

## A DIMENSION RESULT ARISING FROM THE $L^q$ -SPECTRUM OF A MEASURE

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ABSTRACT. We give a rigorous proof of the following heuristic result: Let  $\mu$  be a Borel probability measure and let  $\tau(q)$  be the  $L^q$ -spectrum of  $\mu$ . If  $\tau(q)$  is differentiable at  $q = 1$ , then the Hausdorff dimension and the entropy dimension of  $\mu$  equal  $\tau'(1)$ . Our result improves significantly some recent results of a similar nature; it is also of particular interest for computing the Hausdorff and entropy dimensions of the class of self-similar measures defined by maps which do not satisfy the open set condition.

### 1. INTRODUCTION

Let  $\mu$  be a Borel probability measure on  $\mathbb{R}^d$  with bounded support and let  $\text{supp}(\mu)$  denote the support of  $\mu$ . For a finite Borel partition  $\mathcal{P}$  of  $\text{supp}(\mu)$ , we let  $|\mathcal{P}|$  be the maximum of the diameters of elements of  $\mathcal{P}$ . Define

$$h(\mu, \mathcal{P}) = - \sum_{A \in \mathcal{P}} \mu(A) \ln \mu(A).$$

For  $\delta > 0$ , let

$$h(\mu, \delta) = \inf \{ h(\mu, \mathcal{P}) : \mathcal{P} \text{ is a finite Borel partition of } \text{supp}(\mu), |\mathcal{P}| \leq \delta \}.$$

The *entropy dimension* (or *Rényi dimension* [Re]) of  $\mu$  is defined as

$$\dim_e(\mu) = \lim_{\delta \rightarrow 0^+} \frac{h(\mu, \delta)}{-\ln \delta}.$$

Also, we let  $\dim_H(E)$  denote the Hausdorff dimension of a set  $E$  and define the *Hausdorff dimension* of  $\mu$  as

$$\dim_H(\mu) = \inf \{ \dim_H(E) : \mu(\mathbb{R}^d \setminus E) = 0 \}.$$

Young [Y] proved that if

$$(1.1) \quad \lim_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha \quad \text{for } \mu \text{ a.e. } x \in \text{supp}(\mu),$$

then

$$(1.2) \quad \dim_H(\mu) = \dim_e(\mu) = \alpha.$$

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An important sufficient condition for (1.1) to hold is when  $\mu$  is a self-similar measure defined by

$$\mu = \sum_{i=1}^m p_i \mu \circ S_i^{-1},$$

where  $\{S_i\}_{i=1}^m$  is a family of contractive similitudes satisfying the *open set condition* ([Hut], [F]), and the  $p_i$ 's are the probability weights satisfying  $p_i > 0$  and  $\sum_{i=1}^m p_i = 1$ . In this case (1.1) holds for

$$(1.3) \quad \alpha = \sum_{i=1}^m p_i \ln p_i / \sum_{i=1}^m p_i \ln \rho_i,$$

where  $\rho_i$  is the contraction ratio of  $S_i$ . If we let

$$G = \left\{ x \in \text{supp}(\mu) : \lim_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha \right\},$$

then  $\dim_H(G) = \alpha$  also. This theorem was proved by Geronimo and Hardin [GH] for  $\{S_i\}_{i=1}^m$  satisfying the *strong open set condition* (and also implicitly by Cawley and Mauldin [CM]). It was also proved by Strichartz [S] by using the law of iterated algorithm for  $\{S_i\}_{i=1}^m$  satisfying the open set condition.

