

INTEGRATION OF THE INTERTWINING OPERATOR  
FOR  $h$ -HARMONIC POLYNOMIALS  
ASSOCIATED TO REFLECTION GROUPS

YUAN XU

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let  $V$  be the intertwining operator with respect to the reflection invariant measure  $h_\alpha^2 d\omega$  on the unit sphere  $S^{d-1}$  in Dunkl's theory on spherical  $h$ -harmonics associated with reflection groups. Although a closed form of  $V$  is unknown in general, we prove that

$$\int_{S^{d-1}} V f(\mathbf{y}) h_\alpha^2(\mathbf{y}) d\omega = A_\alpha \int_{B^d} f(\mathbf{x}) (1 - |\mathbf{x}|^2)^{|\alpha|_1 - 1} d\mathbf{x},$$

where  $B^d$  is the unit ball of  $\mathbb{R}^d$  and  $A_\alpha$  is a constant. The result is used to show that the expansion of a continuous function as Fourier series in  $h$ -harmonics with respect to  $h_\alpha^2 d\omega$  is uniformly Cesàro  $(C, \delta)$  summable on the sphere if  $\delta > |\alpha|_1 + (d - 2)/2$ , provided that the intertwining operator is positive.

1. INTRODUCTION AND PRELIMINARIES

The main result of this paper is a formula for the integration of the intertwining operator in Dunkl's theory on  $h$ -harmonics associated with finite reflection groups. We start with background on Dunkl's theory (cf. [2]-[5]).

For  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$  we let  $\langle \mathbf{x}, \mathbf{y} \rangle$  denote the usual inner product of  $\mathbb{R}^d$  and  $|\mathbf{x}| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$  the Euclidean norm. Let  $S^{d-1} = \{\mathbf{x} : |\mathbf{x}| = 1\}$  be the unit sphere in  $\mathbb{R}^d$ . For a nonzero vector  $\mathbf{v} \in \mathbb{R}^d$  define the reflection  $\sigma_{\mathbf{v}}$  by

$$\mathbf{x}\sigma_{\mathbf{v}} := \mathbf{x} - 2(\langle \mathbf{x}, \mathbf{v} \rangle / |\mathbf{v}|^2) \mathbf{v}, \quad \mathbf{x} \in \mathbb{R}^d.$$

Suppose that  $G$  is a finite reflection group on  $\mathbb{R}^d$  with the set  $\{\mathbf{v}_i : i = 1, 2, \dots, m\}$  of positive roots; assume that  $|\mathbf{v}_i| = |\mathbf{v}_j|$  whenever  $\sigma_i$  is conjugate to  $\sigma_j$  in  $G$ , where we write  $\sigma_i = \sigma_{\mathbf{v}_i}$ ,  $1 \leq i \leq m$ . Then  $G$  is a subgroup of the orthogonal group generated by the reflections  $\{\sigma_i : 1 \leq i \leq m\}$ .

Let  $d\omega$  be the normalized surface measure on  $S^{d-1}$ . We consider weight functions of the form  $h_\alpha^2 d\omega$  on  $S^{d-1}$ , where

$$(1.1) \quad h_\alpha(\mathbf{x}) := \prod_{i=1}^m |\langle \mathbf{x}, \mathbf{v}_i \rangle|^{\alpha_i}, \quad \alpha_i \geq 0,$$

---

Received by the editors May 7, 1996.

1991 *Mathematics Subject Classification*. Primary 33C50, 33C45, 42C10.

*Key words and phrases*. Orthogonal polynomials in several variables, sphere,  $h$ -harmonics, reflection groups, intertwining operator.

Supported by the National Science Foundation under Grant DMS-9500532.

with  $\alpha_i = \alpha_j$  whenever  $\sigma_i$  is conjugate to  $\sigma_j$  in  $G$ . Then  $h_\alpha$  is a positively homogeneous  $G$ -invariant function of degree  $|\alpha|_1 = \alpha_1 + \dots + \alpha_m$ .

The  $h$ -harmonics are orthogonal polynomials on  $S^{d-1}$  with respect to  $h_\alpha^2 d\omega$ . The key ingredient of the theory is a family of commuting first-order differential-difference operators,  $\mathcal{D}_i$  (Dunkl's operators), defined by

$$(1.2) \quad \mathcal{D}_i f(\mathbf{x}) := \partial_i + \sum_{j=1}^m \alpha_j \frac{f(\mathbf{x}) - f(\mathbf{x}\sigma_j)}{\langle \mathbf{x}, \mathbf{v}_j \rangle} \langle \mathbf{v}_j, \mathbf{e}_i \rangle, \quad 1 \leq i \leq d,$$

where  $\partial_i$  is the ordinary partial derivative with respect to  $x_i$  and  $\mathbf{e}_1, \dots, \mathbf{e}_d$  are the standard unit vectors of  $\mathbb{R}^d$ . The  $h$ -Laplacian, which plays the role similar to that of the ordinary Laplacian, is defined by

$$(1.3) \quad \Delta_h = \mathcal{D}_1^2 + \dots + \mathcal{D}_d^2.$$

We keep the notation  $\Delta$  for the ordinary Laplacian. Let  $\mathcal{P}_n^d$  denote the space of homogeneous polynomials of degree  $n$  in  $\mathbf{x} = (x_1, \dots, x_d)$ . Then  $\mathcal{D}_i \mathcal{P}_n^d \subset \mathcal{P}_{n-1}^d$ ,  $\Delta_h \mathcal{P}_n^d \subset \mathcal{P}_{n-2}^d$ ; moreover, if  $P \in \mathcal{P}_n$ , then

$$\int_{S^{d-1}} PQ h_\alpha^2 d\omega = 0, \quad \forall Q \in \bigcup_{k=0}^{n-1} \mathcal{P}_k^d,$$

if and only if  $\Delta_h P = 0$ . The space  $\mathcal{H}_n^h = \mathcal{H}_n^{h,d} := \mathcal{P}_n^d \cap \ker \Delta_h$  is called the space of  $h$ -harmonic polynomials of degree  $n$ . We denote by  $\mathcal{H}_n$  the space of ordinary harmonic polynomials, which corresponds to  $h_\alpha = 1$ . The dimension of  $\mathcal{H}_n^h$  is the same as that of  $\mathcal{H}_n$ , which we denote by  $N(n, d)$ ; thus,

$$N(n, d) = \dim \mathcal{H}_n^h = \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d = \binom{n+d-1}{n} - \binom{n+d-3}{n-2}.$$

