ which is just continuous and changes at $x = 0$ an isolated minimum (in x) to isolated maximum when λ passes, say, zero. The proofs of the theorems, as well as the the theorems themselves, are new, in certain important aspects, even when applied to smooth functions.

1. Introduction

In what follows H is a real Hilbert space, $B_H(\alpha)$, or just $B(\alpha)$, is the closed ball of radius α around the origin in H and $(\cdot|\cdot)$ is the inner product in H. Consider the function

$$f_\lambda(x) = (1/2)(Ax|x) + \varphi_\lambda(x)$$

on $[-\alpha, \alpha] \times B(\alpha)$ assuming that

- (a_1)**:** A is a bounded self–adjoint operator on H;
- (a_2)**:** $\varphi(\lambda, x) = \varphi_\lambda(x)$ is a continuous function as well as its gradient $\nabla\varphi_\lambda(x)$ (with respect to x);
- (a_3)**:** $\nabla\varphi_\lambda(x)$ satisfies the Lipschitz condition with respect to x with the constant not depending on λ.

We denote the spectrum of A by $\sigma(A)$. Consider the equation

$$\nabla f_\lambda(x) = \lambda x. \tag{1}$$

Assuming that $x = 0$ satisfies the equation for all λ, we say that μ is a *bifurcation point* of (1) if there are sequences of $\lambda_n \to \mu$ and $x_n \to 0$, $x_n \neq 0$ such that (λ_n, x_n) satisfies (1) for all n. Our primary goal is to prove the following theorem.

Theorem 1. *Suppose that μ is an isolated eigenvalue of A of finite multiplicity and the Lipschitz constant of $\nabla\varphi$ with respect to x is strictly smaller than*

$$r := \inf\{|\lambda - \mu| : \lambda \in \sigma(A), \lambda \neq \mu\}. \tag{2}$$

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Suppose further that for all λ sufficiently close to μ

(3)
$$\nabla\varphi_\lambda(0) = 0; \quad \limsup_{x\to 0} \|x\|^{-2}(r^{-1}\|\nabla\varphi_\lambda(x)\|^2 + |\varphi_\lambda(x)|) < (1/2)|\lambda-\mu| \quad \text{if } \lambda \neq \mu.$$

Then μ is a bifurcation point of (1) and one of the following three possibilities holds:

(i) *$x = 0$ is not an isolated critical point of $f_\mu(\cdot)$;*
(ii) *for every λ in a neighborhood of μ there is a nontrivial solution $x(\lambda)$ of (1) converging to 0 as $\lambda \to \mu$;*
(iii) *there is a one-sided (right or left) neighborhood of μ such that for any $\lambda \neq \mu$ in the neighborhood, (1) has at least two nontrivial solutions converging to zero as $\lambda \to \mu$.*

For φ of class C^2 and not depending on λ this is the same statement as in the classical theorem of Rabinowitz [1] which concluded a series of studies initiated by Krasnosel'ski in the 50s [2]. Rabinowitz's work was preceded by Böhme [3] and Marino [4] who showed that for all sufficiently small $\rho > 0$ there are at least two nontrivial solutions of (1) satisfying $|x| = \rho$, $|\lambda - \mu| \leq \rho$. The latter result was extended by McLeod and Turner [5] to functions φ of class $C^{1,1}$ depending linearly on two parameters and such that the Lipschitz constant of the gradient going to zero as the parameters and x go to zero. (Their paper also contains interesting examples when a Lipschitz rather than smooth perturbation appears in (1).) In all these results it was assumed that the Lipschitz constant of $\nabla\varphi$ goes to zero as $x \to 0$. The latter property is strictly stronger than the last relation in (3) even if φ does not depend on λ.

We mention further a more recent paper by Kielhöffer [6] in which a bifurcation theorem was proved (for a C^2 case) for f in which the operator A also may depend on λ, that is to say, for C^1 potential operators which may depend arbitrarily on λ. However, unlike the theorem of Rabinowitz, the theorem of Kielhöffer only states the existence of a bifurcation and says nothing about the behavior of bifurcating solutions.

The novelty of this work lies not only in weakening of differentiability requirements on f and the requirements on the mode of the dependence of f on λ but also in the technique we use in the proofs which is new for the smooth case as well.