Another sufficient condition to obtain (1.1) comes from the  $L^q$ -spectrum. For  $\delta > 0$  and  $q \in \mathbb{R}$ , the  $L^q$ -(*moment*) *spectrum* of  $\mu$  is defined as

$$(1.4) \quad \tau(q) = \underline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \sup \sum_i \mu(B_\delta(x_i))^q}{\ln \delta},$$

where  $\{B_\delta(x_i)\}_i$  is a family of disjoint closed  $\delta$ -balls with center  $x_i \in \text{supp}(\mu)$  and the supremum is taken over all such families. The function  $\tau(q)$  is an important function in multifractal theory; under suitable conditions, its Legendre transform equals the *dimension spectrum* of the measure  $\mu$  ([H], [F]). Moreover, it is suggested in the physics literature that  $\tau'(1)$  is equal to the entropy dimension of the measure ([HP], [H], [F]). Falconer [F] gives a heuristic argument for such equality. The purpose of this note is to give a rigorous proof of such a folklore theorem. Specifically, we prove

**Theorem 1.1.** *Let  $\mu$  be a Borel probability measure on  $\mathbb{R}^d$  with bounded support. Then*

(a) *for  $\mu$  a.e.  $x \in \text{supp}(\mu)$ , we have*

$$\tau'_+(1) \leq \underline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \overline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \tau'_-(1).$$

(b) *If  $\tau(q)$  is differentiable at  $q = 1$ , then*

$$\lim_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \tau'(1) \quad \text{for } \mu \text{ a.e. } x \in \text{supp}(\mu).$$

Consequently,  $\mu$  is concentrated on  $G = \left\{ x \in \text{supp}(\mu) : \lim_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \tau'(1) \right\}$ , and

$$\dim_H(G) = \dim_H(\mu) = \dim_e(\mu) = \tau'(1).$$

We will prove Theorem 1.1 in Section 3. The main idea is to show that the set of points  $x \in \text{supp}(\mu)$  such that

$$\liminf_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \tau'_+(1) \quad \text{or} \quad \tau'_-(1) < \limsup_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta}$$

has  $\mu$  measure zero. The proof of this relies on estimations of some counting functions (Lemma 2.2) together with a standard covering lemma. For the special case of self-similar measures defined by contractive similitudes satisfying the open set condition,  $\tau(q)$  is given by

$$\sum_{i=1}^m p_i^q \rho_i^{-\tau(q)} = 1.$$

Moreover,  $\tau(q)$  is differentiable and  $\tau'(1) = \alpha$ , where  $\alpha$  is given by (1.3) (see [CM]). Such results have also been proved for some extensions of the self-similar measures [AP], [R] (with the open set condition), and for equilibrium measures of Hölder continuous conformal expanding maps [PW]. The equality of  $\dim_H(\mu)$  and  $\tau'(1)$ , under the assumption that  $\tau(q)$  is differentiable at  $q = 1$ , was recently studied by Fan for a certain class of infinite product measures [Fa]. An additional example is the infinitely convolved Bernoulli measure associated with the golden ratio. This is a good illustration and the main motivation for our result because the open set condition fails. This will be discussed in Section 4.

## 2. PRELIMINARIES

Let  $\tau : \mathbb{R} \rightarrow [-\infty, \infty)$  be a concave function. We define the *effective domain* of  $\tau$  as

$$\text{Dom } \tau = \{x : -\infty < \tau(x) < \infty\}.$$

The *concave conjugate* (or the *Legendre transform*) of  $\tau$  is the function  $\tau^* : \mathbb{R} \rightarrow [-\infty, \infty)$  defined by

$$\tau^*(\alpha) = \inf\{\alpha x - \tau(x) : x \in \mathbb{R}\}.$$

For  $x \in \text{Dom } \tau$ , we let  $\partial\tau(x) \subseteq \mathbb{R}$  be the *subdifferential* of  $\tau$  at  $x$ , i.e.,

$$\partial\tau(x) = \{\alpha : \tau(y) \leq \tau(x) + \alpha(y - x) \quad \text{for all } y \in \mathbb{R}\}.$$

Then  $\tau^*(\alpha) + \tau(x) = \alpha x$  for  $\alpha \in \partial\tau(x)$  [Ro]. If  $\tau(x)$  is differentiable at  $x$ , then  $\partial\tau(x)$  is the singleton  $\tau'(x)$ . Otherwise,  $\partial\tau(x)$  is a closed interval. We will denote the special subdifferentials  $\partial\tau(0)$  and  $\partial\tau(1)$  respectively by  $[\alpha_0^-, \alpha_0^+]$  and  $[\alpha_1^-, \alpha_1^+]$ .

