

## ON DUALS OF WEAKLY ACYCLIC $(LF)$ -SPACES

JUAN CARLOS DÍAZ AND SUSANNE DIEROLF

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ABSTRACT. For countable inductive limits of Fréchet spaces ( $(LF)$ -spaces) the property of being weakly acyclic in the sense of Palamodov (or, equivalently, having condition  $(M_0)$  in the terminology of Retakh) is useful to avoid some important pathologies and in relation to the problem of well-located subspaces. In this note we consider if weak acyclicity is enough for a  $(LF)$ -space  $E := \text{ind } E_n$  to ensure that its strong dual is canonically homeomorphic to the projective limit of the strong duals of the spaces  $E_n$ . First we give an elementary proof of a known result by Vogt and obtain that the answer to this question is positive if the steps  $E_n$  are distinguished or weakly sequentially complete. Then we construct a weakly acyclic  $(LF)$ -space for which the answer is negative.

### INTRODUCTION

Countable inductive limits of Fréchet spaces ( $(LF)$ -spaces) arise in many fields of functional analysis and its applications; e.g., in distribution theory, linear partial differential equations, convolution equations, Fourier analysis, complex analysis. Since their topological structure presents severe pathologies (see [11, 4], [9], [2, 3 and Appendix]) it is interesting to study those conditions which ensure a good behaviour with respect to some particular problems. Here we are concerned with deciding whether the property of being weakly acyclic in the sense of Palamodov (equivalent to the condition  $(M_0)$  of Retakh) is enough for a  $(LF)$ -space  $E$  to identify (in a canonical way) its strong dual with a projective limit of  $(DF)$ -spaces.

Let us fix some notation before going on with the introduction. For a locally convex space  $E$  let  $E'$  denote the topological dual and  $E^*$  its algebraic dual. Given a dual pair  $\langle E, F \rangle$ , let  $\sigma(E, F)$  and  $\beta(E, F)$  denote the corresponding weak and strong topologies, respectively. The bidual of  $E$  is  $E'' := (E', \beta(E', E))'$ , moreover we set  $E'_\beta := (E', \beta(E', E))$  and  $E'_{\beta^2} := (E', \beta(E', E''))$ . The space  $E$  is said to be *distinguished* if every bounded subset of  $(E'', \sigma(E'', E'))$  lies in the  $\sigma(E'', E')$ -closure of some bounded subset of  $E$ . This is equivalent to  $\beta(E', E) = \beta(E', E'')$ ; it is also equivalent with requiring that  $E'_\beta$  is barrelled ([13, 23.7]). If  $E$  is metrizable then it is distinguished if and only if  $E'_\beta$  is bornological, cf. [13, 29.4].

A sequence  $(E_n)_{n \in \mathbb{N}}$  of locally convex spaces is inductive if  $E_n$  is continuously included into  $E_{n+1}$  ( $n \in \mathbb{N}$ ); then  $E := \text{ind } E_n$  is its inductive limit whenever  $E = \bigcup_{n \in \mathbb{N}} E_n$  and  $E$  is endowed with the finest locally convex topology such that the injections  $E_n \rightarrow E$  are continuous.  $E$  is an  $(LF)$ -space if the  $E_n$ 's are Fréchet.

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(We refer to [2], [19] for more information about inductive limits.) Recall that  $E := \text{ind } E_n$  is said to be *weakly acyclic* if the map

$$\oplus E_n \rightarrow \oplus E_n, \quad (x_n)_{n \in \mathbb{N}} \mapsto (x_n - x_{n-1})_{n \in \mathbb{N}}, \quad (x_0 := 0)$$

is a weak topological isomorphism onto its range. Weak acyclicity is equivalent to the property  $(M_0)$  introduced by Retakh in connection with the problem of well-located subspaces, which applies to the study of surjectivity and normal solvability of linear operators in duals of  $(LF)$ -spaces (in particular of linear partial differential operators in the space  $\mathcal{D}'(\Omega)$  of distributions on  $\Omega$ ); see [14], [2, Appendix] and [9] for details.

