SMOOTH KUMMER SURFACES
IN PROJECTIVE THREE-SPACE

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Abstract. In this note we prove the existence of smooth Kummer surfaces in projective three-space containing sixteen mutually disjoint smooth rational curves of any given degree.

Introduction

Let $X$ be a smooth quartic surface in projective three-space $\mathbb{P}^3$. As a consequence of Nikulin's theorem [6] $X$ is a Kummer surface if and only if it contains sixteen mutually disjoint smooth rational curves. The classical examples of smooth Kummer surfaces in $\mathbb{P}^3$ are due to Traynard (see [8] and [4]). They were rediscovered by Barth and Nieto [2] and independently by Naruki [5]. These quartic surfaces contain sixteen skew lines. In [1] it was shown by different methods that there also exist smooth quartic surfaces in $\mathbb{P}^3$ containing sixteen mutually disjoint smooth conics.

Motivated by these results it is then natural to ask if, for any given integer $d \geq 1$, there exist smooth quartic surfaces in $\mathbb{P}^3$ containing sixteen mutually disjoint smooth rational curves of degree $d$. The aim of this note is to show that the method of [1] can be generalized to answer this question in the affirmative. We show:

Theorem. For any integer $d \geq 1$ there is a three-dimensional family of smooth quartic surfaces in $\mathbb{P}^3$ containing sixteen mutually disjoint smooth rational curves of degree $d$.

We work throughout over the field $\mathbb{C}$ of complex numbers.

1. Preliminaries

Let $(A, L)$ be a polarized abelian surface of type $(1, 2d^2 + 1)$, $d \geq 1$, and let $L$ be symmetric. Denote by $e_1, \ldots, e_{16}$ the halfperiods of $A$. We are going to consider the non-complete linear system

$$(*) \quad \left| O_A(2L) \otimes \bigotimes_{i=1}^{16} m_{e_i} \right|_{\pm}$$

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of even respectively odd sections of $O_A(2L)$ vanishing in $e_1, \ldots, e_{16}$ to the order $d$. (As for the sign $\pm$ we will always use the following convention: we take $+$ if $d$ is even, and $-$ if $d$ is odd.) A parameter count shows that the expected dimension of this linear system is 4. In fact, we will show that it yields an embedding of the smooth Kummer surface $X$ of $A$ into $\mathbb{P}_3$ in the generic case. The linear system $(\ast)$ corresponds to a line bundle $M_L$ on $X$ such that

$$
\pi^*M_L = O_{\hat{A}}(2\sigma^*L - d \sum_{i=1}^{16} E_i)
$$

and

$$
H^0(X, M_L) \cong H^0 \left( A, O_A(2L) \otimes \bigotimes_{i=1}^{16} m_i^{d_i} \right)^{\pm}.
$$

Here $\sigma : \hat{A} \rightarrow A$ is the blow-up of $A$ in the halfperiods, $E_1, \ldots, E_{16} \subset \hat{A}$ are the exceptional curves and $\pi : \hat{A} \rightarrow X$ is the canonical projection. The images of $E_1, \ldots, E_{16}$ under $\pi$ will be denoted by $D_1, \ldots, D_{16}$.

We will need the following lemma:

**Lemma 1.1.** Let the surfaces $A$ and $X$ and the line bundles $L$ and $M_L$ be as above. Further, let $C \subset X$ be an irreducible curve, different from $D_1, \ldots, D_{16}$, and let $F = \sigma \pi^*C$ be the corresponding symmetric curve on $A$. Then

(a) $M_L^2 = 4$ and $M_L D_i = d$ for $1 \leq i \leq 16$,

(b) $F^2 = 2C^2 + \sum_{i=1}^{16} \text{mult}_{e_i}(F)^2$, and

(c) $LF = M_L C + \frac{d}{2} \sum_{i=1}^{16} \text{mult}_{e_i}(F)$.

The proof consists in an obvious calculation.

### 2. Bounding degrees and multiplicities

Here we show two technical statements on the degrees and multiplicities of symmetric curves. We start with a lemma which bounds the degree of a symmetric curve on $A$ in terms of the degree of the corresponding curve on the smooth Kummer surface of $A$:

**Lemma 2.1.** Let $C \subset X$ be an irreducible curve, different from $D_1, \ldots, D_{16}$, and let $F = \sigma \pi^*C$.

(a) If $M_L C = 0$, then $LF \leq 2 \left( 1 - C^2 \right) d^2 + 16$.