It is known (e.g. [LN1, Proposition 2.3]) that  $\text{Dom } \tau^*$  is an interval and  $(\text{Dom } \tau^*)^o = (\alpha_{\min}, \alpha_{\max})$ , where

$$\begin{aligned} \alpha_{\min} &:= \inf\{\alpha : \alpha \in \partial\tau(x), x \in \text{Dom } \tau\}, \\ \alpha_{\max} &:= \sup\{\alpha : \alpha \in \partial\tau(x), x \in \text{Dom } \tau\}. \end{aligned}$$

For the rest of this note, we assume that  $\tau(q)$  is the  $L^q$ -spectrum of a Borel probability measure  $\mu$  defined by (1.4). It is known that  $\tau(q)$  is increasing, concave and  $\tau(1) = 0$  (see Figure 1). Moreover, it is proved in [LN1] that

$$(2.1) \quad \alpha_{\min} = \liminf_{\delta \rightarrow 0^+} \frac{\ln(\sup_x \mu(B_\delta(x)))}{\ln \delta} \quad \text{and} \quad \alpha_{\max} = \limsup_{\delta \rightarrow 0^+} \frac{\ln(\inf_x \mu(B_\delta(x)))}{\ln \delta},$$

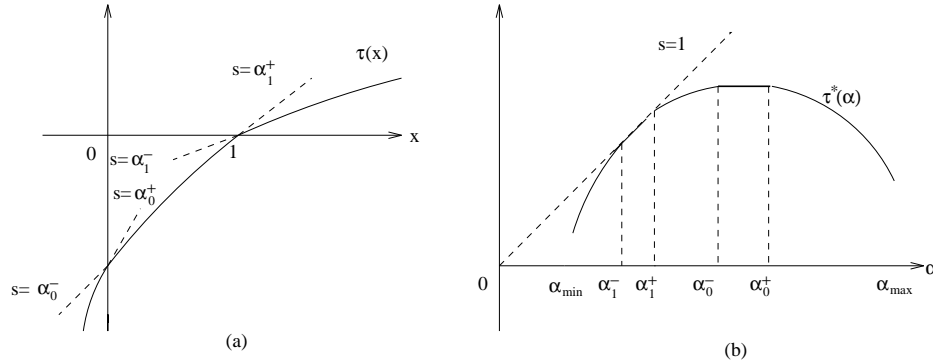


FIGURE 1. A concave function  $\tau$  and its concave conjugate  $\tau^*$  ( $s$  means slope)

where the supremum and infimum are taken over all  $x \in \text{supp}(\mu)$ . Define

$$\tau^*(\alpha_{\min}) := \lim_{q \rightarrow \infty} \tau^*(\alpha),$$

where  $\alpha \in \partial\tau(q)$ . The following proposition will be used in the proof of Lemma 3.1.

**Proposition 2.1.** *Assume that  $\alpha_{\min} < \alpha_1^-$ . Then  $\alpha_{\min} > \tau^*(\alpha_{\min})$ .*

*Proof.* Let  $\alpha_{\min} < \tilde{\alpha} < \alpha_1^-$  and  $q \in \partial\tau^*(\tilde{\alpha})$  (i.e.,  $\tilde{\alpha} \in \partial\tau(q)$ ). Consider the line with slope  $\tilde{\alpha}$  passing through the point  $(q, \tau(q))$ . This line intersects the vertical line  $q = 1$  at  $(1, \tau(q) - (q - 1)\tilde{\alpha})$ . By using the identity  $\tau(q) + \tau^*(\tilde{\alpha}) = q\tilde{\alpha}$  together with the facts that  $\tau$  is concave with  $\tau(1) = 0$  and  $\tilde{\alpha} < \alpha_1^-$ , we have

$$\tilde{\alpha} - \tau^*(\tilde{\alpha}) = \tau(q) - (q - 1)\tilde{\alpha} > 0.$$

