

**THE AR-PROPERTY FOR ROBERTS' EXAMPLE  
OF A COMPACT CONVEX SET WITH NO EXTREME POINTS  
PART 2: APPLICATION TO THE EXAMPLE**

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ABSTRACT. In this second part of our paper, we apply the result of Part 1 to show that the compact convex set with no extreme points, constructed by Roberts (1977), is an AR.

This is the second part of our paper. In this part, we prove that the compact convex set with no extreme points, constructed by Roberts [R1], is an AR.

All notation, used in this part, comes from Part 1.

5. ROBERTS' EXAMPLE

We are going to describe the linear metric space which contains a compact convex set with no extreme points, constructed by Roberts [R1].

Let  $\{d_n\}$  be a sequence of natural numbers. Put  $m(1) = 1$  and inductively define  $m(n+1) = d_n m(n)$ . Let  $\pi_1 = \{[0, 1)\}$ . Assume that  $\pi_n$  is a partition of  $[0, 1)$  into  $m(n)$  equal length intervals of the form  $[a, b)$ . For each  $S \in \pi_n$ , let  $\pi_{n+1}(S)$  denote the partition of  $S$  into  $d_n$  equal length subintervals. Then we define the partition  $\pi_{n+1} = \cup\{\pi_{n+1}(S) : S \in \pi_n\}$  of  $[0, 1)$  into  $m(n+1)$  equal length intervals. Consider the linear space consisting of all functions on  $[0, 1]$  which are finite linear combinations of characteristic functions of the form  $\chi_{[a,b)}$ . We denote

$$(30) \quad E_n = \text{span}\{\chi_S : S \in \pi_n\}; \quad E = \bigcup_{n=1}^{\infty} E_n$$

( $\chi_S$  denotes the characteristic function of  $S$ );

$$E_{n+1}(S) = \text{span}\{\chi_T : T \in \pi_{n+1}(S)\};$$

$$(31) \quad A_n = \{m(n)\chi_S : S \in \pi_n\} = \{a_i^n : i = 1, \dots, m(n)\} \text{ (compare with (10))},$$

where  $a_i^n = m(n)\chi_{S_i^n}$ ,  $i = 1, \dots, m(n)$ ;  $\pi_n = \{S_1^n, \dots, S_{m(n)}^n\}$ . Observe that

$$E_1 \subset E_2 \subset \dots \subset E, \text{ see (30).}$$

Note that paranorms constructed in [R1] are monotone, norm bounded and total, hence they are monotone  $F$ -norms. Then "paranorms" can be replaced by

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“monotone  $F$ -norms”. Observe that a paranorm is an  $F$ -norm if it is norm bounded and total. The following theorem was proved by Roberts [R1, §3].

**Theorem 4.** *For a suitable sequence of natural numbers  $\{d_n\}$ , there exist sequences of paranorms  $\{N_n\}$  and  $\{N_{n+1}(S) : S \in \pi_n\}$  on  $E_n$  and on  $E_{n+1}(S)$  respectively, such that the following conditions hold:*

- (i)  $N_1(x) = \int_0^1 |x(t)|dt$  for every  $x \in E_1$ ;
- (ii)  $N_{n+1}(x) = \inf \{N_n(y) + \sum \{N_{n+1}(S)(x(S)) : S \in \pi_n\}\}$  for every  $x \in E_{n+1}$ , where the infimum is taken over all the expressions of  $x$  of the form:

$$x = y + \sum \{x(S) : S \in \pi_n\}, \quad y \in E_n, \quad x(S) \in E_{n+1}(S);$$

- (iii)  $\{N_n(x)\}$  is a decreasing sequence for every  $x \in E$  and the formula

$$N(x) = \lim_{n \rightarrow \infty} N_n(x) \text{ for } x \in E$$

defines a monotone  $F$ -norm of  $E$ ;

- (iv) If  $x \in E_n$  and  $N_n(x) < 4$ , then  $N_m(x) = N_n(x)$  for every  $m > n$ ;
- (v) The sequence  $\{A_n\}$ , defined by (31), satisfies the conditions (4) and (5) of Theorem 1, Part 1, where  $A_{n+1}(a) = \{m(n+1)\chi_{S'} : S' \in \pi_{n+1}(S)\}$  if  $a = m(n)\chi_S \in A_n, S \in \pi_n$ .

