

ON THE PRODUCT PROPERTY OF THE PLURICOMPLEX GREEN FUNCTION

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ABSTRACT. We prove that the pluricomplex Green function has the product property $g_{D_1 \times D_2} = \max\{g_{D_1}, g_{D_2}\}$ for any domains $D_1 \subset \mathbb{C}^n$ and $D_2 \subset \mathbb{C}^m$.

Let E denote the unit disc in \mathbb{C} . For any domain $G \subset \mathbb{C}^n$ define

$$g_D(a, z) := \inf_{\substack{\varphi \in \mathcal{O}(E, D) \\ \varphi(0) = z \\ a \in \varphi(E)}} \left\{ \prod_{\lambda \in \varphi^{-1}(a)} |\lambda|^{\text{ord}_\lambda(\varphi - a)} \right\}, \quad a, z \in D,$$

where $\mathcal{O}(E, D)$ denotes the set of all holomorphic mappings $E \rightarrow D$ and $\text{ord}_\lambda(\varphi - a)$ denotes multiplicity of $\varphi - a$ at λ .

The function g_D is proposed by Poletsky (cf. [Pol]) and is called the *pluricomplex Green function for D* . We have that (see [Jar-Pff1], Chapter IV)

$$(a) \quad g_D(a, z) = \inf_{\substack{\varphi \in \mathcal{O}(\bar{E}, D) \\ \varphi(0) = z \\ a \in \varphi(E)}} \left\{ \prod_{\lambda \in \varphi^{-1}(a)} |\lambda|^{\text{ord}_\lambda(\varphi - a)} \right\}, \quad a, z \in D.$$

Note that in the formula (a) we take only $\lambda \in \varphi^{-1}(a)$ such that $\lambda \in E$.

(b) For any domains D_1, D_2 and any holomorphic mapping $f : D_1 \rightarrow D_2$ we have the following contractible property: $g_{D_2}(f(z), f(w)) \leq g_{D_1}(z, w)$, $z, w \in D_1$.

The main result of the paper is the following product property.

Theorem. *Let $D_1 \subset \mathbb{C}^n$ and $D_2 \subset \mathbb{C}^m$ be domains. Then*

$$g_{D_1 \times D_2}((z_1, w_1), (z_2, w_2)) = \max\{g_{D_1}(z_1, z_2), g_{D_2}(w_1, w_2)\}, \\ (z_1, w_1), (z_2, w_2) \in D_1 \times D_2.$$

Remark. The product property for $D_1 \times D_2$ for the pluricomplex Green function in the case when D_1 or D_2 is pseudoconvex was proved in [Jar-Pff2]. Note that in [Jar-Pff2] the authors used the description of the pluricomplex Green function given by M. Klimek.

Proof. The inequality “ \geq ” follows from the property (b). So, we have to prove “ \leq ”.

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Let $(a_1, b_1), (a_2, b_2) \in D_1 \times D_2$. If $a_1 = a_2$ or $b_1 = b_2$, then the required inequality follows from the property (b). So, we may assume that $a_1 \neq a_2$ and $b_1 \neq b_2$.

Suppose that $N \in (0, 1)$ is such that

$$(1) \quad \max\{g_{D_1}(a_1, a_2), g_{D_2}(b_1, b_2)\} < N.$$

It is sufficient to prove that

$$g_{D_1 \times D_2}((a_1, b_1), (a_2, b_2)) < N.$$

There are holomorphic mappings $\varphi_1 : \bar{E} \rightarrow D_1$ and $\varphi_2 : \bar{E} \rightarrow D_2$ such that $\varphi_1(0) = a_2, \varphi_2(0) = b_2$,

$$\prod_{\lambda \in \varphi_1^{-1}(a_1)} |\lambda|^{\text{ord}_\lambda(\varphi_1 - a_1)} < N \quad \text{and} \quad \prod_{\lambda \in \varphi_2^{-1}(b_1)} |\lambda|^{\text{ord}_\lambda(\varphi_2 - b_1)} < N.$$

Note that $\nu := \#(\varphi_1^{-1}(a_1) \cap E) < \infty$ and $\mu := \#(\varphi_2^{-1}(b_1) \cap E) < \infty$. We may assume that φ_1 and φ_2 are such that ν and μ are minimal.

Let $\varphi_1^{-1}(a_1) \cap E = \{\zeta_1, \dots, \zeta_\nu\}$ and $\varphi_2^{-1}(b_1) \cap E = \{\xi_1, \dots, \xi_\mu\}$, where each point counts with its multiplicity.¹ Since $\varphi_1(E) \Subset D_1$ and $\varphi_2(E) \Subset D_2$, we may assume that $|\zeta_1| < |\zeta_2| < \dots < |\zeta_\nu|$ and $|\xi_1| < |\xi_2| < \dots < |\xi_\mu|$, i.e. each point ζ_j and ξ_j is with multiplicity one.² Then

$$(2) \quad |\zeta_1 \dots \zeta_\nu| \geq N|\zeta_\nu|^\nu \quad \text{and} \quad |\xi_1 \dots \xi_\mu| \geq N|\xi_\mu|^\mu.$$

For, if $|\zeta_1 \dots \zeta_\nu| < N|\zeta_\nu|^\nu$, then we may consider the mapping $\varphi_1(\zeta_\nu \lambda)$, and it contradicts the minimality of ν .

