

SMOOTH KUMMER SURFACES IN PROJECTIVE THREE-SPACE

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ABSTRACT. In this note we prove the existence of smooth Kummer surfaces in projective three-space containing sixteen mutually disjoint smooth rational curves of any given degree.

INTRODUCTION

Let X be a smooth quartic surface in projective three-space \mathbb{P}^3 . As a consequence of Nikulin's theorem [6] X is a Kummer surface if and only if it contains sixteen mutually disjoint smooth rational curves. The classical examples of smooth Kummer surfaces in \mathbb{P}^3 are due to Traynard (see [8] and [4]). They were rediscovered by Barth and Nieto [2] and independently by Naruki [5]. These quartic surfaces contain sixteen skew lines. In [1] it was shown by different methods that there also exist smooth quartic surfaces in \mathbb{P}^3 containing sixteen mutually disjoint smooth *conics*.

Motivated by these results it is then natural to ask if, for any given integer $d \geq 1$, there exist smooth quartic surfaces in \mathbb{P}^3 containing sixteen mutually disjoint smooth rational curves of degree d . The aim of this note is to show that the method of [1] can be generalized to answer this question in the affirmative. We show:

Theorem. *For any integer $d \geq 1$ there is a three-dimensional family of smooth quartic surfaces in \mathbb{P}^3 containing sixteen mutually disjoint smooth rational curves of degree d .*

We work throughout over the field \mathbb{C} of complex numbers.

1. PRELIMINARIES

Let (A, L) be a polarized abelian surface of type $(1, 2d^2 + 1)$, $d \geq 1$, and let L be symmetric. Denote by e_1, \dots, e_{16} the halfperiods of A . We are going to consider the non-complete linear system

$$(*) \quad \left| \mathcal{O}_A(2L) \otimes \bigotimes_{i=1}^{16} \mathfrak{m}_{e_i}^d \right|^{\pm}$$

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of even respectively odd sections of $\mathcal{O}_A(2L)$ vanishing in e_1, \dots, e_{16} to the order d . (As for the sign \pm we will always use the following convention: we take $+$ if d is even, and $-$ if d is odd.) A parameter count shows that the expected dimension of this linear system is 4. In fact, we will show that it yields an embedding of the smooth Kummer surface X of A into \mathbb{P}_3 in the generic case. The linear system $(*)$ corresponds to a line bundle M_L on X such that

$$\pi^*M_L = \mathcal{O}_{\tilde{A}} \left(2\sigma^*L - d \sum_{i=1}^{16} E_i \right)$$

and

$$H^0(X, M_L) \cong H^0 \left(A, \mathcal{O}_A(2L) \otimes \bigotimes_{i=1}^{16} \mathfrak{m}_{e_i}^d \right)^\pm.$$

Here $\sigma : \tilde{A} \rightarrow A$ is the blow-up of A in the halfperiods, $E_1, \dots, E_{16} \subset \tilde{A}$ are the exceptional curves and $\pi : \tilde{A} \rightarrow X$ is the canonical projection. The images of E_1, \dots, E_{16} under π will be denoted by D_1, \dots, D_{16} .

We will need the following lemma:

Lemma 1.1. *Let the surfaces A and X and the line bundles L and M_L be as above. Further, let $C \subset X$ be an irreducible curve, different from D_1, \dots, D_{16} , and let $F = \sigma_*\pi^*C$ be the corresponding symmetric curve on A . Then*

- (a) $M_L^2 = 4$ and $M_L D_i = d$ for $1 \leq i \leq 16$,
- (b) $F^2 = 2C^2 + \sum_{i=1}^{16} \text{mult}_{e_i}(F)^2$, and
- (c) $LF = M_L C + \frac{d}{2} \sum_{i=1}^{16} \text{mult}_{e_i}(F)$.

The proof consists in an obvious calculation.