An important task on inductive limits, which goes back to Grothendieck, is the description of the strong dual of  $E := \text{ind } E_n$  as the projective limit of the duals of  $E_n$  ( $n \in \mathbb{N}$ ). Indeed, the canonical linear map

$$j : E'_\beta \longrightarrow \text{proj } E'_{n,\beta}, \quad f \longmapsto (f|E_n)_{n \in \mathbb{N}}$$

(where the projective limit is formed with respect to the natural restrictions  $E'_{n+1} \rightarrow E'_n$ ,  $g \mapsto g|E_n$ ) is a continuous and surjective isomorphism, but  $j$  does not need to be open. By a standard duality argument,  $j$  is a topological isomorphism if and only if for every bounded subset  $B$  in  $E$  there is  $n \in \mathbb{N}$  and a bounded subset  $A$  in  $E_n$  such that  $B \subset \overline{A}^E$ . We will call an inductive sequence  $(E_n)_{n \in \mathbb{N}}$  *quasiregular* if it satisfies this (very weak regularity) condition. (It is known that the inductive sequence  $(E_n)_{n \in \mathbb{N}}$  is said to be *regular* if for every bounded subset  $B$  in  $E$  there is  $n \in \mathbb{N}$  such that  $B$  is contained and bounded in  $E_n$ . Regularity has been thoroughly studied in the theory of  $(LF)$ -spaces, cf. [2], [19].)

Next we give examples and further information about quasiregular  $(LF)$ -spaces: (i) Every inductive sequence consisting of  $DF$ -spaces  $E_n$  ( $n \in \mathbb{N}$ ), in particular every  $(LB)$ -space, is quasiregular (by Grothendieck [10]). (ii) If an inductive sequence  $(E_n)_{n \in \mathbb{N}}$  consists of semireflexive spaces  $E_n$  and if the inductive limit  $E := \text{ind } E_n$  is Hausdorff, then  $(E_n)_{n \in \mathbb{N}}$  is quasiregular if and only if it is regular. Consequently, an inductive sequence  $(E_n)_{n \in \mathbb{N}}$  consisting of Fréchet Montel spaces with Hausdorff inductive limit is quasiregular if and only if it is complete (see Wengenroth [20, Theorem 3.9]). (A very easy example of an incomplete  $(LF)$ -space with Fréchet Montel steps is obtained in the following way: Take  $X := \omega := \mathbb{K}^{\mathbb{N}}$  (where  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ ) and let  $Y$  be any Fréchet Montel space continuously embedded into  $X$  with proper dense range. Then the inductive sequence  $(\prod_{k < n} X \times \prod_{k \geq n} Y)_{n \in \mathbb{N}}$  consists of Fréchet Montel spaces and, according to Dierolf [8, Prop.], its inductive limit is topologically isomorphic to a proper dense linear subspace of  $X^{\mathbb{N}}$  hence incomplete, metrizable and provided with a weak topology.) (iii) Quasiregular  $(LF)$ -spaces of generalized Moscatelli type of the shape  $\oplus X + \ell^\infty(Y)$ , where  $Y, X$  are Fréchet spaces with continuous inclusion  $Y \hookrightarrow X$ , were characterized by Bonet-Dierolf-Fernández [6].

It was demonstrated by Vogt ([19, 5], see also [15] and [3] for related results) that weakly acyclic  $(LF)$ -spaces need not be regular. So it is natural to ask whether they are quasiregular. This question was somehow considered by Vogt in [18] and [19]. The following theorem is a direct consequence of Vogt's Lemma 4.1 of [19], Theorem 5.6 of [18] and the fact that all  $E'_{n,\beta^2}$  are bornological by [13, 29.5]. (Note that this theorem extends a Grothendieck's result [12, 3.6, Theorem 2] asserting that a strict  $(LF)$ -space with distinguished steps is distinguished, from where it can be deduced that the space  $\mathcal{D}'_\beta(\Omega)$  of distributions on  $\Omega$  is barrelled and bornological. On the

other hand, we refer the reader to [4] for negative results about the topological structure of strong duals of strict (LF)-spaces.)

**Theorem A** (Vogt). *Let  $E = \text{ind } E_n$  be a weakly acyclic (LF)-space. Then the natural projective limit  $\text{proj } E'_{n,\beta^2}$  is bornological, hence the linear bijection*

$$j : E'_{\beta^2} \longrightarrow \text{proj } E'_{n,\beta^2}, \quad f \longmapsto (f|_{E_n})_{n \in \mathbb{N}}$$

*is a homeomorphism.*

Therefore, if the (LF)-space  $E = \text{ind } E_n$  is weakly acyclic and we consider the diagram

$$\begin{array}{ccc} E'_{\beta^2} & \xrightarrow{j} & \text{proj } E'_{n,\beta^2} \\ \text{id} \downarrow & & \text{id} \downarrow \\ E'_\beta & \xrightarrow{j} & \text{proj } E'_{n,\beta} \end{array}$$

of linear continuous maps, the upper horizontal arrow is a topological isomorphism. Moreover, if all  $E_n$  are distinguished, also the lower horizontal arrow is a topological isomorphism which means that  $(E_n)_{n \in \mathbb{N}}$  is quasiregular. Thus the following result is an immediate consequence of Theorem A.