The same argument shows that  $\alpha - \tau^*(\alpha)$  is an increasing function of  $q$  and hence

$$\alpha - \tau^*(\alpha) \geq \tilde{\alpha} - \tau^*(\tilde{\alpha}) \quad \text{for all } \alpha \leq \tilde{\alpha}.$$

The result follows by letting  $q \rightarrow \infty$ . □

Let  $\mathcal{B}_\delta$  denote a disjoint family of closed balls of radii  $\delta$  centered at points in  $\text{supp}(\mu)$ . For  $\alpha \in (\text{Dom } \tau^*)^\circ$ , we define the counting functions

$$N_\delta(\alpha) = \sup_{\mathcal{B}_\delta} \#\{B : B \in \mathcal{B}_\delta, \mu(B) \geq \delta^\alpha\},$$

$$\tilde{N}_\delta(\alpha) = \sup_{\mathcal{B}_\delta} \#\{B : B \in \mathcal{B}_\delta, \mu(B) < \delta^\alpha\}.$$

The following lemma is proved in [LN1, Lemma 4.2].

**Lemma 2.2.** *Let  $\alpha_{\min} < \alpha < \alpha_0^+$ ,  $q \in \partial\tau^*(\alpha)$  and  $\xi > 0$ . Then for any  $\epsilon > 0$ , there exists  $\delta_\epsilon > 0$  such that for all  $0 < \delta < \delta_\epsilon$ ,*

$$N_\delta(\alpha \pm \epsilon) \leq \delta^{-\tau^*(\alpha) - (\xi \pm q)\epsilon}.$$

For  $\alpha_0^+ \leq \alpha < \alpha_{\max}$ , the above holds with  $\tilde{N}_\delta$  replacing  $N_\delta$ .

Lemma 2.2 and the counting functions play a key role in the proof of the main theorem.

3. PROOF OF THE MAIN THEOREM

We need two lemmas.

**Lemma 3.1.** *Let  $\mu$  be a Borel probability measure on  $\mathbb{R}^d$  with bounded support. Then*

$$\mu\left\{x \in \text{supp}(\mu) : \alpha_{\min} \leq \liminf_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_1^-\right\} = 0.$$

*Proof. Part 1.* We claim that  $\mu\left\{x \in \text{supp}(\mu) : \alpha_{\min} < \liminf_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_1^-\right\} = 0$ . Let  $\alpha_{\min} < \alpha < \alpha_1^-$  and  $q \in \partial\tau^*(\alpha)$ . Then  $q > 1$ . Since  $\tau$  is increasing, concave,  $\alpha < \alpha_1^-$ , and since  $\tau(1) = 0$ , we have  $(\tau(q) - \tau(1))/(q - 1) \geq \tau'_-(q) \geq \alpha > 0$ . We choose  $\epsilon > 0$  small enough so that

$$(3.1) \quad \sigma := (\tau(q) - (q - 1)\alpha)/2 \leq \tau(q) - (q - 1)\alpha - (2 + q)\epsilon.$$

(This implies that  $\alpha + \epsilon < \alpha_1^-$ .) Define

$$L_\epsilon(\alpha) = \left\{x \in \text{supp}(\mu) : \alpha - \frac{\epsilon}{3} \leq \liminf_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \alpha + \frac{\epsilon}{3}\right\}.$$

We will show that  $\mu(L_\epsilon(\alpha)) = 0$ . Putting  $\xi = 1$  in Lemma 2.2, then there exists  $\delta_\epsilon > 0$  such that for all  $0 < \delta \leq \delta_\epsilon$ ,

$$(3.2) \quad N_\delta(\alpha + \epsilon) \leq \delta^{-\tau^*(\alpha) - (1+q)\epsilon}.$$