The intertwining operator  $V$  is a linear operator uniquely defined by ([4]):

$$V\mathcal{P}_n \subset \mathcal{P}_n, \quad V1 = 1, \quad \mathcal{D}_i V = V\partial_i, \quad 1 \leq i \leq d.$$

Note that  $V\mathcal{H}_n \subset \mathcal{H}_n^h$ ; the intertwining operator allows the transfer of certain results about  $\mathcal{H}_n$  to  $\mathcal{H}_n^h$ . In particular, the Poisson kernel for  $h$ -harmonics in the unit ball is given by [4, Theorem 4.2, p. 1225]

$$(1.4) \quad P^h(\mathbf{x}, \mathbf{y}) = V\left((1 - |\mathbf{x}|^2)(1 - 2\langle \mathbf{x}, \cdot \rangle + |\mathbf{x}|^2)^{-|\alpha|_1 - \frac{d}{2}}\right)(\mathbf{y}), \quad |\mathbf{x}| < |\mathbf{y}| = 1.$$

A closed form of  $V$  is known only for  $h_\alpha(\mathbf{x}) = |x_1|^{\alpha_1} \dots |x_d|^{\alpha_d}$ , associated with the group  $\mathbb{Z}^2 \times \dots \times \mathbb{Z}^2$  ([13], the case  $d = 1$  appeared early in [4]), and for  $h_\alpha(\mathbf{x}) = |(x_1 - x_2)(x_2 - x_3)(x_1 - x_3)|^\alpha$ , associated with the symmetric group  $S_3$  ([5]). We state the first case as

**Example 1.1.** For  $h_\alpha(\mathbf{x}) = |x_1|^{\alpha_1} \dots |x_d|^{\alpha_d}$ ,

$$(1.5) \quad Vf(\mathbf{x}) = \int_{[-1,1]^d} f(x_1 t_1, \dots, x_d t_d) \prod_{i=1}^d (1 + t_i) \prod_{i=1}^d c_{\alpha_i} (1 - t_i^2)^{\alpha_i - 1} dt,$$

where the constant  $c_\lambda$  is defined by

$$(1.6) \quad c_\lambda^{-1} = \int_{-1}^1 (1 - t^2)^{\lambda - 1} dt = \frac{\pi^{\frac{1}{2}} \Gamma(\lambda)}{\Gamma(\lambda + \frac{1}{2})}.$$

The closed form of  $V$  in the case of  $S_3$  in [5] is rather complicated. The problem of finding a closed form of  $V$ , however, is very important but also very difficult. The main result in the present paper is to show that integration of  $Vf$  with respect to  $h_\alpha^2 d\omega$  on  $S^{d-1}$  has a simple closed form. We state and prove the main result in the following section. In Section 3, we use the result to study the summability of Fourier series in terms of  $h$ -harmonics, where we shall prove that the expansion of a continuous function as Fourier series in  $h$ -harmonics is uniformly  $(C, \delta)$  summable on  $S^{d-1}$  if  $\delta > |\alpha|_1 + (d - 2)/2$ , provided  $V$  is a positive operator.

2. INTEGRATION OF THE INTERTWINING OPERATOR

Let  $B^d$  be the unit ball in  $\mathbb{R}^d$ . On  $B^d$  we define the weight function

$$W_\gamma(\mathbf{x}) = c_{\gamma,d}(1 - |\mathbf{x}|^2)^{\gamma-1}, \quad c_{\gamma,d}^{-1} = \int_{B^d} (1 - |\mathbf{x}|^2)^{\gamma-1} d\mathbf{x} = \frac{\pi^{d/2}\Gamma(\gamma)}{\Gamma(\gamma + d/2)},$$

where  $c_{\gamma,d}$  is the normalization constant so that the integral of  $W_\gamma$  over  $B^d$  takes the value 1. We denote by  $H_\alpha$  the normalization constant defined by  $H_\alpha^{-1} = \int_{S^{d-1}} h_\alpha^2 d\omega$ . There is a closed formula for  $H_\alpha$  due to the work of Macdonald, Heckman, Opdam, and others (cf. [7], [8], [9]). We shall not write it down since it is not used in this paper.

**Theorem 2.1.** *Let  $h_\alpha$  be defined as in (1.1). Let  $V$  be the intertwining operator. Then*

$$(2.1) \quad \int_{S^{d-1}} Vf(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega = A_\alpha \int_{B^d} f(x)(1 - |\mathbf{x}|^2)^{\alpha-1} d\mathbf{x},$$

for  $f \in \Pi^d$ , where  $A_\alpha = H_\alpha^{-1}c_{\gamma,d}$ .