**Corollary A** (Vogt). *A weakly acyclic (LF)-space  $E$  with distinguished steps is quasiregular.*

The distinguishedness hypothesis does not appear in the statement of [19, 4.2], but it is necessary to prove [18, Theorem 5.6] (see the introduction of [18, Chapter 4]). The problem remains open whether weakly acyclic (LF)-spaces are always quasiregular. In this paper we solve this question in the negative. Indeed we first give an elementary proof of Corollary A, which even provides additional information and a new insight into quasiregularity of weakly acyclic (LF)-spaces. Then we construct a non-quasiregular weakly acyclic (LF)-space.

We start with a technical lemma.

**Lemma 1.** *Let  $(E_n)_{n \in \mathbb{N}}$  be an inductive sequence of metrizable locally convex spaces such that  $E := \text{ind } E_k$  is Hausdorff. For all  $k, m \in \mathbb{N}$ ,  $m \geq k$ , let  $j_{mk} : E_k \hookrightarrow E_m$  and  $j_k : E_k \hookrightarrow E$  denote the natural inclusions. Moreover, let  $n \in \mathbb{N}$  be given and an absolutely convex subset  $A$  of  $E_n$  such that*

$$\sigma(E_n, E'_n) \cap A = \sigma(E, E') \cap A.$$

*Then*

$$\overline{A}^E = \bigcup_{k \in \mathbb{N}, k \geq n} \overline{A}^{E_k}$$

*and there is  $B \subset E''_n$ ,  $A \subset B$ , such that:*

- (i) *Each  $\psi \in B$  is the  $\sigma(E''_n, E'_n)$ -limit of a sequence in  $A$ ;*
- (ii) *The bitranspose  $j_n^{tt} : E''_n \rightarrow E''$  of  $j_n$  maps*

$$(B, \sigma(E''_n, E'_n) \cap B) \text{ topologically onto } (\overline{A}^E, \sigma(E, E') \cap \overline{A}^E).$$

*Proof.* Let  $\overline{A}^n$  and  $\overline{A}$  denote the closure of  $A$  in

$$(E_n^{I*}, \sigma(E_n^{I*}, E'_n)) =: (E_n^{I*}, \sigma_n^{I*}) \quad \text{and} \quad (E^{I*}, \sigma(E^{I*}, E')) =: (E^{I*}, \sigma^{I*}),$$

respectively. As  $(\bar{A}^n, \sigma_n^{t*} \cap \bar{A}^n)$  and  $(\bar{A}, \sigma^{t*} \cap \bar{A})$  are Hausdorff completions of the uniform spaces  $(A, \sigma(E_n, E'_n) \cap A)$  and  $(A, \sigma(E, E') \cap A)$ , respectively, the bitranspose

$$j^{t*} : E_n^{t*} \longrightarrow E^{t*}, \quad \psi \mapsto (g \mapsto \psi(g \circ j_n))$$

induces a uniform equivalence from  $(\bar{A}^n, \sigma_n^{t*} \cap \bar{A}^n)$  onto  $(\bar{A}, \sigma^{t*} \cap \bar{A})$ . Let  $\hat{A}^n := \{\psi \in \bar{A}^n : j_n^{t*}(\psi) \in E\}$ . Then, clearly,  $j_n^{t*}$  induces a surjective uniform equivalence

$$\hat{j}_n : (\hat{A}^n, \sigma_n^{t*} \cap \hat{A}^n) \longrightarrow (\bar{A}, \sigma(E, E') \cap \bar{A}^E).$$

Analogously, for  $m \geq n$ , the bitranspose  $j_{m,n}^{t*} : E_n^{t*} \rightarrow E_m^{t*}$  induces a surjective uniform equivalence  $\hat{j}_{m,n} : (\hat{A}^n, \sigma_n^{t*} \cap \hat{A}^n) \rightarrow (\hat{A}^m, \sigma_m^{t*} \cap \hat{A}^m)$  and  $\hat{j}_m \circ \hat{j}_{m,n} = \hat{j}_n$ . Let  $z \in \bar{A}^E$  be given. Then there is  $m \in \mathbb{N}$ ,  $m \geq n$  such that  $z \in E_m$ , and hence

$$z = (\hat{j}_m)^{-1}(z) \in \hat{A}^m \cap E_m \subset \bar{A}^{\sigma(E_m, E'_m)} = \bar{A}^{E_m}.$$