Fix  $m \in \mathbb{N}$  satisfying

$$(3.3) \quad 2^{-m} < \delta_\epsilon \quad \text{and} \quad m \geq 3\alpha/\epsilon + 2.$$

For each  $x \in L_\epsilon(\alpha)$ , we let  $n_x$  be the smallest integer satisfying the following conditions:

- (i)  $n_x \geq m$ ;
- (ii)  $\mu(B_\delta(x)) < \delta^{\alpha - \epsilon}$  for all  $0 < \delta \leq 2^{-(n_x - 2)}$ ;
- (iii) there exists  $\delta_x > 0$  such that

$$2^{-(n_x + 1)} < \delta_x \leq 2^{-n_x} \quad \text{and} \quad \mu(B_{\delta_x}(x)) > \delta_x^{\alpha + 2\epsilon/3}.$$

Note that  $n_x$  is uniquely determined by  $x$ . Partition  $L_\epsilon(\alpha)$  into a countable disjoint union of subsets  $L_\epsilon^n(\alpha)$  where  $L_\epsilon^n(\alpha) = \{x \in L_\epsilon(\alpha) : n_x = n\}$ . Then

$$(3.4) \quad L_\epsilon(\alpha) = \bigcup_{n=m}^\infty L_\epsilon^n(\alpha).$$

Clearly for each  $n \geq m$ ,

$$L_\epsilon^n(\alpha) \subseteq \bigcup_{x \in L_\epsilon^n(\alpha)} B_{2^{-n}}(x).$$

By a standard covering lemma (see [F, Lemma 4.8]), there exists a finite sequence  $\{x_i\}_{i=1}^\ell$  in  $L_\epsilon^n(\alpha)$  such that  $\{B_{2^{-n}}(x_i)\}_{i=1}^\ell$  is a disjoint family and

$$(3.5) \quad L_\epsilon^n(\alpha) \subseteq \bigcup_{i=1}^\ell B_{2^{-(n-2)}}(x_i).$$

For  $1 \leq i \leq \ell$ , condition (iii) and (3.3) imply that

$$\mu(B_{2^{-n}}(x_i)) > 2^{-(n+1)(\alpha + 2\epsilon/3)} \geq 2^{-n(\alpha + \epsilon)}.$$

Hence by (3.2),

$$(3.6) \quad \ell \leq 2^{-n(-\tau^*(\alpha)-(1+q)\epsilon)}.$$

Combining condition (ii), (3.5), (3.6) and (3.1), we have

$$\begin{aligned} \mu(L_\epsilon^n(\alpha)) &\leq \sum_{i=1}^{\ell} \mu(B_{2^{-(n-2)}}(x_i)) \leq 2^{-(n-2)(\alpha-\epsilon)} \cdot 2^{-n(-\tau^*(\alpha)-(1+q)\epsilon)} \\ &\leq C \cdot 2^{-n(\tau(q)-(q-1)\alpha-(2+q)\epsilon)} \leq C \cdot 2^{-n\sigma}. \end{aligned}$$

( $C$  is a constant independent of  $n$ .) Using this and (3.4), we have

$$\mu(L_\epsilon(\alpha)) \leq \sum_{n=m}^{\infty} \mu(L_\epsilon^n(\alpha)) \leq C \sum_{n=m}^{\infty} 2^{-n\sigma} = C \frac{2^{-\sigma m}}{1-2^{-\sigma}}.$$

Letting  $m \rightarrow \infty$ , we get  $\mu(L_\epsilon(\alpha)) = 0$ . The claim follows easily by taking a countable cover for  $(\alpha_{\min}, \alpha_1^-)$  by sets of the form  $L_\epsilon(\alpha)$ .

