ON THE PRODUCT PROPERTY
OF THE PLURICOMPLEX GREEN FUNCTION

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Abstract. We prove that the pluricomplex Green function has the product property $g_{D_1 \times D_2} = \max\{g_{D_1}, g_{D_2}\}$ for any domains $D_1 \subset \mathbb{C}^n$ and $D_2 \subset \mathbb{C}^m$.

Let $E$ denote the unit disc in $\mathbb{C}$. For any domain $G \subset \mathbb{C}^n$ define

$$g_D(a, z) := \inf_{\varphi \in \mathcal{O}(E, D), \varphi(0) = z} \left\{ \prod_{\lambda \in \varphi^{-1}(a)} |\lambda|^{\text{ord}_\lambda(\varphi - a)} \right\}, \quad a, z \in D,$$

where $\mathcal{O}(E, D)$ denotes the set of all holomorphic mappings $E \to D$ and $\text{ord}_\lambda(\varphi - a)$ denotes multiplicity of $\varphi - a$ at $\lambda$.

The function $g_D$ is proposed by Poletsky (cf. [Pol]) and is called the pluricomplex Green function for $D$. We have that (see [Jar-Pfl1], Chapter IV)

$$(a) \quad g_D(a, z) = \inf_{\varphi \in \mathcal{O}(E, D), \varphi(0) = z} \left\{ \prod_{\lambda \in \varphi^{-1}(a)} |\lambda|^{\text{ord}_\lambda(\varphi - a)} \right\}, \quad a, z \in D.$$

Note that in the formula (a) we take only $\lambda \in \varphi^{-1}(a)$ such that $\lambda \in E$.

(b) For any domains $D_1$, $D_2$ and any holomorphic mapping $f : D_1 \to D_2$ we have the following contractible property: $g_{D_2}(f(z), f(w)) \leq g_{D_1}(z, w), \quad z, w \in D_1$.

The main result of the paper is the following product property.

Theorem. Let $D_1 \subset \mathbb{C}^n$ and $D_2 \subset \mathbb{C}^m$ be domains. Then

$$g_{D_1 \times D_2}((z_1, w_1), (z_2, w_2)) = \max\{g_{D_1}(z_1, z_2), g_{D_2}(w_1, w_2)\},$$

$$(z_1, w_1), (z_2, w_2) \in D_1 \times D_2.$$

Remark. The product property for $D_1 \times D_2$ for the pluricomplex Green function in the case when $D_1$ or $D_2$ is pseudoconvex was proved in [Jar-Pfl2]. Note that in [Jar-Pfl2] the authors used the description of the pluricomplex Green function given by M. Klimek.

Proof. The inequality “$\geq$” follows from the property (b). So, we have to prove “$\leq$”.

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Let \((a_1, b_1), (a_2, b_2) \in D_1 \times D_2\). If \(a_1 = a_2\) or \(b_1 = b_2\), then the required inequality follows from the property (b). So, we may assume that \(a_1 \neq a_2\) and \(b_1 \neq b_2\).

Suppose that \(N \in (0, 1)\) is such that

\[
\max\{g_{D_1}(a_1, a_2), g_{D_2}(b_1, b_2)\} < N.
\]

It is sufficient to prove that

\[
g_{D_1 \times D_2}\left((a_1, b_1), (a_2, b_2)\right) < N.
\]

There are holomorphic mappings \(\varphi_1 : \tilde{E} \to D_1\) and \(\varphi_2 : \tilde{E} \to D_2\) such that \(\varphi_1(0) = a_2, \varphi_2(0) = b_2\),

\[
\prod_{\lambda \in \varphi_1^{-1}(a_1)} |\lambda|^{\ord_{\lambda}(\varphi_1 - a_1)} < N \quad \text{and} \quad \prod_{\lambda \in \varphi_2^{-1}(b_1)} |\lambda|^{\ord_{\lambda}(\varphi_2 - b_1)} < N.
\]

Note that \(\nu := \#(\varphi_1^{-1}(a_1) \cap E) < \infty\) and \(\mu := \#(\varphi_2^{-1}(b_1) \cap E) < \infty\). We may assume that \(\varphi_1\) and \(\varphi_2\) are such that \(\nu\) and \(\mu\) are minimal.