Consequently, there is a sequence  $(z_k)_{k \in \mathbb{N}}$  in  $A$  which converges to  $z$  in the Fréchet space topology of  $E_m$ , hence in  $(\hat{A}^m, \sigma_m^{t*} \cap \hat{A}^m)$ . Because of the properties of  $\hat{j}_{m,n}$ , the sequence  $(z_k)_{k \in \mathbb{N}}$  converges to  $(\hat{j}_n)^{-1}(z)$  in  $(\hat{A}^n, \sigma_n^{t*} \cap \hat{A}^n)$ . This implies  $(\hat{j}_n)^{-1}(z) \in E_n''$ , hence because of  $\hat{j}_n(\hat{A}^n) = \bar{A}^E$ ,  $\hat{A}^n \subset E_n''$ .

Putting  $B := \hat{A}^n$  finishes the proof of the lemma. □

Next we state and prove our main result about quasiregularity of weakly acyclic  $(LF)$ -spaces.

**Theorem 2.** *Let  $E = \text{ind } E_n$  be a weakly acyclic  $(LF)$ -space.*

(i) *For every separable bounded subset  $C$  of  $E$  there exists  $r \in \mathbb{N}$  and a bounded subset  $B$  in  $E_r$  such that  $C \subset \bar{B}^E$ .*

(ii) *Let us assume that each  $E_n$  (or at least infinitely many of the  $E_n$ 's) satisfies the following condition: For each bounded set  $B \subset E_n''$ , whose elements are  $\sigma(E_n'', E'_n)$ -limits of sequences in  $E_n$ , there is a bounded set  $D$  in  $E_n$  such that  $B$  is contained in the  $\sigma(E_n'', E'_n)$ -closure of  $D$ . Then the inductive sequence  $(E_n)_{n \in \mathbb{N}}$  is quasiregular. In particular, weakly acyclic  $(LF)$ -spaces whose steps are distinguished or weakly sequentially complete, are quasiregular.*

*Proof.* Since  $E = \text{ind } E_n$  satisfies condition  $(M_0)$  of Retakh (see e.g. Vogt [19, Theorem 2.11]), we may assume without loss of generality that there exists an increasing sequence  $(U_n)_{n \in \mathbb{N}}$  of absolutely convex 0-nbhds  $U_n$  in  $E_n$  such that for all  $n \in \mathbb{N}$  one has  $\sigma(E_{n+1}, E'_{n+1}) \cap U_n = \sigma(E, E') \cap U_n$ . Now let  $C \subset E$  be bounded. As  $E$  is in particular barrelled, a standard argument (see [16, 8.1.23]) shows that  $C \subset n\bar{U}_n^E$  for a suitable  $n \in \mathbb{N}$ . Applying the lemma to  $A := nU_n \subset E_{n+1}$ , we obtain that there is a subset

$$D \subset \{\psi \in E_{n+1}'' : \psi \text{ is the } \sigma(E_{n+1}'', E'_{n+1})\text{-limit of a sequence in } A\}$$

such that the bitranspose  $j_{n+1}^{tt} : E_{n+1}'' \rightarrow E''$  of the inclusion  $j_{n+1} : E_{n+1} \hookrightarrow E$  maps  $(D, \sigma(E_{n+1}'', E'_{n+1}) \cap D)$  topologically onto  $(C, \sigma(E, E') \cap C)$ . Clearly,  $D$  is bounded in  $E_{n+1}''$ .

If  $C$  is separable, also  $D$  is separable and there is  $B \subset E_{n+1}$  bounded and absolutely convex such that  $D$  is contained in the  $\sigma(E_{n+1}'', E'_{n+1})$ -closure of  $B$ , whence  $C \subset \bar{B}^{\sigma(E'', E')} \cap E = \bar{B}^E$ . This proves (i).

If the hypothesis in (ii) is satisfied, then there is  $B \subset E_{n+1}$  bounded and absolutely convex, such that  $D \subset \overline{B}^{\sigma(E''_{n+1}, E'_{n+1})}$  and we obtain  $C \subset \overline{B}^E$ .  $\square$

The remainder of the note is devoted to show that the hypotheses of Theorem 2 cannot be dropped. We first give a technical result where conditions to construct a weakly acyclic not quasiregular (LF)-space are settled.