Therefore  $C = \overline{\cup_{n=0}^{\infty} \hat{A}_n} \subset X$ , see (6), is a compact convex set with no extreme points, where  $X$  is the completion of  $(E, N)$ .

Let  $\|\cdot\|$  denote the  $F$ -norm on  $X$  induced by  $N$ . Our aim is to show that the compact convex set  $C$ , defined by Theorem 4, satisfies the conditions of Theorem 2, Part 1. We need the following fact:

**Lemma 6.** *Using the notation (31), for each  $i = 1, \dots, m(n)$ , if  $x \in \text{span } C_n(a_i^n) \cap E$ , see (8), then  $\text{supp } x \subset S_i^n$ .*

*Proof.* Let  $x \in \text{span } C_n(a_i^n) \cap E$ . Take  $q > n$  such that  $x \in E_q$ . Then we have

$$(32) \quad x = \sum_{j=1}^{m(q)} \alpha_j a_j^q, \text{ where } a_j^q = m(q)\chi_{S_j^q}, \quad j = 1, \dots, m(q).$$

Assume on the contrary that  $\alpha_j \neq 0$  for some  $S_j^q \subset [0, 1] \setminus S_i^n$ . We may assume that  $j = 1$ . Take a sequence  $\{x_k\} \subset \text{span } \cup_{j=1}^{\infty} \hat{A}_{n+j}(a_i^n) \cap E$  such that  $x_k \rightarrow x$ , see (8). Then, we have

$$(33) \quad \text{supp } x_k \subset S_i^n \text{ for every } k \in \mathbb{N}, \text{ see (30) and (31).}$$

For any function  $x$  on  $[0, 1]$  and  $A \subset [0, 1]$ , denote  $x|_A = \chi_A x$ . Then from (33) we get

$$x - x_k|_{[0, 1] \setminus S_i^n} = x|_{[0, 1] \setminus S_i^n} \in E_q \text{ for every } k \in \mathbb{N}.$$

Since  $\|x - x_k\| \rightarrow 0$ , by Theorem 4(ii)–(iv), there exists an expression

$$(34) \quad x - x_k = z_q(k) + \sum_{j=q}^{n(k)-1} \sum_{i=1}^{m(j)} y_i^j(k), \quad k \in \mathbb{N},$$

where  $n(k) > q$  is chosen so that  $x_k \in E_{n(k)}$ , and  $z_q(k) \in E_q$ ,  $y_i^j(k) \in E_{j+1}(S_i^j)$  for  $i = 1, \dots, m(j)$ ,  $j = q, \dots, n(k) - 1$ , such that

$$(35) \quad N_q(z_q(k)) + \sum_{j=q}^{n(k)-1} \sum_{i=1}^{m(j)} N_{j+1}(S_i^j)(y_i^j(k)) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Since  $z_q(k) \in E_q$  we have, see (30) and (31),

$$(36) \quad z_q(k) = \sum_{t=1}^{m(q)} y_t(k), \text{ where } y_t(k) = \lambda_t(k)m(q)\chi_{S_t^q}, \ t = 1, \dots, m(q).$$

From (35) we get  $N_q(z_q(k)) \rightarrow 0$  as  $k \rightarrow \infty$ . Since  $\{z_q(k)\}$  is a sequence in the finite dimensional space  $E_q$ , equipped with the Hamel base  $\{\chi_{S_t^q} : t = 1, \dots, m(q)\}$  and the  $F$ -norm  $N_q$ , it follows from (36) that

$$(37) \quad N_q(y_t(k)) \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ for every } t = 1, \dots, m(q).$$

Observe that for every  $j \in \{q, \dots, n(k) - 1\}$  and  $i \in \{1, \dots, m(j)\}$  we have  $S_i^j \subset S_t^q$  for some  $t \in \{1, \dots, m(q)\}$ . Denote

$$(38) \quad u_1(k) = \sum_{S_i^j \subset S_1^q} y_i^j(k).$$

Then we have

*Claim 6.*  $y_1(k) + u_1(k) = \alpha_1 m(q)\chi_{S_1^q} \in E_q$  for every  $k \in \mathbb{N}$ .