(b) If $M_L C > 0$, then $LF \leq 4 \left( M_L C - \frac{C^2}{M_L C} \right) d^2 + 9M_L C$.

**Proof.** For $\gamma \geq 0$ apply Hodge index to the line bundle $M_L$ and the divisor $C + \frac{\gamma}{d} D_i$:

$$
M_L^2 \left( C + \frac{\gamma}{d} D_i \right)^2 \leq \left( M_L C + \frac{\gamma}{d} M_L D_i \right)^2.
$$

Using Lemma 1.1(a) and the equality $CD_i = \text{mult}_{e_i}(F)$ we get

$$
\text{mult}_{e_i}(F) \leq \left( \frac{(M_L C)^2}{8\gamma} + \frac{\gamma}{8} + \frac{M_L C}{4} - \frac{C^2}{2\gamma} \right) d + \frac{\gamma}{d}.
$$
hence by Lemma 1.1(c)

\[ LF \leq \left( \frac{(MLC)^2}{\gamma} + \gamma + 2MLC - \frac{4C^2}{\gamma} \right) \gamma^2 + MLC + 8\gamma. \]

Now the assertion follows by setting \( \gamma = 2 \) in case \( MLC = 0 \) and by setting \( \gamma = MLC \) otherwise.

Further, we will need the following inequality on multiplicities of symmetric curves:

**Lemma 2.2.** Let \( F \subset A \) be a symmetric curve such that \( \mathcal{O}_A(F) \) is of type \((1, e)\) with \( e \) odd. Then

\[ 16 \sum_{i=1}^{16} \text{mult}_{e_i}(F)^2 \geq \frac{1}{16} \left( \sum_{i=1}^{16} \text{mult}_{e_i}(F) \right)^2 + \frac{15}{4}. \]

**Proof.** For \( k \geq 0 \) define the integers \( n_k \) by

\[ n_k = \# \{ i \mid m_i = k, \ 1 \leq i \leq 16 \}. \]

Abbreviating \( m_i = \text{mult}_{e_i}(F) \) we then have

\[ \sum_{i=1}^{16} m_i = \sum_{k \geq 0} k n_k \quad \text{and} \quad \sum_{i=1}^{16} m_i^2 = \sum_{k \geq 0} k^2 n_k. \]

The polarized abelian surface \( (A, \mathcal{O}_A(F)) \) is the pull-back of a principally polarized abelian surface \( (B, P) \) via an isogeny \( \varphi : A \rightarrow B \) of odd degree. The Theta divisor \( \Theta \in |P| \) passes through six halfperiods with multiplicity one and through ten halfperiods with even multiplicity. Therefore the symmetric divisor \( F \in |\varphi^*P| \) is of odd multiplicity in six halfperiods and of even multiplicity in ten halfperiods or vice versa. So we have

\[ \sum_{k \equiv 0(2)} n_k = a \quad \text{and} \quad \sum_{k \equiv 1(2)} n_k = b, \]

where \((a, b) = (6, 10)\) or \((a, b) = (10, 6)\).

Under the restriction (1) the difference

\[ \sum_{k \equiv 0(2)} k^2 n_k - \frac{1}{16} \left( \sum_{k \equiv 0(2)} k n_k \right)^2 \]

is minimal, if for some integer \( k_0 \geq 0 \) we have

\[ n_{k_0} = 10, \ n_{k_0+1} = 6 \ or \ n_{k_0} = 6, \ n_{k_0+1} = 10. \]

In this case we get \( \sum_{k \equiv 0(2)} k^2 n_k - \frac{1}{16} \left( \sum_{k \equiv 0(2)} k n_k \right)^2 = \frac{15}{4} \), which implies the assertion of the lemma.

3. **Kummer surfaces with sixteen skew rational curves of given degree**

The aim of this section is to show:

**Theorem 3.1.** Let \((A, L)\) be a polarized abelian surface of type \((1, 2d^2 + 1), \ d \geq 1.\) Assume \( \rho(A) = 1. \) Then the map \( \varphi_{ML} : X \rightarrow \mathbb{P}^3 \) defined by the linear system \(|ML|\) is an embedding. The image surface \( \varphi_{ML}(X) \) is a smooth quartic surface containing sixteen mutually disjoint smooth rational curves of degree \( d. \)

In particular, this implies the theorem stated in the introduction.
Proof. Using Riemann-Roch, Kodaira vanishing and Lemma 1.1(a), we will be done as soon as we can show that \( M_L \) is very ample. For \( d = 1 \) this follows from [3], whereas for \( d = 2 \) it follows from [1]. So we may assume \( d \geq 3 \) in the sequel.

