SMOOTH KUMMER SURFACES
IN PROJECTIVE THREE-SPACE

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(Communicated by Ron Donagi)

ABSTRACT. In this note we prove the existence of smooth Kummer surfaces in projective three-space containing sixteen mutually disjoint smooth rational curves of any given degree.

INTRODUCTION

Let $X$ be a smooth quartic surface in projective three-space $\mathbb{P}^3$. As a consequence of Nikulin's theorem [6] $X$ is a Kummer surface if and only if it contains sixteen mutually disjoint smooth rational curves. The classical examples of smooth Kummer surfaces in $\mathbb{P}^3$ are due to Traynard (see [8] and [4]). They were rediscovered by Barth and Nieto [2] and independently by Naruki [5]. These quartic surfaces contain sixteen skew lines. In [1] it was shown by different methods that there also exist smooth quartic surfaces in $\mathbb{P}^3$ containing sixteen mutually disjoint smooth conics.

Motivated by these results it is then natural to ask if, for any given integer $d \geq 1$, there exist smooth quartic surfaces in $\mathbb{P}^3$ containing sixteen mutually disjoint smooth rational curves of degree $d$. The aim of this note is to show that the method of [1] can be generalized to answer this question in the affirmative. We show:

Theorem. For any integer $d \geq 1$ there is a three-dimensional family of smooth quartic surfaces in $\mathbb{P}^3$ containing sixteen mutually disjoint smooth rational curves of degree $d$.

We work throughout over the field $\mathbb{C}$ of complex numbers.

1. PRELIMINARIES

Let $(A, L)$ be a polarized abelian surface of type $(1, 2d^2 + 1)$, $d \geq 1$, and let $L$ be symmetric. Denote by $e_1, \ldots, e_{16}$ the halfperiods of $A$. We are going to consider the non-complete linear system

(*) \[ \mathcal{O}_A(2L) \otimes \bigotimes_{i=1}^{16} m_{e_i}^\pm \]

Received by the editors April 6, 1996.

1991 Mathematics Subject Classification. Primary 14J28; Secondary 14E25.

The author was supported by DFG contract Ba 423/7-1.

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of even respectively odd sections of \( \mathcal{O}_A(2L) \) vanishing in \( e_1, \ldots, e_{16} \) to the order \( d \). (As for the sign \( \pm \) we will always use the following convention: we take + if \( d \) is even, and − if \( d \) is odd.) A parameter count shows that the expected dimension of this linear system is 4. In fact, we will show that it yields an embedding of the smooth Kummer surface \( X \) of \( A \) into \( \mathbb{P}_3 \) in the generic case. The linear system \((*)\) corresponds to a line bundle \( M_L \) on \( X \) such that

\[
\pi^* M_L = \mathcal{O}_{\tilde{A}} \left( 2\sigma^* L - d \sum_{i=1}^{16} E_i \right)
\]

and

\[
H^0(X, M_L) \cong H^0 \left( A, \mathcal{O}_A(2L) \otimes \bigotimes_{i=1}^{16} m_i^d \right)^{\pm}.
\]

Here \( \sigma : \tilde{A} \to A \) is the blow-up of \( A \) in the halfperiods, \( E_1, \ldots, E_{16} \subset \tilde{A} \) are the exceptional curves and \( \pi : \tilde{A} \to X \) is the canonical projection. The images of \( E_1, \ldots, E_{16} \) under \( \pi \) will be denoted by \( D_1, \ldots, D_{16} \).

We will need the following lemma:

**Lemma 1.1.** Let the surfaces \( A \) and \( X \) and the line bundles \( L \) and \( M_L \) be as above. Further, let \( C \subset X \) be an irreducible curve, different from \( D_1, \ldots, D_{16} \), and let \( F = \sigma^* \pi^* C \) be the corresponding symmetric curve on \( A \). Then

(a) \( M_L^2 = 4 \) and \( M_L D_i = d \) for \( 1 \leq i \leq 16 \),

(b) \( F^2 = 2C^2 + \sum_{i=1}^{16} \text{mult}_{e_i}(F)^2 \), and

(c) \( LF = M_L C + \frac{d}{2} \sum_{i=1}^{16} \text{mult}_{e_i}(F) \).

The proof consists in an obvious calculation.