2. BOUNDING DEGREES AND MULTIPLICITIES

Here we show two technical statements on the degrees and multiplicities of symmetric curves. We start with a lemma which bounds the degree of a symmetric curve on A in terms of the degree of the corresponding curve on the smooth Kummer surface of A :

Lemma 2.1. *Let $C \subset X$ be an irreducible curve, different from D_1, \dots, D_{16} , and let $F = \sigma_*\pi^*C$.*

- (a) *If $M_L C = 0$, then $LF \leq 2(1 - C^2)d^2 + 16$.*
- (b) *If $M_L C > 0$, then $LF \leq 4 \left(M_L C - \frac{C^2}{M_L C} \right) d^2 + 9M_L C$.*

Proof. For $\gamma \geq 0$ apply Hodge index to the line bundle M_L and the divisor $C + \frac{\gamma}{d}D_i$:

$$M_L^2 \left(C + \frac{\gamma}{d}D_i \right)^2 \leq \left(M_L C + \frac{\gamma}{d}M_L D_i \right)^2.$$

Using Lemma 1.1(a) and the equality $CD_i = \text{mult}_{e_i}(F)$ we get

$$\text{mult}_{e_i}(F) \leq \left(\frac{(M_L C)^2}{8\gamma} + \frac{\gamma}{8} + \frac{M_L C}{4} - \frac{C^2}{2\gamma} \right) d + \frac{\gamma}{d};$$

hence by Lemma 1.1(c)

$$LF \leq \left(\frac{(M_L C)^2}{\gamma} + \gamma + 2M_L C - \frac{4C^2}{\gamma} \right) d^2 + M_L C + 8\gamma .$$

Now the assertion follows by setting $\gamma = 2$ in case $M_L C = 0$ and by setting $\gamma = M_L C$ otherwise. \square

Further, we will need the following inequality on multiplicities of symmetric curves:

Lemma 2.2. *Let $F \subset A$ be a symmetric curve such that $\mathcal{O}_A(F)$ is of type $(1, e)$ with e odd. Then*

$$\sum_{i=1}^{16} \text{mult}_{e_i}(F)^2 \geq \frac{1}{16} \left(\sum_{i=1}^{16} \text{mult}_{e_i}(F) \right)^2 + \frac{15}{4} .$$

Proof. For $k \geq 0$ define the integers n_k by

$$n_k =_{\text{def}} \#\{i \mid m_i = k, 1 \leq i \leq 16\} .$$

Abbreviating $m_i = \text{mult}_{e_i}(F)$ we then have

$$\sum_{i=1}^{16} m_i = \sum_{k \geq 0} kn_k \quad \text{and} \quad \sum_{i=1}^{16} m_i^2 = \sum_{k \geq 0} k^2 n_k .$$

The polarized abelian surface $(A, \mathcal{O}_A(F))$ is the pull-back of a principally polarized abelian surface (B, P) via an isogeny $\varphi : A \rightarrow B$ of odd degree. The Theta divisor $\Theta \in |P|$ passes through six halfperiods with multiplicity one and through ten halfperiods with even multiplicity. Therefore the symmetric divisor $F \in |\varphi^*P|$ is of odd multiplicity in six halfperiods and of even multiplicity in ten halfperiods or vice versa. So we have

$$(1) \quad \sum_{k \equiv 0(2)} n_k = a \quad \text{and} \quad \sum_{k \equiv 1(2)} n_k = b ,$$

where $(a, b) = (6, 10)$ or $(a, b) = (10, 6)$.

Under the restriction (1) the difference

$$\sum k^2 n_k - \frac{1}{16} \left(\sum kn_k \right)^2$$

is minimal, if for some integer $k_0 \geq 0$ we have

$$n_{k_0} = 10, n_{k_0+1} = 6 \text{ or } n_{k_0} = 6, n_{k_0+1} = 10 .$$

In this case we get $\sum k^2 n_k - \frac{1}{16} (\sum kn_k)^2 = \frac{15}{4}$, which implies the assertion of the lemma. \square

3. KUMMER SURFACES WITH SIXTEEN SKEW RATIONAL CURVES OF GIVEN DEGREE

The aim of this section is to show:

Theorem 3.1. *Let (A, L) be a polarized abelian surface of type $(1, 2d^2 + 1)$, $d \geq 1$. Assume $\rho(A) = 1$. Then the map $\varphi_{M_L} : X \rightarrow \mathbb{P}^3$ defined by the linear system $|M_L|$ is an embedding. The image surface $\varphi_{M_L}(X)$ is a smooth quartic surface containing sixteen mutually disjoint smooth rational curves of degree d .*

In particular, this implies the theorem stated in the introduction.