Before we turn to the proof of the theorem, some remarks are in order. First, the condition  $f \in \Pi^d$  is not necessary. The formula may hold for any function as long as the integrals on both sides are finite. In fact, the first indication that such a formula holds true lies in the fact that the integral of  $P^h$  as given in (1.4) is equal to 1. Secondly, we emphasize that the real strength of the theorem lies in the fact that the formula of  $Vf$  for a general reflection group is unknown. Nevertheless, let us point out that in the case of Example 1.1, the equation (2.1) takes the form

$$(2.2) \quad \int_{S^{d-1}} \int_{[-1,1]^d} f(x_1t_1, \dots, x_d t_d) \prod_{i=1}^d (1 + t_i) \prod_{i=1}^d c_{\alpha_i} (1 - t_i^2)^{\alpha_i-1} dt \prod_{i=1}^d |x_i|^{\alpha_i} d\omega \\ = A_\alpha \int_{B^d} f(x)(1 - |\mathbf{x}|^2)^{|\alpha|-1} d\mathbf{x}.$$

This identity can be proved by changing variables and using integration by parts, but such a proof is not a trivial matter.

The following special case of the theorem is of interest in itself.

**Corollary 2.2.** *Let  $g : \mathbb{R} \mapsto \mathbb{R}$  be a polynomial. Then*

$$(2.3) \quad \int_{S^{d-1}} Vg(\langle \mathbf{x}, \cdot \rangle)(\mathbf{y})h_\alpha^2(\mathbf{y})d\omega = B_\alpha \int_{-1}^1 g(t|\mathbf{x}|)(1 - t^2)^{|\alpha|_1 + \frac{d-3}{2}} dt,$$

where  $B_\alpha = H_\alpha^{-1}c_{|\alpha|_1+(d-1)/2}$ .

The corollary shows, in particular, that the integration of  $Vg(\langle \mathbf{x}, \cdot \rangle)$  against  $h_\alpha^2$  yields a radial function. For the Lebesgue measure, this is evident since  $V = id$  and the integral in the left side of (2.3) with  $h_\alpha = 1$  is clearly invariant under orthogonal transformations. For  $h_\alpha \neq 1$  this is far from being obvious since  $h_\alpha$  is invariant under a subgroup of the group of orthogonal transformations. Even in the case of  $h_\alpha$  in Example 1.1 this is not obvious in view of (2.2).

In order to prove the theorem we explore the orthogonal polynomials with respect to the weight function  $W_\gamma$  on  $B^d$ . The following formula is very useful: If  $g$  is a continuous function on  $B^d$ , then

$$(2.4) \quad \int_{B^d} g(\mathbf{x})(1 - |\mathbf{x}|^2)^{\gamma-1} d\mathbf{x} = \int_0^1 \left( \int_{S^{d-1}} g(r\mathbf{x}') d\omega \right) r^{d-1} (1 - r^2)^{\gamma-1} dr,$$

under the standard change of variables  $\mathbf{x} = r\mathbf{x}'$ ,  $\mathbf{x}' \in S^{d-1}$ . An orthogonal basis of  $L^2(B^d, W_\gamma d\mathbf{x})$  can be given in terms of Jacobi polynomials and the ordinary spherical harmonics. We use the standard notation  $P_n^{(\alpha, \beta)}$  for the Jacobi polynomials orthogonal with respect to  $(1 - x)^\alpha (1 + x)^\beta$  on  $[-1, 1]$  (cf. [10, Chapter IV]). Let  $\{S_{n,i}, 1 \leq i \leq N(n, d)\}$  be an orthonormal basis for  $\mathcal{H}_n$ . Let

$$\mathbb{P}_n = \{b_{k,n} P_k^{(\gamma-1, n-2k+\frac{d-2}{2})} (2|\mathbf{x}|^2 - 1) S_{n-2k,i}(\mathbf{x}), \quad 1 \leq i \leq N(n - 2k, d), 0 \leq 2k \leq n\},$$

where  $b_{k,n}$  are the normalization constants so that the square of each element has integral equal to 1 with respect to  $W_\gamma d\mathbf{x}$  over  $B^d$ . Using formula (2.4) and the fact that  $S_m$  is homogeneous of degree  $m$ , it follows from the orthogonality of the Jacobi polynomials and the orthogonality of the (ordinary) spherical harmonics that  $\mathbb{P}_n$  is orthogonal to  $\mathbb{P}_m$  for  $n \neq m$  and the elements in  $\mathbb{P}_n$  are mutually orthogonal to each other. Since the polynomials in  $\mathbb{P}_n$  are of degree exactly  $n$  and there are exactly

$$\sum_{0 \leq 2k \leq n} \dim \mathcal{H}_{n-2k,d} = \sum_{0 \leq 2k \leq n} (\dim \mathcal{P}_{n-2k} - \dim \mathcal{P}_{n-2k-2}) = \dim \mathcal{P}_n^d$$

many polynomials in  $\mathbb{P}_n$ , it follows that  $L^2(B^d, W_\gamma d\mathbf{x}) = \bigoplus_{n=0}^\infty \text{span } \mathbb{P}_n$ . Therefore, the elements of  $\mathbb{P}_n$  form an orthonormal basis of  $L^2(W_\gamma d\mathbf{x}, B^d)$ . We choose a total order for the elements of  $\mathbb{P}_n$  and treat it as a polynomial vector. Such a vector notation has been used to study orthogonal polynomials in several variables quite generally (cf. [11]). In [12], a different orthonormal basis of  $\mathbb{P}_n$  with respect to  $W_\gamma d\mathbf{x}$ , given in terms of Gegenbauer polynomials, has been used to study the summability of Fourier series in terms of  $\mathbb{P}_n$ .