As many earlier proofs, ours consists of two parts: reduction to a function of the form

$$g_\lambda(x) = (\lambda/2)\|x\|^2 + \psi(\lambda, x) \tag{4}$$

and bifurcation analysis for (4). The new technology we apply to bifurcation analysis of (4) leads to a simpler and space saving proof in a very general situation when we actually assume ψ only continuous, even not Lipschitz. A suitable concept of a critical point and necessary techniques are provided by a recently developed "metric" critical point theory for continuous functions in complete metric spaces [7], [8], [9], [10]. In the heart of the techniques lies a Potential Well Theorem of [9] (Proposition 2 below) giving a priori estimates for the size of the potential well associated with a given local minimum, which leads, either jointly with a "nonsmooth" mountain pass theorem of [7], [10], [11] or without it, to short and transparent proofs of the (i) – (iii) alternative.

The next section contains all necessary information from the metric critical point theory. Section 3 is devoted to a bifurcation analysis of $g_\lambda(x)$. In fact even a more general class of functions is considered there: basically this is the bifurcation analysis of a function whose parametric dependence contains a change from a local minimum to a local maximum at the reference critical point. The final section contains a proof of Theorem 1.

2. Preliminaries

Let (X, d) be a complete metric space and $f(x)$ a continuous function on X.

Definition 1. We say that x is a δ–regular point of f if there are a neighborhood U of x and a continuous mapping $\omega_\lambda(u) = \omega(\lambda, u) : [0, 1] \times U \mapsto X$ such that for all $(\lambda, u) \in [0, 1] \times U$

(i) $\omega_0(u) = u$ and $\omega_\lambda(u) \neq u$ if $\lambda > 0$;
(ii) $f(u) - f(\omega_\lambda(u)) \geq \delta \cdot d(u, \omega_\lambda(u))$.

Slightly changing the notation of [8], we denote by $|df|^+(x)$ the least upper bound of all $\delta > 0$ such that f is δ-regular at x and set $|df|^+(x) = 0$ if f is not δ–regular at x for any positive δ. We set further $|df|(x) = \min\{|df|^+(x), |d(-f)|^+(x)\}$ and say that x is a *critical point* of f if $|df|(x) = 0$. Likewise, we call a sequence $\{x_n\}$ *critical* if $|df|(x_n) \to 0$. (Note that $|df|(x)$ is a lower semicontinuous function of x.) A local minimum (or maximum) is obviously a critical point. If X is a Banach space and f is strictly Fréchet differentiable at x (that is, $\|h\|^{-1}\|f(u+h) - f(u) - f'(u)h\| \to 0$ as $u \to x$ and $h \to 0$), then $|df|(x)$ coincides with the norm of the derivative of f at x. In other words, in this case the definition of a critical point reduces to the standard definition. If f satisfies a Lipschitz condition near x, then $|df|(x)$ is not smaller than the distance from zero to $\partial f(x)$ (Clarke's generalized gradient of f at x). Therefore in this case a necessary condition for an x to be a critical point is that $0 \in \partial f(x)$.

Definition 2 ([9])**.** Let $G \subset X$ be an open set and $z \notin G$. A positive nondecreasing function $\delta(t)$ on $(0, \infty)$ is a *modulus of regularity* of f on G with respect to z if $|df|(x) \geq \delta(d(x, z))$ for all $x \in G$.

Definition 3 ([7], [9], [11])**.** We say that f satisfies the *Palais–Smale condition* at the level c if any critical sequence $\{x_n\}$ such that $f(x_n) \to c$ has a converging subsequence. If any critical sequence $\{x_n\}$ such that the sequence of $f(x_n)$ is bounded has a converging subsequence, then we say that f satisfies the Palais–Smale condition (without indicating the level).

Proposition 1. *Let (X, d) be a complete metric space and f a continuous function on X satisfying the Palais–Smale condition. Let B be the closed ball of radius r centered at z such that z is the unique critical point of f in B. Then f has a modulus of regularity with respect to z defined on $G = \{x : 0 < d(x, z) < r\}$.*

It has to be also observed that the properties of being a critical point or a critical sequence, the existence of a modulus of regularity and the Palais–Smale condition are *metric* properties; that is to say, they are invariant with respect to Lipschitz homeomorphisms.

The following plays the crucial role in our proof of the bifurcation theorem.