*Proof.* Recall that  $x|_A = \chi_A x$ . Since  $S_1^q \subset [0, 1] \setminus S_i^n$ , from (33) and (34) we get

$$(39) \quad x|_{S_1^q} = x - x_k|_{S_1^q} = z_q(k)|_{S_1^q} + \sum_{j=q}^{n(k)-1} \sum_{i=1}^{m(j)} y_i^j(k)|_{S_1^q}.$$

From (32) and (36) we have

$$x|_{S_1^q} = \alpha_1 m(q)\chi_{S_1^q} \text{ and } z_q(k)|_{S_1^q} = y_1(k).$$

Since  $y_i^j(k) \in E_{j+1}(S_i^j)$ , we get

$$\sum_{j=q}^{n(k)-1} \sum_{i=1}^{m(j)} y_i^j(k)|_{S_1^q} = \sum_{S_i^j \subset S_1^q} y_i^j(k) = u_1(k), \text{ see (38).}$$

Consequently, the claim follows from (39).

Since  $y_1(k) \in E_q$ , from Claim 6 we get  $u_1(k) \in E_q$  for every  $k \in \mathbb{N}$ . Therefore, from Theorem 4(ii)–(iv) and from (35) we get

$$\begin{aligned} N_q(u_1(k)) &\leq \sum_{S_i^j \subset S_1^q} N_{j+1}(S_i^j)(y_i^j(k)) \leq \sum_{t=1}^{m(q)} \sum_{S_i^j \subset S_t^q} N_{j+1}(S_i^j)(y_i^j(k)) \\ &= \sum_{j=q}^{n(k)-1} \sum_{i=1}^{m(j)} N_{j+1}(S_i^j)(y_i^j(k)) \rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

Hence, from Claim 6 and from (37) we obtain

$$N_q(\alpha_1 m(q)\chi_{S_1^q}) \leq N_q(y_1(k)) + N_q(u_1(k)) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Consequently,  $N_q(\alpha_1 m(q) \chi_{S_1^q}) = 0$  and therefore  $\alpha_1 = 0$ . So, we get a contradiction. The lemma is proved.

6. THE RESULT

Now we come to our result in this paper:

**Theorem 5.** *The compact convex set  $C$ , defined by Theorem 4, is an AR.*

*Proof.* We are going to verify the conditions of Theorem 2. First, observe that Condition (ii) follows from the construction. Let us check Condition (i): if  $x_i \in \text{span } C_n(a_i^n) \setminus \{\theta\}$ ,  $i = 1, \dots, m(n)$ , then the set  $\{x_1, \dots, x_{m(n)}\}$  is linearly independent in  $X$ .

Let  $x_i \in \text{span } C_n(a_i^n) \setminus \{\theta\}$  and  $\lambda_i \in \mathbb{R}$ ,  $i = 1, \dots, m(n)$ , such that

$$(40) \quad \lambda_1 x_1 + \dots + \lambda_{m(n)} x_{m(n)} = \theta \text{ where } \lambda_i \in \mathbb{R}, i = 1, \dots, m(n).$$

We shall prove that  $\lambda_i = 0$  for every  $i = 1, \dots, m(n)$ . Assume on the contrary that  $\lambda_i \neq 0$  for some  $i \in \{1, \dots, m(n)\}$ . We may assume that  $i = 1$ . Then we get

$$x_1 = \sum_{i=2}^{m(n)} \alpha_i x_i, \text{ where } \alpha_i = -\lambda_i / \lambda_1, i = 2, \dots, m(n).$$

Let  $0 < \varepsilon < 4$ . We take  $y_i \in \text{span } C_n(a_i^n) \cap E$  such that

$$(41) \quad \|y_i - x_i\| < (2m(n)\alpha)^{-1}\varepsilon < \varepsilon/2 \text{ for every } i = 1, \dots, m(n),$$

where  $\alpha = \max\{[\alpha_i] + 1 : i = 2, \dots, m(n)\}$ . (Recall that  $[\alpha]$  denotes the greatest integer which is smaller than  $\alpha$ .) From Lemma 6 we get

$$(42) \quad \text{supp } y_i \subset S_i^n \text{ for every } i = 1, \dots, m(n).$$