For  $f \in L^2(B^d, W_\gamma d\mathbf{x})$ , we can write its Fourier expansion with respect to  $\{\mathbb{P}_n\}_{n=0}^\infty$  as (see [11])

$$(2.5) \quad f(\mathbf{x}) = \sum_{n=0}^\infty \mathbf{a}_n^T(f) \mathbb{P}_n(\mathbf{x}), \quad \mathbf{a}_n(f) = \int_{B^d} f(\mathbf{y}) \mathbb{P}_n(\mathbf{y}) W_\gamma(\mathbf{y}) d\mathbf{y}.$$

We want to study the effect of  $V$  applied to  $P_n \in \mathbb{P}_n$ . Because of the formulae [10, p. 62, (4.21.2)] and [10, p. 59, (4.1.4)],

$$(2.6) \quad P_k^{(\alpha, \beta)}(2t^2 - 1) = \frac{(-1)^k}{k!} \sum_{j=0}^k \binom{k}{j} \frac{\Gamma(k + \alpha + \beta + j + 1)}{\Gamma(k + \alpha + \beta + 1)} \frac{\Gamma(k + \beta + 1)}{\Gamma(j + \beta + 1)} (-1)^j t^{2j},$$

we first consider the effect of  $V$  applied to  $|\mathbf{x}|^{2j}S_{n-2k,i}(\mathbf{x})$ . We have

**Lemma 2.3.** *Let  $S_m \in \mathcal{H}_m$ ,  $m \geq 0$ . Then*

$$(2.7) \quad \int_{S^{d-1}} V(|\cdot|^{2j}S_m)(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega = H_\alpha \frac{\Gamma(|\alpha|_1 + d/2)\Gamma(j + d/2)}{\Gamma(|\alpha|_1 + d/2 + j)\Gamma(d/2)}\delta_{m,0}.$$

*Proof.* Since  $|\mathbf{x}|^{2j}S_m$  is homogeneous of degree  $m + 2j$ , so is  $V(|\cdot|^{2j}S_m)$ . Using the canonical decomposition [2, Theorem 1.7, p. 37]

$$\mathcal{P}_n^d = \bigoplus_{j=0}^{[n/2]} |\mathbf{x}|^{2j}\mathcal{H}_{n-2j}^h$$

and the harmonic projection operator  $\text{proj}_{\mathcal{H}_k^h} : \mathcal{P}_k \mapsto \mathcal{H}_k^h$  [2, Theorem 1.11, p. 38], we have

$$\begin{aligned} &V(|\cdot|^{2j}S_m) \\ &= \sum_{i=0}^{[m/2]+j} |\mathbf{x}|^{2i} \frac{1}{4^i i!} \frac{\Gamma(|\alpha|_1 + m + 2j - i + d/2)}{\Gamma(|\alpha|_1 + m + 2j - 2i + d/2)} \text{proj}_{\mathcal{H}_{m+2j-2i}^h}(\Delta_h^i V(|\cdot|^{2j}S_m)). \end{aligned}$$

Since  $\mathcal{H}_k^h \perp 1$ ,  $k > 0$ , with respect to  $h_\alpha^2 d\omega$ , it follows that the integral of the function  $\text{proj}_{\mathcal{H}_{m+2j-2i}^h} \Delta_h^i V(|\cdot|^{2j}S_m)$  is zero unless  $m + 2j - 2i = 0$ . In case  $2i = m + 2j$ , we have

$$\begin{aligned} &\int_{S^{d-1}} V(|\cdot|^{2j}S_m)(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega \\ &= \frac{1}{4^{j+m/2}(j+m/2)!} \frac{\Gamma(|\alpha|_1 + j + (d+m)/2)}{\Gamma(|\alpha|_1 + d/2)} \int_{S^{d-1}} \Delta_h^{j+m/2} V(|\cdot|^{2j}S_m)h_\alpha^2(\mathbf{x})d\omega, \end{aligned}$$

since  $\text{proj}_{\mathcal{H}_0^h} = id$ . By the intertwining property of  $V$ , it follows that

$$\Delta_h^{j+m/2} V(|\cdot|^{2j}S_m) = V\Delta^{j+m/2}(|\cdot|^{2j}S_m).$$

Using the identity

$$\Delta(|\mathbf{x}|^{2j}g_m) = 4j(m+j-1+d/2)|\mathbf{x}|^{2j-2}g_m + |\mathbf{x}|^{2j}\Delta g_m,$$

for  $g_m \in \mathcal{P}_m$ ,  $m = 0, 1, 2, \dots$ , and the fact that  $\Delta S_m = 0$ , we see that

$$\Delta_h^{j+m/2}(|\cdot|^{2j}S_m)(\mathbf{x}) = 4^j j! \frac{\Gamma(j+d/2)}{\Gamma(d/2)} \Delta_h^{m/2} S_m(\mathbf{x}),$$

which is zero if  $m > 0$ . For  $m = 0$ , we use the fact that  $S_0(\mathbf{x}) = 1$  and put the constants together to finish the proof.  $\square$

**Lemma 2.4.** *For  $P_n \in \mathbb{P}_n$ ,*

$$(2.8) \quad \int_{S^{d-1}} V(P_n)(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega = H_\alpha^{-1}\delta_{0,n}.$$