Proposition 2 (Potential Well Theorem [9]). *Suppose X is a locally connected complete metric space, z is a local minimum of f and there are an $r > 0$ and a modulus of regularity $\delta(t)$ of f with respect to z defined on the set $\{x : 0 < d(x, z) < r\}$. Let $0 < \theta < (r/2)\delta(r/4)$, and let B be a connected closed ball of radius $\leq r/4$ on which $f(x) \leq f(z) + \theta$. Then the inequality*

$$f(x) \geq f(z) + \int_0^{d(x,z)} \delta(t/2)dt$$

holds for all $x \in B$.

In case when the reference point z is *not* a local minimum for f, we have the following *upper* estimate for the function.

Proposition 3 ([9], Proposition 6). *Suppose (X, d) is a complete metric space, f is a continuous function on X and there is a modulus of regularity $\delta(t)$ of f on the set $G = \{x : \ 0 < d(x, z) < r\}$ $(r > 0)$. If z is not a local minimum of f, then for any $\tau \in [0, 1]$*

$$\inf\{f(x) : \ d(x, z) \leq \tau r\} \leq f(z) - (\tau^2/2)r\delta(\tau r/4).$$

As a consequence of this fact, we get the property which also follows from the variational principle of Ekeland [14].

Proposition 4. *Let (X, d) be a complete metric space and f a continuous function on X bounded from below. Then for any $\varepsilon > 0$ there is an x such that $f(x) \leq \inf f + \varepsilon$, $|df|(x) \leq \varepsilon$.*

The last result we quote extends the mountain pass theorem of Ambrosetti–Rabinowitz to continuous functions on complete metric spaces. We state it in a simplified form sufficient for our purpose.

Proposition 5 ([8],[11]). *Let X be a complete metric space and f a continuous function on X. Fix two points $u, v \in X$ and consider the collection P of all continuous paths $p(t) : [0, 1] \mapsto X$ joining u and v, say such that $p(0) = u$, $p(1) = v$. Set*

$$c = \inf_{p \in P} \max_{0 \leq t \leq 1} f(p(t)).$$

Assume further that $c > \max\{f(u), f(v)\}$ and that f satisfies the Palais–Smale condition. Then f has a critical point x with $f(x) = c$.

3. An auxiliary problem: a "pure" bifurcation theorem

According to the strategy described in the introduction we have to study the finite dimensional bifurcation problem for functions (4) with nondifferentiable perturbations φ (see [15], p.159; [16], p. 745;[9] for the case of a differentiable perturbation function φ). We shall actually consider a more general situation of a function on an infinite dimensional space (even Banach). So let $g_\lambda(x)$ be defined on the product of the segment $[-\lambda_0, \lambda_0]$ and $B(\alpha)$, where λ_0 and α are positive constants.

Assume that $x = 0$ is a critical point of $g_\lambda(\cdot)$ for all λ. We say that $\lambda = 0$ is a *bifurcation point* for g if there are sequences $\lambda_n \to 0$ and $x_n \to 0$, $x_n \neq 0$ such that x_n is a critical point of $g_{\lambda_n}(\cdot)$ for each n.

Theorem 2. *We assume that*

(a_4)**:** $g_\lambda(x)$ *is defined and continuous on* $[-\lambda_0, \lambda_0] \times B(\alpha)$ *;*

(a_5)**:** $x = 0$ *is a critical point of* $g_0(\cdot)$ *;* $g_\lambda(\cdot)$ *has an isolated local minimum (maximum) at zero for every* $\lambda > 0$ *and an isolated local maximum (minimum) at zero for every* $\lambda < 0$*;*

(a_6)**:** $g(\lambda, \cdot)$ *satisfies the Palais–Smale condition uniformly in* λ*; that is, if a sequence of* (λ_n, x_n) *is such that* $g_{\lambda_n}(x_n)$ *is a bounded sequence and either* $|dg_{\lambda_n}|^+(x_n) \to 0$ *or* $|d(-g_{\lambda_n})|^+(x_n) \to 0$*, then* $\{(\lambda_n, x_n)\}$ *contains a subsequence converging to a certain* (λ, x) *such that* x *is a critical point of* $g_\lambda(\cdot)$*.*