**Proposition 3.** *Assume that there exist Fréchet spaces  $Y, X$  with continuous inclusion  $j : Y \hookrightarrow X$  satisfying the following three conditions:*

- (i) *There is a 0-nbhd.  $U$  in  $Y$  such that  $\sigma(Y, Y') \cap U = \sigma(X, X') \cap U$ .*
- (ii) *The bitranspose  $j^{tt} : Y'' \rightarrow X''$  maps  $Y''$  into  $X$ .*
- (iii)  *$Y$  is not distinguished.*

*Then there exists a weakly acyclic (LF)-space which is not quasiregular.*

*Proof.* For every  $n \in \mathbb{N}$ ,

$$E_n := \prod_{h < n} X \times c_0((Y)_{h \geq n}) := \{(x_h)_{h \in \mathbb{N}} \in \prod_{h < n} X \times \prod_{h \geq n} Y : x_h \xrightarrow{h \geq n} 0 \text{ in } Y\}$$

provided with its natural Fréchet space topology is continuously included into  $E_{n+1}$ . The (LF)-space  $E = \text{ind } E_n$  is weakly acyclic by Müller-Dierolf-Frerick [15, Theorem (2)]. We will prove that  $(E_n)_{n \in \mathbb{N}}$  is not quasiregular. In fact, since  $Y$  is not distinguished, there is a bounded set  $B$  in  $Y''$  such that for all bounded subsets  $A$  in  $Y$ ,  $B$  is not contained in  $\overline{A}^{\sigma(Y'', Y')}$ . We will construct a bounded set  $C$  in  $E$  for which the quasiregularity property fails. By hypothesis (ii) the set  $C := \bigoplus_{h \in \mathbb{N}} j^{tt}(B)$  is a subset of  $E$ .

We first check that  $C$  is bounded in  $E$ . Let  $V$  be a closed 0-nbhd in  $E$ . Then there is an absolutely convex 0-nbhd  $W$  in  $Y$  such that  $W^{\mathbb{N}} \cap c_0(Y) \subset V$ . As  $B$  is  $\sigma(Y'', Y')$ -bounded, there is  $\rho > 0$  such that  $B \subset \rho \overline{W}^{\sigma(Y'', Y')}$  which implies

$$C = \bigoplus_{h \in \mathbb{N}} j^{tt}(B) \subset \rho \bigoplus_{h \in \mathbb{N}} j^{tt}(\overline{W}^{\sigma(Y'', Y')}) \subset_{(ii)} \rho \bigoplus_{h \in \mathbb{N}} \overline{W}^X \subset_{(*)} \rho \overline{\bigoplus_{h \in \mathbb{N}} W}^E \subset \rho V$$

where  $(*)$  follows from the fact that the direct sum  $\bigoplus_{h \in \mathbb{N}} X$  is continuously embedded into  $E$ . Now it remains to show that  $C$  is not contained in the  $E$ -closure of any bounded set of a step. Assuming the contrary, there is a bounded set  $A$  in  $Y$  and  $n \in \mathbb{N}$  such that

$$C \subset \overline{\left( \prod_{h < n} X \times \prod_{h \geq n} A \right) \cap E_n}^E.$$

Looking at the  $n$ -th component one obtains that  $j^{tt}(B) \subset \overline{A}^X$ . On the other hand, by (i) and (ii) the bitranspose  $j^{tt} : Y'' \rightarrow X$  maps  $\overline{U}^{\sigma(Y'', Y')}$  provided with  $\sigma(Y'', Y') \cap \overline{U}^{\sigma(Y'', Y')}$  topologically onto  $(\overline{U}^X, \sigma(X, X') \cap \overline{U}^X)$ . In particular,  $j^{tt}$  is injective. Since  $A$  is absorbed by  $U$ , we may assume that  $A \subset U$  and obtain that  $j^{tt}(\overline{A}^{\sigma(Y'', Y')}) = \overline{A}^{\sigma(X, X')}$ . From the above  $j^{tt}(B) \subset \overline{A}^X$ , we now deduce that  $B \subset \overline{A}^{\sigma(Y'', Y')}$ , which is a contradiction to the choice of  $B$ .  $\square$

*Observations 4.* 1) The statement of Proposition 3 holds if  $Y, X$  are assumed to be just metrizable locally convex spaces. With the same proof we get a corresponding inductive sequence  $(E_n)_{n \in \mathbb{N}}$  of metrizable locally convex spaces which is again weakly acyclic (by [15, Theorem (2)]) but not quasiregular. For this (less interest-

ing) situation suitable entries  $Y \hookrightarrow X$  are obtained quite easily. According to [4, p. 208], see also [1], there exists a reflexive Fréchet space  $X$  containing a dense linear subspace  $Y$  which is not distinguished. In that case the corresponding  $LM$ -space  $(E_n)_{n \in \mathbb{N}}$  (of generalized Moscatelli type) is even strict without being quasiregular.