*Proof.* By the definition of  $\mathbb{P}_n$ , we need to show that

$$\int_{S^{d-1}} V(P_k^{(|\alpha|_1-1, n-2k+(d-2)/2)}(2|\cdot|^2-1)S_{n-2k,i})(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega = H_\alpha^{-1}\delta_{0,n}$$

for  $0 \leq 2k \leq n$  and  $S_{n-2k,i} \in \mathcal{H}_{n-2k}$ . From Lemma 2.3 and (2.6) this follows readily if  $2k < n$ . For the remaining case  $2k = n$ , we use (2.6) and (2.7) to derive that

$$\begin{aligned} & \int_{S^{d-1}} V(P_k^{(|\alpha|_1-1, n-2k+\frac{d-2}{2})})(2|\cdot|^2 - 1)S_{n-2k,i}(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega \\ &= \frac{(-1)^k}{k!} \sum_{j=0}^k \binom{k}{j} \frac{\Gamma(k + |\alpha|_1 + j - 1 + d/2)\Gamma(k + d/2)}{\Gamma(k + |\alpha|_1 - 1 + d/2)\Gamma(j + d/2)} (-1)^j \int_{S^{d-1}} V(|\cdot|^{2j})h_\alpha^2 d\omega \\ &= H_\alpha^{-1} \frac{\Gamma(|\alpha|_1 + d/2)}{\Gamma(d/2)} \frac{(-1)^k}{k!} \sum_{j=0}^k \binom{k}{j} \frac{\Gamma(k + |\alpha|_1 + j - 1 + d/2)\Gamma(k + d/2)}{\Gamma(k + |\alpha|_1 - 1 + d/2)\Gamma(|\alpha|_1 + j + d/2)} (-1)^j \\ &= H_\alpha \frac{\Gamma(|\alpha|_1 + d/2)}{\Gamma(d/2)} \frac{\Gamma(k + d/2)}{\Gamma(k + |\alpha|_1 + d/2)} P_k^{(-1, |\alpha|_1-1+d/2)}(1), \end{aligned}$$

where we note that the definition of  $P_k^{(\alpha,\beta)}$  can be extended to  $\alpha = -1$  by using (2.6). From [10, (4.22.2), p. 64],

$$P_k^{(-1,\beta)}(x) = \frac{k + \beta}{k} \frac{x - 1}{2} P_{k-1}^{(1,\beta)}(x),$$

it follows that  $P_k^{(-1, |\alpha|_1-1+d/2)}(1) = 0$  for  $k > 0$  or  $n = 2k > 0$ . For  $n = 0$  we use the fact that  $P_0^{(\alpha,\beta)}(x) = 1$  and  $V1 = 1$ . □

*Proof of Theorem 2.1.* We expand  $f$  into the Fourier series with respect to the orthonormal basis  $\mathbb{P}_n$  as in (2.5). Since  $V$  is a linear operator, we have that

$$\int_{S^{d-1}} Vf(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega = \sum_{n=0}^\infty \mathbf{a}_n^T(f) \int_{S^{d-1}} V(\mathbb{P}_n)(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega.$$

Thus, Lemma 2.4 implies that

$$\int_{S^{d-1}} Vf(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega = H_\alpha^{-1} \mathbf{a}_0(f) = A_\alpha \int_{B^d} f(\mathbf{x})(1 - |\mathbf{x}|^2)^{|\alpha|_1-1} d\mathbf{x},$$

which is the desired result. □

*Proof of Corollary 2.2.* The following formula is known to hold for  $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ :

$$\int_{S^{d-1}} f(\langle \mathbf{x}, \mathbf{y} \rangle) d\omega(\mathbf{y}) = \omega_{d-2} \int_{-1}^1 f(s|\mathbf{x}|)(1 - s^2)^{\frac{d-3}{2}} ds, \quad \mathbf{x} \in \mathbb{R}^d.$$

Using this formula and Theorem 2.1 we have

$$\begin{aligned} I(\mathbf{x}) &:= \int_{S^{d-1}} Vg(\langle \mathbf{x}, \cdot \rangle)(\mathbf{y})h_\alpha^2(\mathbf{y})d\omega = A_\alpha \int_{B^d} g(\langle \mathbf{x}, \mathbf{y} \rangle)(1 - |\mathbf{y}|^2)^{|\alpha|_1-1} d\mathbf{y} \\ &= c \int_0^1 r^{d-1} \int_{S^{d-1}} g(r\langle \mathbf{x}, \mathbf{y}' \rangle) d\omega(\mathbf{y}') (1 - r^2)^{|\alpha|_1-1} dr \\ &= c \int_0^1 \int_{-1}^1 g(sr|\mathbf{x}|)(1 - s^2)^{\frac{d-3}{2}} ds (1 - r^2)^{|\alpha|_1-1} dr, \end{aligned}$$

where  $c = A_\alpha \omega_{d-2}$ . Changing the variable  $s \mapsto t/r$  in the last formula and interchanging the order of the integrations we have

$$\begin{aligned} I(\mathbf{x}) &= c \int_{-1}^1 g(t|\mathbf{x}|) \int_{|t|}^1 (r^2 - t^2)^{\frac{d-3}{2}} r(1 - r^2)^{|\alpha|_1 - 1} dr dt \\ &= c \frac{1}{2} \int_0^1 u^{\frac{d-3}{2}} (1 - u)^{|\alpha|_1 - 1} du \int_{-1}^1 g(t|\mathbf{x}|) (1 - t^2)^{|\alpha|_1 + \frac{d-3}{2}} dt, \end{aligned}$$

where the last step follows from another change of variable. This is the desired formula. Instead of keeping track of the constant we can determine it by setting  $g = 1$  and using the fact that  $V1 = 1$ .  $\square$

We end this section with the following remark: Formula (2.8) in Lemma 2.4 shows that  $V(\mathbb{P}_n)$ ,  $n > 0$ , is orthogonal to  $\mathcal{H}_0^h$  with respect to  $h_\alpha^2 d\omega$ . In fact, with little more effort, the same approach can be used to show that  $V(\mathbb{P}_n)$  is orthogonal to  $\mathcal{H}_k^h$  for  $k \neq n$ . This may suggest a possible way of studying the intertwining operator  $V$ . However, we do not know how to compute the integral of  $V(P)Q$  for  $P \in \mathbb{P}_n$  and  $Q \in \mathcal{H}_n^h$ .