Let \(\varphi_1^{-1}(a_1) \cap E = \{\zeta_1, \ldots, \zeta_\nu\}\) and \(\varphi_2^{-1}(b_1) \cap E = \{\xi_1, \ldots, \xi_\mu\}\), where each point counts with its multiplicity.\(^1\) Since \(\varphi_1(E) \subset D_1\) and \(\varphi_2(E) \subset D_2\), we may assume that \(|\zeta_1| < |\zeta_2| < \cdots < |\zeta_{\nu}|\) and \(|\xi_1| < |\xi_2| < \cdots < |\xi_{\mu}|\), i.e. each point \(\zeta_j\) and \(\xi_j\) is with multiplicity one.\(^2\) Then

\[
|\zeta_1 \cdots \zeta_{\nu}| \geq N|\zeta_{\nu}|^\nu \quad \text{and} \quad |\xi_1 \cdots \xi_{\mu}| \geq N|\xi_{\mu}|^\mu.
\]

For, if \(|\zeta_1 \cdots \zeta_{\nu}| < N|\zeta_{\nu}|^\nu\), then we may consider the mapping \(\varphi_1(\zeta, \lambda)\), and it contradicts the minimality of \(\nu\).

If \(|\zeta_1 \cdots \zeta_{\nu}| < |\xi_1 \cdots \xi_{\mu}|\), then we replace \(\varphi_1\) with the mapping \(\tilde{\varphi}_1(\lambda) = \varphi_1(t\lambda)\), where \(t := \left(\frac{|\zeta_1 \cdots \zeta_{\nu}|}{|\xi_1 \cdots \xi_{\mu}|}\right)^{\frac{1}{\nu}}\). Then \(t\zeta_j < 1, j = 1, \ldots, \nu\) (use (2)), and

\[
\left|\left(\frac{\zeta_1}{t}\right) \cdots \left(\frac{\zeta_{\nu}}{t}\right)\right| = |\zeta_1 \cdots \zeta_{\nu}|.
\]

Hence, we may assume that

\[
|\zeta_1 \cdots \zeta_{\nu}| = |\zeta_1 \cdots \zeta_{\nu}| = C < N.
\]

Moreover, replacing \(\varphi_1(\lambda)\) with \(e^{-i\theta_1}\lambda\) and \(\varphi_2(\lambda)\) with \(e^{-i\theta_2}\lambda\), where \(\theta_1, \theta_2\) are chosen such that \(e^{i\theta_1}\zeta_1 \cdots e^{i\theta_1}\zeta_{\nu} = C\) and \(e^{i\theta_2}\xi_1 \cdots e^{i\theta_2}\xi_{\mu} = C\), we may assume that 

\[
\zeta_1 \cdots \zeta_{\nu} = \xi_1 \cdots \xi_{\mu} = C.
\]

We consider Blaschke products

\[
B_1(\lambda) := \prod_{j=1}^{\nu} \frac{\zeta_j - \lambda}{1 - \zeta_j \lambda}
\]

and

\[
\tilde{B}_1(\lambda) = \frac{B_1(\lambda) - B_1(0)}{1 - B_1(0)B_1(\lambda)} = e^{i\theta} \prod_{j=1}^{\nu} \frac{\lambda - w_j}{1 - w_j \lambda}, \quad \lambda \in \tilde{E}.
\]

\(^1\)Note that mappings \(\varphi_1\) and \(\varphi_2\) are holomorphic in some neighborhood of \(\overline{E}\), and the sets \(\varphi_1^{-1}(a_1)\) and \(\varphi_2^{-1}(b_1)\) may contain points outside of \(E\).