Then $\lambda = 0$ *is a bifurcation point of* g*. Moreover, at least one of the following three possibilities is always valid.*

(i) $x = 0$ *is not an isolated critical point of* $g_0(\cdot)$*;*

(ii) *for every* λ *of a neighborhood of zero there is a nontrivial critical point of* $g_\lambda(\cdot)$ *converging to zero as* $\lambda \to 0$*;*

(iii) *there is a one–sided (right or left) neighborhood of zero such that for every* $\lambda \neq 0$ *in the neighborhood there are two distinct nontrivial critical points of* $g_\lambda(\cdot)$ *converging to zero as* $\lambda \to 0$*.*

Proof. We can assume of course that $g_\lambda(0) \equiv 0$. Suppose (i) does not hold, that is, $x = 0$ is an isolated critical point of $g_0(\cdot)$. Then either zero is a local minimum or a local maximum of $g_0(\cdot)$ or $g_0(\cdot)$ assumes values of both signs in any neighborhood of zero. We shall show that in the last case (ii) holds while (iii) takes place in the first case.

1. Let $g_0(\cdot)$ assume values of both signs in any neighborhood of zero. Assuming that (ii) does not hold, we conclude that there are $\varepsilon > 0$ and a sequence of $\lambda_n \to 0$ such that $g_{\lambda_n}(\cdot)$ does not have nontrivial critical points in $B(\varepsilon)$. Set $G(\varepsilon) = \{x : 0 < \|x\| < \varepsilon\}$. We claim that there is a modulus of regularity $\delta(t)$ on G w.r.t. zero common to all $g_{\lambda_n}(\cdot)$. If this were not true, we would find a $t > 0$, a sequence of indices $\{n_k\}$ (not necessarily different) and a sequence $x_{n_k} \in B(\varepsilon)$, $\|x_{n_k}\| \geq t$ such that $d|g_{\lambda_{n_k}}(\cdot)|(x_{n_k}) \to 0$. By ($a_6$), this means that either $g_0(\cdot)$ or a certain $g_{\lambda_n}(\cdot)$ has a critical point with $t \leq \|x\| \leq \varepsilon$ in contradiction with our assumption.

Assume that $\lambda_n > 0$ for infinitely many n (or for all of them which is the same). Then 0 is a local minimum of $g_{\lambda_n}(\cdot)$ by (a_5) (if n is sufficiently large, to be precise). By the Potential Well Theorem (Proposition 2) there is $r > 0$ such that for such n

$$g_{\lambda_n}(x) \geq \int_0^{\|x\|} \delta(t/2)dt > 0 \quad \text{if } 0 < \|x\| \leq r$$

which by continuity implies the same inequality for $g_0(\cdot)$ in contradiction with the assumption that $g_0(\cdot)$ assumes values of both signs.

2. We turn now to the case when zero is a local minimum of $g_0(\cdot)$. (The case of a maximum is similar, just replace g by $-g$; see also the remark after the proof.) We first observe that by (a_6) every $g_\lambda(\cdot)$ satisfies the Palais–Smale condition. So by Propositions 1 and 2 there are $r > 0$ and $\beta > 0$ such that $g_0(x) > g_0(0) = 0$ if $\|x\| \leq r$ and $g_0(x) \geq \beta$ if $\|x\| = r$. Choose a $\mu > 0$ such that $|g_\lambda(x) - g_0(x)| < \beta/2$ if $|\lambda| < \mu$, $\|x\| \leq r$. Take a $\lambda \in (-\mu, 0)$ and set

$$\beta(\lambda) = \frac{|\lambda|\beta}{2\mu}, \; r(\lambda) = \inf\{\rho > 0 : g_\lambda(x) \geq \beta(\lambda) \text{ if } \|x\| = \rho\}.$$

It is clear that $r(\lambda) \to 0$ as $\lambda \to 0$. As $\lambda < 0$, zero is a local maximum of $g_\lambda(\cdot)$ by (a_5) (for $\lambda < 0$ sufficiently close to zero). As every $g_\lambda(\cdot)$ satisfies the Palais–Smale condition, this together with Proposition 3 implies that the minimal value of $g_\lambda(\cdot)$ on $B(r(\lambda))$ is attained at a certain $x_1 = x_1(\lambda)$ with $\|x_1\| < r(\lambda)$ and

$$\varepsilon(\lambda) = g_\lambda(x_1) = \min\{g_\lambda(x) \ : \|x\| \le r(\lambda)\} < 0.$$