3. SUMMABILITY OF ORTHOGONAL SERIES IN  $h$ -HARMONICS

As an application of Theorem 2.1, we consider the Cesàro summability of the Fourier expansion of a function with respect to  $h$ -harmonics on the sphere. Let  $\{S_{n,i}^h, 1 \leq i \leq N(n, d)\}$  be an orthonormal basis for  $\mathcal{H}_n^h$ . For  $f \in L^2(S^{d-1}, h_\alpha^2 d\omega)$ , its Fourier orthogonal expansion in terms of  $\{S_{n,i}^h\}$  is given by

$$f \sim \sum_{n=0}^\infty \sum_{i=1}^{N(n,d)} a_i^n(f) S_{n,i}^h, \quad a_i^n(f) = \int_{S^{d-1}} f S_{n,i}^h h_\alpha^2 d\omega.$$

We denote the  $n$ -th partial sum of this expansion by  $S_n(h_\alpha^2; f)$ ; thus,

$$S_n(h_\alpha^2; f, \mathbf{x}) = \sum_{k=0}^n \sum_{i=1}^{N(k,d)} a_i^k(f) S_{k,i}^h(\mathbf{x}).$$

Let us recall the definition of Cesàro summability. The sequence  $\{s_n\}$  is summable by Cesàro’s method of order  $\delta$ ,  $(C, \delta)$ , to  $s$  if

$$\frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^n \binom{n-k+\delta-1}{n-k} s_k$$

converges to  $s$  as  $n \rightarrow \infty$ . If, for each  $n \in \mathbb{N}_0$ ,  $s_n$  is the  $n$ -th partial sum of the series  $\sum_{k=0}^\infty c_k$ , the Cesàro means can be rewritten as

$$\frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^n \binom{n-k+\delta}{n-k} c_k.$$

For the basic properties of Cesàro summability see [14, Chapter III]. We are interested in the  $(C, \delta)$  means of  $S_n(h_\alpha^2, f)$ . Our main result in this section is the following.

**Theorem 3.1.** *Assume the intertwining operator  $V$  with respect to the weight function  $h_\alpha$  is positive. Let  $f$  be continuous on  $S^{d-1}$ . Then the expansion of  $f$  as Fourier series with respect to  $h_\alpha^2$  is uniformly  $(C, \delta)$  summable over  $S^{d-1}$  provided  $\delta > |\alpha|_1 + (d - 2)/2$ .*

We note that the intertwining operator is known to be positive in the case of Example 1.1 and for some parameters in the  $S_3$  case ([5]). Dunkl has conjectured that the intertwining operator is positive for all weight functions invariant under reflection groups. When  $\alpha = 0$  the theorem reduces to the classical case of Lebesgue measure.

The proof of this theorem depends on Theorem 2.1 and a compact formula of the reproducing kernel for  $\mathcal{H}_n^h$ . For each  $n \in \mathbb{N}_0$ , the reproducing kernel,  $P_n^h$ , for  $\mathcal{H}_n^h$  is defined by the property

$$H_\alpha \int_{S^{d-1}} Q(\mathbf{y}) P_n^h(\mathbf{x}, \mathbf{y}) h_\alpha^2 d\omega(\mathbf{y}) = Q(\mathbf{x}), \quad Q \in \mathcal{H}_n^h.$$

In terms of an orthonormal basis of  $\mathcal{H}_n^h$ , the function  $P_n^h$  is given by

$$P_n^h(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{N(n,d)} S_{n,i}^h(\mathbf{x}) S_{n,i}^h(\mathbf{y}).$$

Using the intertwining operator  $V$ , we can express  $P_n^h$  in a closed form in terms of the Gegenbauer polynomials  $C_n^{(\lambda)}$ .

**Theorem 3.2.** For  $n \in \mathbb{N}_0$ ,

(3.1)

$$P_n^h(\mathbf{x}, \mathbf{y}) = \frac{n + |\alpha|_1 + (d - 2)/2}{|\alpha|_1 + (d - 2)/2} [V C_n^{(|\alpha|_1 + (d-2)/2)}(\langle \cdot, \mathbf{y} \rangle)](\mathbf{x}), \quad |\mathbf{y}| \leq |\mathbf{x}| = 1.$$

*Proof.* Introducing the notation  $K_m(\mathbf{x}, \mathbf{y}) = [V(\langle \cdot, \mathbf{y} \rangle^m / m!)](\mathbf{x})$ , Dunkl [4, Theorem 4.1, p. 1224] has proved that  $P_n^h$  can be written as

$$P_n^h(\mathbf{x}, \mathbf{y}) = \sum_{j \leq n/2} \frac{(|\alpha|_1 + d/2)_n 2^{n-2j}}{(2 - n - |\alpha|_1 - d/2)_j j!} |\mathbf{x}|^{2j} |\mathbf{y}|^{2j} K_{n-2j}(\mathbf{x}, \mathbf{y})$$

for  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ . Since  $V$  is linear and  $\langle \cdot, \mathbf{y} \rangle^m$  is homogeneous of degree  $m$  in  $\mathbf{y}$ , for  $|\mathbf{x}| = 1$  we can write

$$P_n^h(\mathbf{x}, \mathbf{y}) = V \left\{ \sum_{j \leq n/2} \frac{(|\alpha|_1 + d/2)_n 2^{n-2j}}{(2 - n - |\alpha|_1 - d/2)_j j! (n - 2j)!} \langle \cdot, \mathbf{y}/|\mathbf{y}| \rangle^{n-2j} \right\}(\mathbf{x}) |\mathbf{y}|^n.$$