2) It is also easy to present a pair  $Y \hookrightarrow X$  even of Banach spaces satisfying (i) and (ii) of the proposition: Take  $Y := c_0$  and  $X := \{(x_n)_{n \in \mathbb{N}} : (\frac{1}{n}x_n)_{n \in \mathbb{N}} \in c_0\}$ . Then the Moscatelli construction leads to a weakly acyclic but not regular  $LB$ -space (see [15, p. 158]).

3) In order to find Fréchet entries  $Y \hookrightarrow X$  satisfying (i), (ii), (iii) of Proposition 3, one must at least find a Fréchet space  $Y$  which is nondistinguished to such an extent that its bidual  $Y''$  contains a bounded set whose elements are limits of weak Cauchy sequences in  $Y$  and which is not contained in the  $\sigma(Y'', Y')$ -closure of a bounded set in  $Y$ . A Fréchet space with that special property had been constructed by Díaz [7]. In fact, this space is weak\*-sequentially dense in its bidual. An appropriate use of that construction leads to the following example:

**Example 5.** (Fréchet spaces  $Y \hookrightarrow X$  satisfying (i), (ii) and (iii) of Proposition 3.) We start with some Banach background. The classical quasireflexive James space  $J$  is defined as

$$J := \{(x_i)_{i \geq 1}; \sup_{0=n_0 < n_1 < \dots < n_k} \left( \sum_{j=0}^{k-1} \left( \sum_{i=n_j+1}^{n_{j+1}} x_i \right)^2 \right)^{1/2} < \infty\}.$$

If  $e_i$  denotes the sequence taking the value 1 in the  $i$ -th component and 0 elsewhere then  $(e_i)_{i \geq 1}$  is a boundedly complete basis (by basis we mean Schauder basis) of  $J$ . The dual space  $J'$  has a basis given by  $(e_i^*)_{i \geq 0}$  where  $(e_i^*)_{i \geq 1}$  is the sequence of biorthogonal coefficient functionals associated to  $(e_i)_{i \geq 1}$  and  $e_0^*$  is defined by

$$e_0^* \left( \sum_{i=1}^{\infty} x_i e_i \right) := \sum_{i=1}^{\infty} x_i, \quad \forall x = (x_i) \in J.$$

(This element should be denoted  $e_\omega^*$  but we prefer to use 0 instead of  $\omega$  in our context.) Thus  $J' \equiv \text{sp}[e_0^*] \oplus \overline{\text{sp}}[e_i^*; i \geq 1]$ . The dual of  $\overline{\text{sp}}[e_i^*; i \geq 1]$  is  $J$ ; therefore if  $e_0$  denotes the element in  $J''$  such that  $e_0(e_i^*) = \delta_{0,i}$ , ( $i \geq 0$ ), then  $J'' \equiv \text{sp}[e_0] \oplus J$  and  $(e_i)_{i \geq 0}$  is a basis of  $J''$ . On account of this information if we consider the continuous linear map

$$\phi : J \rightarrow \mathbb{K} \oplus \ell_2, \quad (x_i)_{i \geq 1} \mapsto \left( \sum_{i=1}^{\infty} x_i, (x_i)_{i \geq 1} \right),$$

then an easy computation shows that the transpose  $\phi^t$  and bitranspose  $\phi^{tt}$  are given by

$$\begin{aligned} \phi^t : \mathbb{K} \oplus \ell_2 &\rightarrow J', & (\alpha, (y_n)) &\mapsto \alpha e_0^* + \sum_{n=1}^{\infty} y_n e_n^*, \\ \phi^{tt} : J'' &\rightarrow \mathbb{K} \oplus \ell_2, & (x_n)_{n \geq 0} &\mapsto \left( \sum_{n=0}^{\infty} x_n, (x_n)_{n \geq 1} \right); \end{aligned}$$

in particular  $\phi^{tt}$  is injective.