If $|\zeta_1 \dots \zeta_\nu| < |\xi_1 \dots \xi_\mu|$, then we replace φ_1 with the mapping $\tilde{\varphi}_1(\lambda) = \varphi_1(t\lambda)$, where $t := \left(\frac{|\zeta_1 \dots \zeta_\nu|}{|\xi_1 \dots \xi_\mu|}\right)^{\frac{1}{\nu}}$. Then $|\frac{1}{t}\zeta_j| < 1, j = 1, \dots, \nu$ (use (2)), and

$$\left| \left(\frac{\zeta_1}{t}\right) \dots \left(\frac{\zeta_\nu}{t}\right) \right| = |\xi_1 \dots \xi_\mu|.$$

Hence, we may assume that

$$|\zeta_1 \dots \zeta_\nu| = |\xi_1 \dots \xi_\mu| = C < N.$$

Moreover, replacing $\varphi_1(\lambda)$ with $\varphi_1(e^{-i\theta_1}\lambda)$ and $\varphi_2(\lambda)$ with $\varphi_2(e^{-i\theta_2}\lambda)$, where θ_1, θ_2 are chosen such that $e^{i\theta_1}\zeta_1 \dots e^{i\theta_1}\zeta_\nu = C$ and $e^{i\theta_2}\xi_1 \dots e^{i\theta_2}\xi_\mu = C$, we may assume that $\zeta_1 \dots \zeta_\nu = \xi_1 \dots \xi_\mu = C$.

We consider Blaschke products

$$B_1(\lambda) := \prod_{j=1}^{\nu} \frac{\zeta_j - \lambda}{1 - \bar{\zeta}_j \lambda}$$

and

$$\tilde{B}_1(\lambda) = \frac{B_1(\lambda) - B_1(0)}{1 - \bar{B}_1(0)B_1(\lambda)} = e^{i\theta} \prod_{j=1}^{\nu} \frac{\lambda - w_j}{1 - \bar{w}_j \lambda}, \quad \lambda \in \bar{E}.$$

¹Note that mappings φ_1 and φ_2 are holomorphic in some neighborhood of \bar{E} , and the sets $\varphi_1^{-1}(a_1)$ and $\varphi_2^{-1}(b_1)$ may contain points outside of E .

²For example, it is enough to change very little the mappings φ_1 and φ_2 by the formula (3) given below.

We choose different w'_j , $1 \leq j \leq \nu$, as close to w_j as we want such that $0 \in \{w'_1, \dots, w'_\nu\}$. Define

$$G_1(\lambda) = e^{i\theta} \prod_{j=1}^{\nu} \frac{\lambda - w'_j}{1 - \bar{w}'_j \lambda}.$$

Note that $\tilde{B}_1^{-1}(-C) = \{\zeta_1, \dots, \zeta_\nu\}$. We can find w'_1, \dots, w'_ν such that $G_1^{-1}(-C)$ consists of ν different points ζ'_j , $1 \leq j \leq \nu$, as close to points ζ_j as we want. Let us replace the mapping φ_1 with the mapping

$$(3) \quad \tilde{\varphi}_1(\lambda) := (\varphi(\lambda) - a_1) \frac{\prod_{j=1}^{\nu} \zeta_j(\lambda - \zeta'_j)}{\prod_{j=1}^{\nu} \zeta'_j(\lambda - \zeta_j)} + a_1.$$

Clearly, when ζ'_j , $1 \leq j \leq \nu$, are sufficiently close to ζ_j , $\tilde{\varphi}_1$ maps E into D_1 (recall that φ_1 maps some neighborhood of \bar{E} into D_1 , hence $\varphi_1(E) \Subset D_1$), and $\tilde{\varphi}_1(0) = \varphi_1(0)$, $\tilde{\varphi}_1(\zeta'_j) = \varphi_1(\zeta_j)$.

Repeating this process for φ_2 , we may assume that for Blaschke products B_1 and B_2 derivatives are not equal to 0 either on preimages of C or at points ζ_j or ξ_j respectively.

Let A be the union of images of singular points under mappings B_1 and B_2 . Note that neither 0 nor C is in A . Let π be a holomorphic universal covering of $E \setminus A$ by E with $\pi(0) = C$. There are liftings ψ_1 and ψ_2 mapping E into E such that $\pi = B_1 \circ \psi_1 = B_2 \circ \psi_2$ and $\psi_1(0) = \psi_2(0) = 0$. If $\pi^{-1}(0) = \{\eta_1, \eta_2, \dots\}$, then mappings $\varphi_1 \circ \psi_1$ and $\varphi_2 \circ \psi_2$ map 0 into a_2 and b_2 , and all points η_j into a_1 and b_1 respectively.

Note that π has all radial limits either in ∂E or in A . Since A is finite, π is an inner function. By Theorem 2 of Ch. III in [Nos], every inner function which has no zero radial limits is a Blaschke product. Thus

$$\pi(\lambda) = e^{i\alpha} \prod_{j=1}^{\infty} \frac{\bar{\eta}_j}{|\eta_j|} \frac{\eta_j - \lambda}{1 - \bar{\eta}_j \lambda}$$

and

$$|\prod_{j=1}^{\infty} \eta_j| = \pi(0) = C < N.$$

Since $(\varphi_1 \circ \psi_1, \varphi_2 \circ \psi_2)$ maps E into $D_1 \times D_2$,

$$g_{D_1 \times D_2}((a_1, b_1), (a_2, b_2)) \leq \prod_{j=1}^{\infty} |\eta_j| < N.$$

□

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