Setting  $t = \langle \cdot, \mathbf{y}/|\mathbf{y}| \rangle$  we now derive a closed formula for the expression inside the curly bracket, which we denote by  $J(t)$ . If  $n$  is even, say,  $n = 2m$ , then it follows from the definition of  $(a)_j = a(a + 1) \cdots (a + j - 1)$  that

$$(2 - 2m - |\alpha|_1 - d/2)_{m-k} = (-1)^{m-k} \frac{\Gamma(2m + |\alpha|_1 + d/2 - 1)}{\Gamma(m + k + |\alpha|_1 + d/2 - 1)},$$

from which and the formula

$$(2k)! = \Gamma(2k + 1) = 2^{2k} \Gamma(k + 1/2) \Gamma(k + 1) / \Gamma(1/2),$$

we conclude that

$$\begin{aligned} J(t) &= \sum_{k=0}^m \frac{(|\alpha|_1 + d/2)_{2m} 2^{2k}}{(2 - 2m - |\alpha|_1 - d/2)_{m-k} (m - k)! (2k)!} t^{2k} \\ &= \frac{\Gamma(1/2)(-1)^m}{m! \Gamma(m + 1/2)} \sum_{k=0}^m \binom{m}{k} \frac{\Gamma(m + k + |\alpha|_1 + d/2 - 1) \Gamma(m + 1/2)}{\Gamma(2m + |\alpha|_1 + d/2 - 1) \Gamma(k + 1/2)} (-1)^k t^{2k} \\ &= \frac{\Gamma(m + |\alpha|_1 + d/2 - 1) \Gamma(1/2)}{\Gamma(2m + |\alpha|_1 + d/2 - 1) \Gamma(m + 1/2)} P_m^{(|\alpha|_1 + \frac{d-3}{2}, -\frac{1}{2})}(2t^2 - 1) \end{aligned}$$

by formula (2.6). Upon using formula [10, (4.1.5), p. 59]

$$C_{2m}^{(|\alpha|_1)}(t) = \frac{\Gamma(|\alpha|_1 + m) \Gamma(1/2)}{\Gamma(|\alpha|_1) \Gamma(m + 1/2)} P_m^{(|\alpha|_1 - \frac{1}{2}, -\frac{1}{2})}(2t^2 - 1),$$

we obtain a compact formula for  $J(t)$  in terms of Gegenbauer polynomials for  $n = 2m$ , which leads to

$$P_n^h(\mathbf{x}, \mathbf{y}) = \frac{n + |\alpha|_1 + (d - 2)/2}{|\alpha|_1 + (d - 2)/2} [VC_n^{(|\alpha|_1 + (d-2)/2)}(\langle \cdot, \mathbf{y}/|\mathbf{y}| \rangle)](\mathbf{x}) |\mathbf{y}|^n.$$

Since each  $S_{n,i}$  is homogeneous of degree  $n$ ,  $P_n^h$  is homogeneous of degree  $n$  in  $\mathbf{y}$ . Hence, replacing  $\mathbf{y}/|\mathbf{y}|$  by  $\mathbf{y}$  in the last identity leads to the desired result for  $n = 2m$ . The case  $n = 2m - 1$  is proved similarly.  $\square$

We note that this formula can be viewed as the addition formula for the  $h$ -harmonics; indeed, it reduces to the addition formula for the ordinary spherical harmonics [6, Vol. II, p. 244, (2)]. The formula agrees with Dunkl’s formula for the Poisson kernel (1.4) because

$$\sum_{n=0}^{\infty} \frac{n + \lambda}{\lambda} C_n^{(\lambda)}(t) r^n = \frac{1 - r^2}{(1 - 2tr + r^2)^{\lambda+1}},$$

which follows easily from the expansion of the generating function of Gegenbauer polynomials ([10, (4.7.23), p.82]).

The proof of Theorem 3.1 depends also on the following fact. Let  $w_\lambda$  be the normalized weight function on  $[-1, 1]$ :

$$w_\lambda = c_{\lambda-1/2} (1 - x^2)^{\lambda-1/2}, \quad \int_{-1}^1 w_\lambda(t) dt = 1,$$

where  $c_{\lambda-1/2}$  is the constant defined in (1.6). A function  $f \in L^2(w_\lambda, [-1, 1])$  can be expanded into Gegenbauer series

$$f \sim \sum_{k=0}^{\infty} b_k(f) \tilde{C}_n^{(\lambda)}, \quad b_k(f) = \int_{-1}^1 \tilde{C}_n^{(\lambda)}(x) w_\lambda(x) dx,$$

where  $\tilde{C}_n^{(\lambda)}$  is the normalized Gegenbauer polynomial, which differs from  $C_n^{(\lambda)}$  by a normalization constant chosen so that  $\{\tilde{C}_n^{(\lambda)}\}$  forms an orthonormal basis with respect to  $w_\lambda$  (cf. [10, (4.7.15), p.81]). The  $n$ -th partial sum of this expansion can be written as

$$s_n(w_\lambda; f) = \int_{-1}^1 f(y) K_n(w_\lambda; x, y) w_\lambda(y) dy,$$

where the  $n$ -th reproducing kernel  $K_n(w_\lambda)$  is given by

$$K_n(w_\lambda; x, y) = \sum_{k=0}^n \tilde{C}_k^{(\lambda)}(x) \tilde{C}_k^{(\lambda)}(y) = \sum_{k=0}^n \frac{k + \lambda}{\lambda} C_k^{(\lambda)}(x) C_k^{(\lambda)}(y) / C_k^{(\lambda)}(1);$$

here in the second equation we have switched back to the usual Gegenbauer polynomials.