(a) First we show that \( M_L \) is globally generated. A possible base part \( B \) of the system \( \left| O_A (2L) \otimes \bigotimes_{i=1}^{16} m_i \right| \) is totally symmetric, so \( B \) is algebraically equivalent to some even multiple of \( L \), which is impossible for dimensional reasons. It remains the possibility that one – hence all – of the curves \( D_i \) is fixed in \( |M_L| \). So \( M_L - \mu \sum D_i \) is free for some \( \mu \geq 1 \). But \( (M_L - \mu \sum D_i)^2 = 4 - 32\mu d - 32\mu^2 < 0 \), a contradiction.

(b) Our next claim is that \( M_L \) is ample. Otherwise there is an irreducible \((-2)\)-curve \( C \subset X \) such that \( M_L C = 0 \). Lemma 1.1 shows that we have

\[
LF = \frac{d}{2} \sum m_i \quad \text{and} \quad F^2 = -4 + \sum m_i^2
\]

for the symmetric curve \( F = \sigma^* \pi^* C \) with multiplicities \( m_i = \text{mult}_{x_i} (F) \). According to Lemma 2.1 the degree of \( F \) is bounded by

\[(2) \quad LF \leq 6d^2 + 16 . \]

Since \( L \) is a primitive line bundle, the assumption on the Néron-Severi group of \( A \) implies that \( O_A (F) \) is algebraically equivalent to some multiple \( pL \), \( p \geq 1 \), thus we have \( LF = pL^2 = p(4d^2 + 2) \), and then (2) implies \( p = 1 \) because of our assumption \( d \geq 3 \). So we find

\[
8d^2 + 4 = 2LF = d \sum m_i
\]

and reduction mod \( d \) shows that necessarily \( d = 4 \). But in this case \( \sum m_i \) would be odd, which is impossible (cf. [3]).

(c) Finally we prove that \( M_L \) is very ample. Suppose the contrary. Saint-Donat’s criterion [7, Theorem 5.2 and Theorem 6.1(iii)] then implies the existence of an irreducible curve \( C \subset X \) with \( M_L C = 2 \) and \( C^2 = 0 \). So we have

\[
LF = 2 + \frac{d}{2} \sum m_i \quad \text{and} \quad F^2 = \sum m_i^2
\]

for the corresponding symmetric curve \( F = \sigma^* \pi^* C \). Lemma 2.1 yields the estimate

\[
LF \leq 8d^2 + 18 .
\]

As above \( O_A (F) \) is algebraically equivalent to some multiple \( pL \), \( p \geq 1 \), hence we get

\[
p \left( 4d^2 + 2 \right) = pL^2 \leq 8d^2 + 18 ,
\]

which implies \( p \leq 2 \). If we had \( p = 2 \) then reduction mod \( d \) of the equation

\[
2 \left( 4d^2 + 2 \right) = 2 + \frac{d}{2} \sum m_i
\]

would give \( d = 4 \). But in this case we have \( \sum m_i = 65 \), which is impossible.

So the only remaining possibility is \( p = 1 \), thus

\[
4d^2 + 2 = 2 + \frac{d}{2} \sum m_i = \sum m_i^2 .
\]

But a numerical check shows that this contradicts Lemma 2.2. This completes the proof of the theorem. \( \square \)
Remark 3.2. We conclude with a remark on the genericity assumption on the abelian surface $A$. It is certainly not true that the line bundle $M_L$ is very ample for every polarized abelian surface $(A, L)$ of type $(1, 2d^2 + 1)$. Consider for instance the case where $A = E_1 \times E_2$ is a product of elliptic curves and $L = \mathcal{O}_A \left( \{0\} \times E_2 + (2d^2 + 1)E_1 \times \{0\} \right)$. Here, taking $C \subset X$ to be curve corresponding to $E_1 \times \{0\}$, we have

$$M_L C = 1 - 2d < 0,$$

so in this case $M_L$ is not even ample or globally generated.

References
2. Barth, W., Nieto, I.: Abelian surfaces of type $(1, 3)$ and quartic surfaces with 16 skew lines. J. Algebraic Geometry 3, 173-222 (1994) MR 95e:14033