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# CONSTRUCTION OF ANR TOPOLOGIES ON CERTAIN GROUPS

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Abstract. A method is shown to construct ANR-topologies on topological groups of suitable homotopy type.

It is well known that every simple countable CW-complex with vanishing Postnikov invariants has the homotopy type of a topological Abelian group $G$, that can be taken as realization of a countable simplicial Abelian group and hence is a countable CW-complex [5, Ch.V]. In particular this means that the homotopy groups $\pi_n(G)$ can be chosen as an arbitrarily prescribed sequence of countable Abelian groups. However, there are situations where it would be preferable to have $G$ as an ANR-space instead of a CW-complex. One would guess that a suitable topology exists on $G$, because by subdivision it can be turned into a polyhedron and then can get the metric topology, but the subdivision process necessarily leaves the range of simplicial Abelian groups and need not produce a group topology. Instead, we follow a method developed by Cauty in [1].

**Theorem.** *Let $G$ be a locally contractible, not necessarily Abelian group, every open subset of which is $\sigma$-compact, such that $G$ is the union of a sequence of finite dimensional compact metric spaces and has the homotopy type of a CW-complex. Then $G$ carries a metrizable group topology, coarser than the original one but of the same homotopy type, which turns $G$ into an ANR-space.*

All these requirements are satisfied, for instance, if $G$ itself is a countable CW-complex.

*Proof.* We construct a decreasing sequence of open neighborhoods $V_n$ of the identity $1 \in G$ such that

1. $\bigcap_{n=1}^{\infty} V_n = \{1\}$,
2. the inclusion map $V_{n+1} \hookrightarrow V_n$ is null homotopic,
3. if $H_m : V_{m+1} \times I \to V_m$ is the null homotopy from 2, then for any $g \in V_{m+1}$ and each $n \geq m$ there exist $N \geq n$ and $\varepsilon > 0$ such that $g' \in gV_N,\ |t' - t| < \varepsilon \Rightarrow H_m(g', t') \in H_m(g, t)V_n$,
4. $V_n^{-1} = V_n$ and $V_{n+1}V_{n+1} \subseteq V_n$,
5. for each $n$ and every $g \in G$ there exists $N \geq n$ with $g^{-1}V_N g \subseteq V_n$.

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All of these properties are immediately clear with the exception of 3 and 5. We construct $V_n$ inductively. Let $U_n$ be a sequence of neighborhoods of the identity with $\bigcap_{n=1}^{\infty} U_n = \{1\}$; it exists because the open set $G \setminus \{1\}$ is $\sigma$-compact and therefore $\{1\}$ is a $G_\delta$. Choose $V_1$ with $V_1 = V_1^{-1} \subseteq U_1$, otherwise arbitrarily. Now suppose that $V_1, \dots, V_n$ are already constructed such that 2, 4, $V_k \subseteq U_k$ and the following properties hold:

3a. For $k < m \leq n$ there exist a sequence of neighborhoods of the identity $\{W_i^{km} \mid i \in \mathbb{N}\}$ and a sequence of numbers $\varepsilon_i^{km} > 0$ such that for each $g \in V_{k+1}$ there exists $i$ with $g' \in gW_i^{km}$, $|t' - t| < \varepsilon_i^{km} \Rightarrow H_k(g', t') \in H_k(g, t)V_m$.

3b. $V_n \subseteq \bigcap_{k<m<n, i \leq n} W_i^{km}$.

5a. For every $g \in K_n$ we have $g^{-1}V_{n+1}g \subseteq V_n$, where $K_m$ is an increasing sequence of compacta covering $G$.

To pass from $n$ to $n+1$ choose $V_{n+1}$ subject to conditions 2, 3b, 4, 5a and $V_{n+1} \subseteq V_n \cap U_{n+1}$. It now remains to construct the neighborhoods $W_i^{k,n+1}$ for $k \leq n$. By assumption we can find a sequence of compact sets $C_i \subseteq V_{k+1}$ with $\bigcup_{i=1}^{\infty} C_i = V_{k+1}$. For each of them we choose a neighborhood of the identity $W_i^{k,n+1}$ with $C_i W_i^{k,n+1} \subseteq V_{k+1}$ and a number $\varepsilon_i^{k,n+1} > 0$ such that $g \in C_i$, $g' \in gW_i^{k,n+1}$ and $|t' - t| < \varepsilon_i^{k,n+1}$ implies $H_k(g', t') \in H_k(g, t)V_m$. It is clear that the whole sequence of neighborhoods $V_n$ satisfies 1–5 and hence defines a metrizable group topology on $G$, which is coarser than the original topology. Denote by $\tilde{G}$ the group $G$ topologized by our neighborhood system $V_n$. 3 ensures that the homotopies $H_n$ are continuous with respect to our metrizable topology, and hence that $\tilde{G}$ is locally contractible. It now follows from [3] that $\tilde{G}$ is an ANR-space and from [4, Theorem 1] that the identity map $G \to \tilde{G}$ is a homotopy equivalence. □

In a similar way it can be proved that the construction is "almost" functorial in the sense that for any family of continuous homomorphisms $f_\lambda : G_\lambda \to H_\lambda$ between groups satisfying the assumptions of our theorem, such that each group appears at most countably often among the source groups $G_\lambda$ the ANR group topology can be chosen in such a way that all homomorphisms $f_\lambda : \tilde{G}_\lambda \to \tilde{H}_\lambda$ are continuous. The countability assumption seems to be essential.

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