\(^2\)For example, it is enough to change very little the mappings \(\varphi_1\) and \(\varphi_2\) by the formula (3) given below.
We choose different \( w'_j, 1 \leq j \leq \nu \), as close to \( w_j \) as we want such that \( 0 \in \{w'_1, \ldots, w'_\nu\} \). Define

\[
G_1(\lambda) = e^{i \theta} \prod_{j=1}^{\nu} \frac{\lambda - w'_j}{1 - w'_j \lambda}.
\]

Note that \( B_1^{-1}(-C) = \{\zeta_1, \ldots, \zeta_\nu\} \). We can find \( w'_1, \ldots, w'_\nu \) such that \( G_1^{-1}(-C) \) consists of \( \nu \) different points \( \zeta'_j, 1 \leq j \leq \nu \), as close to points \( \zeta_j \) as we want. Let us replace the mapping \( \varphi_1 \) with the mapping

\[
\tilde{\varphi}_1(\lambda) := (\varphi(\lambda) - a_1) \prod_{j=1}^{\nu} \frac{\zeta_j(\lambda - \zeta'_j)}{\zeta'_j(\lambda - \zeta_j)} + a_1.
\]

Clearly, when \( \zeta'_j, 1 \leq j \leq \nu \), are sufficiently close to \( \zeta_j \), \( \tilde{\varphi}_1 \) maps \( E \) into \( D_1 \) (recall that \( \varphi_1 \) maps some neighborhood of \( E \) into \( D_1 \), hence \( \varphi_1(E) \subseteq D_1 \), and \( \tilde{\varphi}_1(0) = \varphi_1(0), \tilde{\varphi}_1(\zeta'_j) = \varphi_1(\zeta_j) \).

Repeating this process for \( \varphi_2 \), we may assume that for Blaschke products \( B_1 \) and \( B_2 \) derivatives are not equal to 0 either on preimages of \( C \) or at points \( \zeta_j \) or \( \xi_j \) respectively.

Let \( A \) be the union of images of singular points under mappings \( B_1 \) and \( B_2 \). Note that neither 0 nor \( C \) is in \( A \). Let \( \pi \) be a holomorphic universal covering of \( E \setminus A \) by \( E \) with \( \pi(0) = C \). There are liftings \( \psi_1 \) and \( \psi_2 \) mapping \( E \) into \( E \) such that \( \pi = B_1 \circ \psi_1 = B_2 \circ \psi_2 \) and \( \psi_1(0) = \psi_2(0) = 0 \). If \( \pi^{-1}(0) = \{\eta_1, \eta_2, \ldots\} \), then mappings \( \varphi_1 \circ \psi_1 \) and \( \varphi_2 \circ \psi_2 \) map 0 into \( a_2 \) and \( b_2 \), and all points \( \eta_j \) into \( a_1 \) and \( b_1 \) respectively.

Note that \( \pi \) has all radial limits either in \( \partial E \) or in \( A \). Since \( A \) is finite, \( \pi \) is an inner function. By Theorem 2 of Ch. III in [Nos], every inner function which has no zero radial limits is a Blaschke product. Thus

\[
\pi(\lambda) = e^{i \alpha} \prod_{j=1}^{\infty} \frac{\bar{\eta}_j - \lambda}{|\eta_j| - \bar{\eta}_j \lambda}
\]

and

\[
\prod_{j=1}^{\infty} |\eta_j| = \pi(0) = C < N.
\]

Since \( (\varphi_1 \circ \psi_1, \varphi_2 \circ \psi_2) \) maps \( E \) into \( D_1 \times D_2 \),

\[
g_{D_1 \times D_2}((a_1, b_1), (a_2, b_2)) \leq \prod_{j=1}^{\infty} |\eta_j| < N.
\]

\[\square\]

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