We now come to the Fréchet space framework. Denote by  $\Gamma$  the set of all real valued increasing sequences  $(\gamma(n))_{n \in \mathbb{N}}$  with  $\gamma(1) \geq 1$ . Given the space  $\mathbb{K}^{\Gamma \times \mathbb{N}}$  and  $k \in \mathbb{N}$ ,  $P_k$  denotes the canonical linear projection onto the  $n \geq k$  rows. We also

denote by  $P_k$  the projection in  $J$  defined by  $P_k(\sum_{i=1}^\infty x_i e_i) := \sum_{i=k}^\infty x_i e_i$  ( $x = (x_i) \in J$ ); note that  $\|P_k\| = 1$ ,  $k \in \mathbb{N}$ . The following space was introduced in [7]:

$$Y := \{(z_\gamma) = (z_{\gamma,j})_{\gamma,(j \geq 1)} \in J^\Gamma;$$

$$\|(z_\gamma)\|_k := \sum_{j=1}^{k-1} (\sum_{\gamma} |z_{\gamma,j}|^2 \gamma(j))^{1/2} + (\sum_{\gamma} \|P_k(z_\gamma)\|_J^2)^{1/2} < \infty, k \in \mathbb{N}\}.$$

Observe that the  $k$ -th local Banach space  $Y_k := (Y/\|\cdot\|_k^{-1}(0), \|\cdot\|_k)$  is isometric to

$$\ell_2(\Gamma, \gamma(1)) \oplus \dots \oplus \ell_2(\Gamma, \gamma(k-1)) \oplus P_k(\ell_2(\Gamma, J))$$

where

$$\ell_2(\Gamma, \gamma(i)) := \{(x_\gamma) \in \mathbb{K}^\Gamma; \|(x_\gamma)\| = (\sum_{\gamma \in \Gamma} |x_\gamma|^2 \gamma(i))^{1/2} < \infty\},$$

$$\ell_2(\Gamma, J) := \{(z_\gamma) \in J^\Gamma; \|(z_\gamma)\| = (\sum_{\gamma \in \Gamma} \|z_\gamma\|_J^2)^{1/2} < \infty\}.$$

Moreover for every  $k \in \mathbb{N}$  the linking map  $I_{k,k+1} : Y_{k+1} \rightarrow Y_k$  is defined, when restricted to the  $k$ -th row, as the canonical continuous inclusion from  $\ell_2(\Gamma, \gamma(k))$  into  $(P_k - P_{k-1})(\ell_2(\Gamma, J))$  and it is the identity on the rest. The space  $Y \equiv \text{proj}(Y_k, I_{k,k+1})$  is not distinguished (see [7]) but every linear form defined on  $Y'_\beta$  and bounded on the bounded sets is continuous: In fact, if  $f$  is a linear form defined on  $Y'$  and continuous on  $Y'_k$  ( $k \in \mathbb{N}$ ), there is countable subset  $N \subset \Gamma$  such that, if  $Y_N := \{(z_\gamma) \in Y : z_\gamma = 0 \text{ if } \gamma \notin N\}$  then  $f$  annihilates outside  $(Y_N)'$ . Now  $(Y_N)'_\beta$  is separable (cf. [7, Theorem 5.(i)]) hence barrelled and bornological ([13, 29.3 and 29.4]). Thus  $f$  is continuous on  $(Y_N)'_\beta$ ; since  $(Y_N)'_\beta$  is complemented in  $Y'_\beta$  it follows that  $f$  is continuous. This in particular implies that the bidual of  $Y$  is  $Y'' \equiv \text{proj}(Y''_k, I_{k+1,k}^{tt})$ . Also observe that  $I_{k+1,k}^{tt}$  is injective for every  $k \in \mathbb{N}$  whence the seminorm induced by  $Y''_1$  is a norm on  $Y''$ .