*Part 2.* We will show that if  $\alpha_{\min} < \alpha_1^-$ , then

$$\mu\left\{x \in \text{supp}(\mu) : \lim_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha_{\min}\right\} = 0.$$

By Proposition 2.1, we may choose  $\epsilon > 0$  sufficiently small and  $\alpha \in (\text{Dom } \tau^*)^\circ$  sufficiently close to  $\alpha_{\min}$  such that

$$0 < \sigma := (\alpha_{\min} - \tau^*(\alpha))/2 \leq \alpha_{\min} - \tau^*(\alpha) - (2+q)\epsilon,$$

where  $q \in \partial\tau^*(\alpha)$ . By Lemma 2.2, there exists  $\delta_\epsilon > 0$  such that for all  $0 < \delta \leq \delta_\epsilon$ ,

$$N_\delta(\alpha + \epsilon) \leq \delta^{-\tau^*(\alpha)-(1+q)\epsilon}.$$

Now choose  $m$  and  $n_x$  as in the proof of Part 1 but replace conditions (ii) and (iii) respectively by

- (ii)'  $\mu(B_\delta(x)) < \delta^{\alpha_{\min}-\epsilon}$  for all  $0 < \delta \leq 2^{-(n_x-2)}$ ;
- (iii)' there exists  $\delta_x > 0$  such that

$$2^{-(n_x+1)} < \delta_x \leq 2^{-n_x} \quad \text{and} \quad \mu(B_{\delta_x}(x)) > \delta_x^{\alpha_{\min}+\epsilon/2}.$$

The same proof yields the result and the lemma follows by combining the above two parts. □

**Lemma 3.2.** *Under the same hypotheses of Lemma 3.1, then*

$$\mu\left\{x \in \text{supp}(\mu) : \alpha_1^+ < \overline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta}\right\} = 0.$$

*Proof.* Again we divide the proof into two parts.

*Part 1.*  $\mu\left\{x \in \text{supp}(\mu) : \alpha_1^+ < \overline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_0^+\right\} = 0$ . Let  $\alpha_1^+ < \alpha < \alpha_0^+$  and  $q \in \partial\tau^*(\alpha)$ . The condition  $\tau(q) - (q-1)\alpha > 0$  still holds by the assumption  $\alpha > \alpha_1^+$  and by the fact that  $\tau$  is increasing and concave. Instead of  $L_\epsilon(\alpha)$ , we define

$$U_\epsilon(\alpha) = \left\{x \in \text{supp}(\mu) : \alpha - \frac{\epsilon}{3} \leq \overline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \alpha + \frac{\epsilon}{3}\right\}.$$

Let  $\delta_\epsilon > 0$  and  $m \in \mathbb{N}$  be as in the proof of Lemma 3.1. For each  $x \in U_\epsilon(\alpha)$ , we let  $n_x$  be chosen as in Lemma 3.1 but replace conditions (ii) and (iii) by (ii)' and (iii)' respectively as follows:

- (ii)' For all  $0 < \delta \leq 2^{-(n_x-1)}$ ,  $\mu(B_\delta(x)) \geq \delta^{\alpha+\epsilon}$ ;
- (iii)' there exists  $\delta_x > 0$  such that

$$2^{-n_x} < \delta_x \leq 2^{-(n_x-1)} \quad \text{and} \quad \mu(B_{\delta_x}(x)) \leq \delta_x^{\alpha-\epsilon}.$$

Then apply the same technique.

*Part 2.* We need to show that if  $\alpha > \alpha_1^+$  and  $\alpha_0^+ \leq \alpha \leq \alpha_{\max}$ , then

$$\mu\left\{x \in \text{supp}(\mu) : \overline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} > \alpha\right\} = 0.$$

Choose  $\epsilon > 0$  as in the proof of Part 1 of Lemma 3.1 and define

$$U'_\epsilon(\alpha) = \left\{x \in \text{supp}(\mu) : \alpha - \frac{\epsilon}{3} \leq \overline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta}\right\}.$$

Using Lemma 2.2, we can replace inequality (3.2) by  $\tilde{N}_\epsilon(\alpha - \epsilon) \leq \delta^{-\tau^*(\alpha) - (1-q)\epsilon}$ . A similar argument yields  $\mu(U'_\epsilon(\alpha)) = 0$  and the result follows.  $\square$

We now proof the main theorem by combining Lemmas 3.1 and 3.2.