2. **Bounding degrees and multiplicities**

Here we show two technical statements on the degrees and multiplicities of symmetric curves. We start with a lemma which bounds the degree of a symmetric curve on \( A \) in terms of the degree of the corresponding curve on the smooth Kummer surface of \( A \):

**Lemma 2.1.** Let \( C \subset X \) be an irreducible curve, different from \( D_1, \ldots, D_{16} \), and let \( F = \sigma^* \pi^* C \) be the corresponding symmetric curve on \( A \). Then

(a) If \( M_L C = 0 \), then \( LF \leq 2 \left( 1 - C^2 \right) d^2 + 16 \).

(b) If \( M_L C > 0 \), then \( LF \leq 4 \left( M_L C - \frac{C^2}{M_L C} \right) d^2 + 9M_L C \).

**Proof.** For \( \gamma \geq 0 \) apply Hodge index to the line bundle \( M_L \) and the divisor \( C + \frac{\gamma}{d} D_i \):

\[
M_L^2 \left( C + \frac{\gamma}{d} D_i \right)^2 \leq \left( M_L C + \frac{\gamma}{d} M_L D_i \right)^2.
\]

Using Lemma 1.1(a) and the equality \( CD_i = \text{mult}_{e_i}(F) \) we get

\[
\text{mult}_{e_i}(F) \leq \left( \frac{(M_L C)^2}{8\gamma} + \frac{\gamma}{8} + \frac{M_L C}{4} - \frac{C^2}{2\gamma} \right) d + \frac{\gamma}{d} ;
\]
hence by Lemma 1.1(c)
\[ LF \leq \left( \frac{(MLC)^2 + \gamma + 2MLC - \frac{4C^2}{\gamma}}{\gamma} \right) d^2 + MLC + 8\gamma. \]
Now the assertion follows by setting \( \gamma = 2 \) in case \( MLC = 0 \) and by setting \( \gamma = MLC \) otherwise. \( \square \)

Further, we will need the following inequality on multiplicities of symmetric curves:

**Lemma 2.2.** Let \( F \subset A \) be a symmetric curve such that \( \mathcal{O}_A(F) \) is of type \((1, e)\) with \( e \) odd. Then
\[ \sum_{i=1}^{16} \text{mult}_{e_i}(F)^2 \geq \frac{1}{16} \left( \sum_{i=1}^{16} \text{mult}_{e_i}(F) \right)^2 + \frac{15}{4}. \]

**Proof.** For \( k \geq 0 \) define the integers \( n_k \) by
\[ n_k = \# \{ i \mid m_i = k, \ 1 \leq i \leq 16 \}. \]
Abbreviating \( m_i = \text{mult}_{e_i}(F) \) we then have
\[ \sum_{i=1}^{16} m_i = \sum_{k \geq 0} kn_k \quad \text{and} \quad \sum_{i=1}^{16} m_i^2 = \sum_{k \geq 0} k^2 n_k. \]
The polarized abelian surface \((A, \mathcal{O}_A(F))\) is the pull-back of a principally polarized abelian surface \((B, P)\) via an isogeny \( \varphi : A \to B \) of odd degree. The Theta divisor \( \Theta \in |P| \) passes through six halfperiods with multiplicity one and through ten halfperiods with even multiplicity. Therefore the symmetric divisor \( F \in |\varphi^*P| \) is of odd multiplicity in six halfperiods and of even multiplicity in ten halfperiods or vice versa. So we have
\[ \sum_{k \equiv 0(2)} n_k = a \quad \text{and} \quad \sum_{k \equiv 1(2)} n_k = b, \]
where \((a, b) = (6, 10)\) or \((a, b) = (10, 6)\).
Under the restriction (1) the difference
\[ \sum k^2 n_k - \frac{1}{16} \left( \sum kn_k \right)^2 \]
is minimal, if for some integer \( k_0 \geq 0 \) we have
\[ n_{k_0} = 10, \ n_{k_0+1} = 6 \quad \text{or} \quad n_{k_0} = 6, \ n_{k_0+1} = 10. \]
In this case we get
\[ \sum k^2 n_k - \frac{1}{16} \left( \sum kn_k \right)^2 = \frac{15}{4}, \]
which implies the assertion of the lemma. \( \square \)

## 3. Kummer surfaces with sixteen skew rational curves of given degree

The aim of this section is to show:

**Theorem 3.1.** Let \((A, L)\) be a polarized abelian surface of type \((1, 2d^2 + 1)\), \( d \geq 1 \). Assume \( \rho(A) = 1 \). Then the map \( \varphi_{ML} : X \to \mathbb{P}^3 \) defined by the linear system \(|ML|\) is an embedding. The image surface \( \varphi_{ML}(X) \) is a smooth quartic surface containing sixteen mutually disjoint smooth rational curves of degree \( d \).