We now construct the space  $X$ . First we need the following non-separable Köthe sequence space of order 2 on a Köthe matrix  $A$  implicitly defined below,

$$\lambda_2(\Gamma \times \mathbb{N}, A) := \{(x_{\gamma,n}) \in \mathbb{K}^{\Gamma \times \mathbb{N}};$$

$$\|(x_{\gamma,n})\|_k := (\sum_{\gamma,(n < k)} |x_{\gamma,n}|^2 \gamma(n) + \sum_{\gamma,(n \geq k)} |x_{\gamma,n}|^2)^{1/2} < \infty, k \in \mathbb{N}\}.$$

Then we consider the space  $X := \ell_2(\Gamma) \oplus \lambda_2(\Gamma \times \mathbb{N}, A)$  and define the following linear inclusion:

$$\Phi : Y \hookrightarrow X, \quad (x_\gamma) \mapsto ((\sum_{n=1}^\infty x_{\gamma,n})_\gamma, (x_{\gamma,n})_{\gamma,(n \geq 1)}).$$

Let us check that  $\Phi$  is continuous.

$$\|\Phi(x)\|_k = (\sum_{\gamma} (\sum_{n=1}^\infty x_{\gamma,n})^2)^{1/2} + (\sum_{j=1}^{k-1} (\sum_{\gamma} |x_{\gamma,j}|^2 \gamma(j)) + \sum_{\gamma} \sum_{j \geq k} |x_{\gamma,j}|^2)^{1/2}$$

$$\leq (\sum_{\gamma} \|x_\gamma\|_J^2)^{1/2} + (\sum_{j=1}^{k-1} (\sum_{\gamma} |x_{\gamma,j}|^2 \gamma(j)) + \sum_{\gamma} \|P_k(x_\gamma)\|_J^2)^{1/2}$$

$$\leq \|x\|_1 + T_k \|x\|_k$$

for some constant  $T_k$  that only depends on  $k$ .

Condition (ii) of Proposition 3 is obvious and condition (iii) was proved in [7] as already stated. To show (i) we will check the following two properties:

(i') Let  $U_1$  denote the unit ball of  $\|\cdot\|_1$  in  $Y$ . If  $B \subset U_1$  and  $\Phi(B)$  is bounded in  $X$  then  $B$  is bounded in  $Y$ .

(i'') The bitranspose  $\Phi^{tt}$  is injective.

(It is readily checked that (i') + (i'') is equivalent to (i) with  $U = U_1$ .)

*Proof of (i').* Take any  $B \subset Y$  with  $B \subset U_1$ , such that  $\Phi(B)$  is bounded, and let us show that  $B$  is bounded in  $Y$ . Indeed let  $M_k := \sup \{\|\Phi(z)\|_k; z \in B\}$  and let us estimate  $\|z\|_k$ ,  $z = (z_\gamma) \in B$ .

$$\begin{aligned} \|z\|_k &= \sum_{j=1}^{k-1} \left( \sum_{\gamma} |z_{\gamma,j}|^2 \gamma(j) \right)^{1/2} + \left( \sum_{\gamma} \|P_k(z_\gamma)\|_J^2 \right)^{1/2} \\ &\leq \|\Phi(z)\|_k + \left( \sum_{\gamma} \|z_\gamma\|_J^2 \right)^{1/2} \\ &\leq M_k + \|z\|_1 \leq M_k + 1. \end{aligned}$$

□

*Proof of (i'').* From the proof of continuity it follows that  $\Phi$  factorizes through the  $k$ -th local Banach spaces; in particular we have

$$\begin{array}{ccc} Y & \xrightarrow{\Phi} & X \\ \downarrow & & \downarrow \\ Y_1 & \xrightarrow{\phi} & X_1 \end{array}$$

Bidualizing this diagram we get

$$\begin{array}{ccc} Y'' & \xrightarrow{\Phi^{tt}} & X \\ \downarrow & & \downarrow \\ Y_1'' & \xrightarrow{\phi^{tt}} & X_1 \end{array}$$

Since the first seminorm is a norm on  $Y''$  it suffices to show that  $\phi^{tt}$  is injective. In this case we have (recall the Banach case)

$$\begin{aligned} \phi^{tt} : \ell_2(\Gamma, J'') &\rightarrow \ell_2(\Gamma) \oplus \ell_2(\Gamma \times \mathbb{N}), \\ (x_\gamma) &= (x_{\gamma,n})_{\gamma, (n \geq 0)} \mapsto \left( \left( \sum_{n=0}^{\infty} x_{\gamma,n} \right)_\gamma, (x_{\gamma,n})_{\gamma, (n \geq 1)} \right), \end{aligned}$$

which is clearly injective. This finishes the construction of the example. □

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DEPARTAMENTO DE MATEMÁTICAS, E.T.S.I.A.M., UNIVERSIDAD DE CÓRDOBA, 14004 CÓRDOBA, SPAIN

*E-mail address:* ma1dialj@lucano.uco.es

FBIV-MATHEMATIK, UNIVERSITÄT TRIER, D-54286 TRIER, GERMANY