Proof. Using Riemann-Roch, Kodaira vanishing and Lemma 1.1(a), we will be done as soon as we can show that M_L is very ample. For $d = 1$ this follows from [3], whereas for $d = 2$ it follows from [1]. So we may assume $d \geq 3$ in the sequel.

(a) First we show that M_L is globally generated. A possible base part B of the system $|\mathcal{O}_A(2L) \otimes \bigotimes_{i=1}^{16} m_{e_i}^d|^\pm$ is totally symmetric, so B is algebraically equivalent to some even multiple of L , which is impossible for dimensional reasons. It remains the possibility that one – hence all – of the curves D_i is fixed in $|M_L|$. So $M_L - \mu \sum D_i$ is free for some $\mu \geq 1$. But $(M_L - \mu \sum D_i)^2 = 4 - 32\mu d - 32\mu^2 < 0$, a contradiction.

(b) Our next claim is that M_L is ample. Otherwise there is an irreducible (-2) -curve $C \subset X$ such that $M_L C = 0$. Lemma 1.1 shows that we have

$$LF = \frac{d}{2} \sum m_i \quad \text{and} \quad F^2 = -4 + \sum m_i^2$$

for the symmetric curve $F = \sigma_* \pi^* C$ with multiplicities $m_i = \text{mult}_{e_i}(F)$. According to Lemma 2.1 the degree of F is bounded by

$$(2) \quad LF \leq 6d^2 + 16 .$$

Since L is a primitive line bundle, the assumption on the Néron-Severi group of A implies that $\mathcal{O}_A(F)$ is algebraically equivalent to some multiple pL , $p \geq 1$, thus we have $LF = pL^2 = p(4d^2 + 2)$, and then (2) implies $p = 1$ because of our assumption $d \geq 3$. So we find

$$8d^2 + 4 = 2LF = d \sum m_i$$

and reduction mod d shows that necessarily $d = 4$. But in this case $\sum m_i$ would be odd, which is impossible (cf. [3]).

(c) Finally we prove that M_L is very ample. Suppose the contrary. Saint-Donat’s criterion [7, Theorem 5.2 and Theorem 6.1(iii)] then implies the existence of an irreducible curve $C \subset X$ with $M_L C = 2$ and $C^2 = 0$. So we have

$$LF = 2 + \frac{d}{2} \sum m_i \quad \text{and} \quad F^2 = \sum m_i^2$$

for the corresponding symmetric curve $F = \sigma_* \pi^* C$. Lemma 2.1 yields the estimate

$$LF \leq 8d^2 + 18 .$$

As above $\mathcal{O}_A(F)$ is algebraically equivalent to some multiple pL , $p \geq 1$, hence we get

$$p(4d^2 + 2) = pL^2 \leq 8d^2 + 18 ,$$

which implies $p \leq 2$. If we had $p = 2$ then reduction mod d of the equation

$$2(4d^2 + 2) = 2 + \frac{d}{2} \sum m_i$$

would give $d = 4$. But in this case we have $\sum m_i = 65$, which is impossible.

So the only remaining possibility is $p = 1$, thus

$$4d^2 + 2 = 2 + \frac{d}{2} \sum m_i = \sum m_i^2 .$$

But a numerical check shows that this contradicts Lemma 2.2. This completes the proof of the theorem. \square

Remark 3.2. We conclude with a remark on the genericity assumption on the abelian surface A . It is certainly not true that the line bundle M_L is very ample for *every* polarized abelian surface (A, L) of type $(1, 2d^2 + 1)$. Consider for instance the case where $A = E_1 \times E_2$ is a product of elliptic curves and $L = \mathcal{O}_A(\{0\} \times E_2 + (2d^2 + 1)E_1 \times \{0\})$. Here, taking $C \subset X$ to be curve corresponding to $E_1 \times \{0\}$, we have

$$M_L C = 1 - 2d < 0 ,$$

so in this case M_L is not even ample or globally generated.

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