In particular, this implies the theorem stated in the introduction.
Proof. Using Riemann-Roch, Kodaira vanishing and Lemma 1.1(a), we will be done as soon as we can show that $M_L$ is very ample. For $d = 1$ this follows from [3], whereas for $d = 2$ it follows from [1]. So we may assume $d \geq 3$ in the sequel.

(a) First we show that $M_L$ is globally generated. A possible base part $B$ of the system $\left| O_A (2L) \otimes \bigotimes_{i=1}^{16} m_i^d \right|$ is totally symmetric, so $B$ is algebraically equivalent to some even multiple of $L$, which is impossible for dimensional reasons. It remains the possibility that one – hence all – of the curves $D_i$ is fixed in $|M_L|$. So $M_L - \mu \sum D_i$ is free for some $\mu \geq 1$. But $(M_L - \mu \sum D_i)^2 = 4 - 32\mu d - 32\mu^2 < 0$, a contradiction.

(b) Our next claim is that $M_L$ is ample. Otherwise there is an irreducible $(-2)$-curve $C \subset X$ such that $M_L C = 0$. Lemma 1.1 shows that we have

$$LF = \frac{d}{2} \sum m_i \quad \text{and} \quad F^2 = -4 + \sum m_i^2$$

for the symmetric curve $F = \sigma_* \pi^* C$ with multiplicities $m_i = \text{mult}_{e_i}(F)$. According to Lemma 2.1 the degree of $F$ is bounded by

$$LF \leq 6d^2 + 16.$$  

Since $L$ is a primitive line bundle, the assumption on the Néron-Severi group of $A$ implies that $O_A (F)$ is algebraically equivalent to some multiple $pL$, $p \geq 1$, thus we have $LF = pL^2 = p(4d^2 + 2)$, and then (2) implies $p = 1$ because of our assumption $d \geq 3$. So we find

$$8d^2 + 4 = 2LF = d \sum m_i$$

and reduction mod $d$ shows that necessarily $d = 4$. But in this case $\sum m_i$ would be odd, which is impossible (cf. [3]).

(c) Finally we prove that $M_L$ is very ample. Suppose the contrary. Saint-Donat’s criterion [7, Theorem 5.2 and Theorem 6.1(iii)] then implies the existence of an irreducible curve $C \subset X$ with $M_L C = 2$ and $C^2 = 0$. So we have

$$LF = 2 + \frac{d}{2} \sum m_i \quad \text{and} \quad F^2 = \sum m_i^2$$

for the corresponding symmetric curve $F = \sigma_* \pi^* C$. Lemma 2.1 yields the estimate

$$LF \leq 8d^2 + 18.$$  

As above $O_A (F)$ is algebraically equivalent to some multiple $pL$, $p \geq 1$, hence we get

$$p \left( 4d^2 + 2 \right) = pL^2 \leq 8d^2 + 18,$$

which implies $p \leq 2$. If we had $p = 2$ then reduction mod $d$ of the equation

$$2 \left( 4d^2 + 2 \right) = 2 + \frac{d}{2} \sum m_i$$

would give $d = 4$. But in this case we have $\sum m_i = 65$, which is impossible.

So the only remaining possibility is $p = 1$, thus

$$4d^2 + 2 = 2 + \frac{d}{2} \sum m_i = \sum m_i^2.$$  

But a numerical check shows that this contradicts Lemma 2.2. This completes the proof of the theorem.
Remark 3.2. We conclude with a remark on the genericity assumption on the abelian surface $A$. It is certainly not true that the line bundle $M_L$ is very ample for every polarized abelian surface $(A, L)$ of type $(1, 2d^2 + 1)$. Consider for instance the case where $A = E_1 \times E_2$ is a product of elliptic curves and $L = O_A \left( \{0\} \times E_2 + (2d^2 + 1) E_1 \times \{0\} \right)$. Here, taking $C \subset X$ to be curve corresponding to $E_1 \times \{0\}$, we have

$$M_L C = 1 - 2d < 0,$$

so in this case $M_L$ is not even ample or globally generated.

REFERENCES

2. Barth, W., Nieto, I.: Abelian surfaces of type $(1, 3)$ and quartic surfaces with 16 skew lines. J. Algebraic Geometry 3, 173-222 (1994) MR 95e:14033