*Proof of Theorem 3.1.* From the definition of  $P_n^h$  it is easy to see that we can write the partial sum  $S_n(h_\alpha^2; f)$  as

$$S_n(h_\alpha^2; f, \mathbf{x}) = \int_{S^{d-1}} \sum_{k=0}^n P_k^h(\mathbf{x}, \cdot) f h_\alpha^2 d\omega = \int_{S^{d-1}} G_n(\mathbf{x}, \cdot) f h_\alpha^2 d\omega$$

where, according to Theorem 3.2 and using the notation of  $K_n(w_\lambda)$ , we have

$$(3.2) \quad G_n^h(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^n P_k^h(\mathbf{x}, \mathbf{y}) = V[K_n(w_{|\alpha|_1 + (d-2)/2}; 1, \langle \cdot, \mathbf{y} \rangle)](\mathbf{x}).$$

Let us denote by  $S_n^\delta(h_\alpha^2; f)$  the  $(C, \delta)$  means of  $S_n(h_\alpha^2; f)$ . Then we have

$$S_n^\delta(h_\alpha^2; f, \mathbf{x}) = \int_{S^{d-1}} G_n^\delta(\mathbf{x}, \cdot) f h_\alpha^2 d\omega,$$

$$G_n^\delta(\mathbf{x}, \mathbf{y}) = V[K_n^\delta(w_{|\alpha|_1 + (d-2)/2}; 1, \langle \cdot, \mathbf{x} \rangle)](\mathbf{y}),$$

where  $G_n^\delta$  denotes the  $(C, \delta)$  means of  $G_n$  and  $K_n^\delta(w_\lambda)$  the means of  $K_n(w_\lambda)$ . To prove the theorem it's enough to show that

$$F_n(\mathbf{x}) := \int_{S^{d-1}} |G_n^\delta(\mathbf{x}, \mathbf{y})| h_\alpha^2(\mathbf{y}) d\omega(\mathbf{y}) < \infty$$

uniformly for  $\mathbf{x} \in S^{d-1}$  if  $\delta > \gamma + (d-2)/2$ . If  $V$  is positive, we have

$$|V[K_n^\delta(w_{|\alpha|_1 + (d-2)/2}; 1, \langle \cdot, \mathbf{x} \rangle)](\mathbf{y})| \leq V[|K_n^\delta(w_{|\alpha|_1 + (d-2)/2}; 1, \langle \cdot, \mathbf{x} \rangle)](\mathbf{y}).$$

Therefore, we can apply Corollary 2.2 to conclude that

$$F_n(\mathbf{x}) \leq \int_{S^{d-1}} V[|K_n^\delta(w_{|\alpha|_1 + (d-2)/2}; 1, \langle \cdot, \mathbf{x} \rangle)](\mathbf{y}) h_\alpha^2(\mathbf{y}) d\omega$$

$$\leq B_\alpha \int_{-1}^1 |K_n^\delta(w_{|\alpha|_1 + (d-2)/2}; 1, t)| (1-t^2)^{|\alpha|_1 + \frac{d-3}{2}} dt.$$

The last integral is bounded if and only if  $\delta > |\alpha|_1 + (d-2)/2$ , since it is precisely the condition on the  $(C, \delta)$  summability of the Gegenbauer series with index  $|\alpha|_1 + (d-2)/2$  at the point  $x = 1$  and the conclusion follows from [10, Theorem 9.1.3, p. 246].  $\square$

We conclude the paper with the following remark: From Theorem 3.2 and an inequality of Kogbetliantz (see [1, p. 71]) for the Gegenbauer polynomials, it follows that the  $(C, 2|\alpha|_1 + d - 1)$  means of the reproducing kernels  $P_n^h$  are nonnegative over  $S^{d-1}$  provided that  $V$  is positive. In particular, the  $(C, 2|\alpha|_1 + d - 1)$  means of the Fourier series with respect to  $h_\alpha^2 d\omega$  on  $S^{d-1}$  define a positive linear operator, provided that  $V$  is positive. Other related results on the positivity of the  $(C, \delta)$  means can be derived from an inequality of Askey and Gasper (see [1, p. 74]).

## REFERENCES

1. R. Askey, *Orthogonal polynomials and special functions*, SIAM, Philadelphia, 1975. MR **58**:1288
2. C. Dunkl, *Reflection groups and orthogonal polynomials on the sphere*, Math. Z. **197** (1988), 33-60. MR **89b**:42016
3. C. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311** (1989), 167-183. MR **90k**:33027
4. C. Dunkl, *Integral kernels with reflection group invariance*, Can. J. Math. **43** (1991), 1213-1227. MR **93g**:33012
5. C. Dunkl, *Intertwining operators associated to the group  $S_3$* , Trans. Amer. Math. Soc. **347** (1995), 3347-3374. CMP 95:14
6. A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions*, vol. 2, McGraw-Hill, New York, 1953. MR **84h**:33001b
7. G. J. Heckman, *Hypergeometric and Spherical Functions*, Harmonic Analysis and Special Functions on Symmetric Spaces, Academic Press, San Diego, 1994.
8. I.G. Macdonald, *Some conjectures for root systems*, SIAM. J. Math. Anal. **13** (1982), 988-1007. MR **84h**:17006a
9. E.M. Opdam, *Some applications of hypergeometric shift operators*, Invent. Math. **98** (1989), 1-18. MR **91h**:33024
10. H. Szegő, *Orthogonal polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ. vol.23, Providence, RI, 1975.
11. Y. Xu, *On multivariate orthogonal polynomials*, SIAM J. Math. Anal. **24** (1993), 783-794. MR **94i**:42031
12. Y. Xu, *Summability of Fourier orthogonal series for Jacobi weight on a ball in  $\mathbb{R}^d$*  (to appear).
13. Y. Xu, *Orthogonal polynomials for a family of product weight functions on the spheres*, Canadian J. Math. **49** (1997), 175-192.
14. A. Zygmund, *Trigonometric Series*, 2th ed., Cambridge Univ. Press, Cambridge, 1968. MR **38**:4882

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON 97403-1222  
E-mail address: yuan@math.uoregon.edu