*Proof of Theorem 1.1.* (a) It follows easily from (2.1) that for each  $x \in \text{supp}(\mu)$ ,

$$\alpha_{\min} \leq \underline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \overline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \alpha_{\max}.$$

Consequently, Lemma 3.1 implies that

$$\mu\left\{x \in \text{supp}(\mu) : \underline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_1^-\right\} = 0.$$

By Lemma 3.2,

$$\mu\left\{x \in \text{supp}(\mu) : \overline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} > \alpha_1^+\right\} = 0.$$

Part (a) now follows.

(b) The assumption that  $\tau(q)$  is differentiable at  $q = 1$  implies that  $\partial\tau(1)$  is a singleton, i.e.,  $\alpha_1^- = \alpha_1^+ = \tau'(1)$ . Part (a) now implies that for  $\mu$  a.e.  $x \in \text{supp}(\mu)$ ,

$$\underline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \tau'(1).$$

The result follows from Theorem 4.4 in [Y].  $\square$

#### 4. INFINITE BERNOULLI CONVOLUTIONS

Let  $0 < \rho < 1$ ,  $S_1(x) = \rho x$ ,  $S_2(x) = \rho x + (1 - \rho)$ , and let  $\mu_\rho$  be the self-similar measure defined by  $S_1, S_2$ , i.e.,

$$\mu_\rho = \frac{1}{2}\mu_\rho \circ S_1^{-1} + \frac{1}{2}\mu_\rho \circ S_2^{-1}.$$

$\mu_\rho$  is known as an *infinitely convolved Bernoulli measure* (ICBM) because it can be identified with the distribution of the random variable  $(1 - \rho) \sum_{n=0}^\infty \rho^n \epsilon_n$  where  $\{\epsilon_n\}$  are i.i.d. random variables each taking values 0 or 1 with probability 1/2. Such measures have been studied extensively since the 30's. For  $1/2 < \rho < 1$ ,  $\{S_1, S_2\}$  does not satisfy the open set condition and hence the dimension result stated in (1.2) (with  $\alpha = \tau'(1)$ ) does not cover such measures. An important result of Erdős says that if  $\rho^{-1}$  is a P.V. number, then  $\mu$  is singular [E]. (Recall that an algebraic integer  $\beta > 1$  is a *P.V. number* if all of its conjugates have moduli strictly less than 1.)

We will consider the special P.V. number  $\rho_o^{-1} = (\sqrt{5} + 1)/2$  (the golden ratio), which is so far the best understood case. The Hausdorff and entropy dimensions of this particular measure have been studied by a number of authors ([AY], [AZ], [LP], [La]). It is known that these two dimensions are equal and it is conjectured that they are equal to 0.99571312... [AZ]. In [LN2], a closed formula which defines the corresponding  $\tau(q)$  for all  $q > 0$  is derived. Moreover, it is proved that  $\tau(q)$  is differentiable on  $(0, \infty)$  and

$$(4.1) \quad \tau'(1) = \frac{1}{9 \ln \rho_o} \sum_{k=0}^{\infty} \sum_{|J|=k} c_J \ln c_J,$$

where

$$c_J = \frac{1}{8 \cdot 4^k} [1, 1] P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

and  $P_J = P_{j_1} \cdots P_{j_k}$ , with  $j_i = 0$  or  $1$ . Theorem 1.1 implies that  $\tau'(1)$  is equal to the Hausdorff and entropy dimensions of the measure. Numerical calculations using (4.1) suggest that  $\tau'(1) \approx 0.9957$ , agreeing with the result obtained in [AY], [AZ] and [La]. It is an open question how to obtain the  $L^q$ -spectrum  $\tau(q)$  for other P.V. numbers.

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