Thus we have proved that $g_\lambda(\cdot)$ has at least one nontrivial critical point. If there are more, then (iii) is proved. Otherwise x_1 is a unique global minimum of $g_\lambda(\cdot)$ on $B(r(\lambda))$. Consider the linear segment joining 0 and $u = -r(\lambda)x_1/\|x_1\|$. Then the lower bound of g_λ on the segment is strictly greater than $\varepsilon(\lambda)$ since this segment does not contain x_1. If we consider now the collection $\mathcal{P}$ of all continuous paths $x(t) : [0,1] \mapsto B(r(\lambda))$ joining zero and u (that is, such that $x(0) = 0, x(1) = u$), then we have to conclude, taking into account that zero is a strict local maximum of $g_\lambda(\cdot)$, that

$$0 > c = \sup_{x(\cdot)\in\mathcal{P}} \min_{0\le t\le 1} g_\lambda(x(t)) > \varepsilon(\lambda).$$

(Indeed, as there is no critical point in a neighborhood of zero, the left inequality follows from Propositions 1 and 2 (applied to $-g_\lambda(\cdot)$) which imply that the upper bound of $g_\lambda(\cdot)$ on small spheres centered at zero must be negative.)

Consider now $B(r(\lambda))$ with the induced metric as a separate metric space. Then the mountain pass theorem of Proposition 5 implies that the restriction of $g_\lambda(\cdot)$ on $B(r(\lambda))$ must have a critical point at level c. The above inequality shows that this can be neither zero nor x_1. On the other hand, as $g_\lambda(\cdot)$ is positive on the boundary of $B(r(\lambda))$ the point must be strictly inside the ball, thus being a critical point of $g_\lambda(\cdot)$ itself. This completes the proof of the theorem.

Remark. It is possible to give a slightly longer proof of the second part of the theorem but using only the potential well theorem with no reference to a mountain pass theorem. Here is the proof for the case when zero is an isolated local *maximum* of $g_0(\cdot)$ and a positive open neighborhood of zero is considered.

As above, for any $\lambda \in (0,\mu)$ we find $r(\lambda)$ and $x_1(\lambda) \ne 0$ in the interior of $B(r(\lambda))$ such that $g_\lambda(\cdot)$ attains a maximum on $B(r(\lambda))$ at $x_1(\lambda)$ and $g_\lambda(x_1(\lambda)) = \varepsilon(\lambda) > 0$. In particular, every $g_\lambda(\cdot)$ assumes values of both signs in $B(r(\lambda))$. Denote by $\varepsilon'(\lambda)$ the upper bound of $g_\lambda(x)$ on the line segment joining zero and $u = -r(\lambda)\|x_1(\lambda)\|^{-1}x_1(\lambda)$. Then $\varepsilon'(\lambda) < \varepsilon(\lambda)$ if there is no other critical point of $g_\lambda(\cdot)$ in $B(r(\lambda))$. Consider the set

$$X(\lambda) = \{x \in H : \|x\| \ge r(\lambda) \quad \text{or } \|x\| < r(\lambda) \ \&\ g_\lambda(x) \ \le \max\{\varepsilon', \varepsilon/2\}\}.$$

(In other words, we cut away a neighborhood of the local maximum.) Consider the metric space $(X(\lambda), d)$ with the induced metric and observe that the regularity constants $|dg_\lambda(\cdot)\,|^+(x)$ at any point of $X(\lambda)$ with $\|x\| < r(\lambda)$ are the same when calculated in $X(\lambda)$ or in the whole of X, so that $g_\lambda(\cdot)$ satisfies the Palais–Smale condition on $X(\lambda)$ as well. This implies that $X(\lambda)$ is a connected space. Indeed, if $X(\lambda)$ has a component other than the main component containing zero and the exterior of $B(r(\lambda))$, then $g_\lambda(\cdot)$ must attain a minimum in that component which is a critical point of $g_\lambda(\cdot)$ on X different from both zero and $x_1(\lambda)$.

The rest of the proof is similar to the proof of the first part of the theorem: suppose there is a sequence of positive $\lambda_n \to 0$ such that $x_1(\lambda)$ is the only nontrivial critical point of every $g_{\lambda_n}(\cdot)$ on $B(r)$. Then by (a_6) there must be a modulus of regularity with respect to zero, common to the restrictions of all $g(\lambda_n, \cdot)$ to $X(\lambda)$

and this, by way of the potential well theorem through the fact that $g_\lambda(\cdot)$ attains a local minimum at zero if $\lambda > 0$, will lead to a contradiction with the fact that $g_0(\cdot)$ has a local maximum at zero.

4. Proof of Theorem 1

The first step of the proof is standard: the Lyapunov–Schmidt reduction. Let U be the eigenspace of A corresponding to μ and $V = U^\perp$. The equation (1) means that x is a critical point of the function $F_\lambda(x) = f_\lambda(x) - (\lambda/2)\|x\|^2$. We can represent this function in the following form:

$$F_\lambda(x) = (1/2)(Tv|v) + \varphi_\lambda(u+v) - (1/2)(\lambda-\mu)(\|u\|^2 + \|v\|^2),$$

where T is the restriction of $A - \mu I$ to V and $x = u + v$. Let P_U and P_V denote the projections to U and V respectively. Then (1) is equivalent to the following two equations the left–hand parts of which are the u-component and the v-component of the gradient of F_λ:

$$\left.\begin{array}{l} P_U\nabla\varphi_\lambda(u+v) - (\lambda-\mu)u = 0; \\ Tv + P_V\nabla\varphi_\lambda(u+v) - (\lambda-\mu)v = 0. \end{array}\right\} \tag{5}$$

As the Lipschitz constant of $\nabla\varphi_\lambda$ is strictly smaller than $r = \|T^{-1}\|^{-1}$, the second equation can be resolved with respect to v (for λ sufficiently close to μ), that is, there is a Lipschitz continuous mapping $v_\lambda(u)$ from a neighborhood of zero in U into V such that

$$v_\lambda(u) = -(T - (\lambda-\mu)I)^{-1}P_V\nabla\varphi_\lambda(u + v_\lambda(u)\ )\ , \quad v_\lambda(0) = 0. \tag{6}$$

Set

$$g_\lambda(u) = F_\lambda(u + v_\lambda(u)).$$

As by definition of $v_\lambda(u)$ the gradient of F_λ is orthogonal to V at $u + v_\lambda(u)$, it is clear that g_λ is differentiable with the derivative

$$\nabla g_\lambda(u)\ = P_U\nabla\varphi_\lambda(u + v_\lambda(u)\ )\ - (\lambda-\mu)u$$

continuous jointly in λ and u. This means that u is a critical point of $g_\lambda(\cdot)$ if and only if $u + v_\lambda(u)$ is a critical point of F_λ, so the problem reduces to the analysis of critical points of $g_\lambda(\cdot)$. This is the end of the reduction step.

The short second step consists in verification that $g_\lambda(\cdot)$ satisfies the conditions of Theorem 3 in which case application of the theorem immediately completes the proof.

Replacing $\lambda - \mu$ by λ and dividing (1) by r, we reduce the situation to the case $\mu = 0$. Condition (a_4) of Theorem 3 is clearly valid. We see also from (5),(6) that $x = 0$ is a critical point of all $g_\lambda(\cdot)$. Furthermore, by (6)

$$\begin{array}{rcl} |(Tv_\lambda(u)\ |v_\lambda(u)\ )| & \leq & \|T(T-\lambda I)^{-1}\|\|(T-\lambda I)^{-1}\|\|\nabla\varphi_\lambda(u + v_\lambda(u)\ )\ \|^2 \\ & \leq & (1 + O(\lambda))r^{-1}\|\nabla\varphi_\lambda(u + v_\lambda(u)\ )\ \|^2. \end{array}$$

Together with (3) this shows that

$$|(Tv_\lambda(u)\ |v_\lambda(u)\ )| + 2|\varphi_\lambda(u + v_\lambda(u)\ )\ | \leq \lambda\|u + v_\lambda(u)\ \|^2$$

if $\lambda \neq 0$ and u is sufficiently close to zero. The latter implies that (a_5) also holds. Finally (a_6) follows from the fact that the derivative of $g_\lambda(\cdot)$ is jointly continuous in λ and u. This completes the proof of the theorem.

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