Take  $k \in \mathbb{N}$  such that  $y_i \in E_{n+k}$  for every  $i = 1, \dots, m(n)$ . Let  $z_i^1 = \alpha_i y_i$  for  $i = 2, \dots, m(n)$ . Then we can denote

$$z_1 = y_1 - \sum_{i=2}^{m(n)} \alpha_i y_i = y_1 - \sum_{i=2}^{m(n)} z_i^1 \in E_{n+k},$$

whence

$$(43) \quad y_1 = z_1 + \sum_{i=2}^{m(n)} z_i^1.$$

Note that

$$(44) \quad \text{supp } y_1 \subset S_1^n \text{ and } \text{supp } z_i^1 \subset S_i^n \text{ for } i = 2, \dots, m(n).$$

Then from (1) and (41) we have

$$\begin{aligned} \|z_1\| &= \left\| y_1 - \sum_{i=2}^{m(n)} \alpha_i y_i \right\| \leq \|y_1 - x_1\| + \left\| x_1 - \sum_{i=2}^{m(n)} \alpha_i y_i \right\| \\ &< 2^{-1}\varepsilon + \left\| \sum_{i=2}^{m(n)} \alpha_i (x_i - y_i) \right\| \leq 2^{-1}\varepsilon + \sum_{i=2}^{m(n)} \|\alpha_i (x_i - y_i)\| \\ &\leq 2^{-1}\varepsilon + \sum_{i=2}^{m(n)} \max\{\alpha_i + 1 : i = 2, \dots, m(n)\} \|x_i - y_i\| \\ &< 2^{-1}\varepsilon + m(n)\alpha(2m(n)\alpha)^{-1}\varepsilon = \varepsilon. \end{aligned}$$

Now, recall that  $y_1 \in E_{n+k}$ . Our strategy is to “cut off” the “top” of  $y_1$  step by step until we get an element in the space  $E_n$ .

**The first step.** In this step we cut a small element  $u_1$  off from  $y_1$  so that  $y_1 - u_1 \in E_{n+k-1}$ .

Since  $\|z_1\| < \varepsilon < 4$ , it follows from Theorem 4(iii) that there exists an  $r \in \mathbb{N}$  such that  $N_r(z_1) < 4$ . Replacing  $r$  or  $n + k$  by  $\max\{r, n + k\}$  if necessary, we may assume that  $r = n + k$ . Hence from Theorem 4(iv) we get  $N_{n+k}(z_1) = \|z_1\| < \varepsilon$ . Therefore by Theorem 4(ii) there exist

$$y_i^{n+k-1} \in E_{n+k}(S_i^{n+k-1}), \quad i = 1, \dots, m(n+k-1), \quad \text{and } z_2 \in E_{n+k-1} \text{ such that}$$

$$z_1 = z_2 + \sum_{i=1}^{m(n+k-1)} y_i^{n+k-1};$$

and

$$(45) \quad N_{n+k-1}(z_2) + \sum_{i=1}^{m(n+k-1)} N_{n+k}(S_i^{n+k-1})(y_i^{n+k-1}) < \varepsilon.$$

Denote

$$(46) \quad \varepsilon_1 = \sum_{i=1}^{m(n+k-1)} N_{n+k}(S_i^{n+k-1})(y_i^{n+k-1}).$$

Then from (45) we have

$$(47) \quad N_{n+k-1}(z_2) < \varepsilon - \varepsilon_1.$$

Let

$$(48) \quad v_i = \sum_{j \in T(i)} y_j^{n+k-1}, \quad \text{where } T(i) = \{j : S_j^{n+k-1} \subset S_i^n\} \text{ for } i = 1, \dots, m(n).$$

Let  $u_1 = v_1$ . Then we have  $\text{supp}(y_1 - u_1) \subset S_1^n$ . We prove:

*Claim 7.*  $y_1 - u_1 \in E_{n+k-1}$ .

*Proof.* Observe that

$$\begin{aligned} z_2 &= z_1 - \sum_{i=1}^{m(n+k-1)} y_i^{n+k-1} = y_1 - \sum_{i=2}^{m(n)} \alpha_i y_i - \sum_{i=1}^{m(n+k-1)} y_i^{n+k-1} \\ &= y_1 - \sum_{i=2}^{m(n)} \alpha_i y_i - \sum_{i=1}^{m(n)} \sum_{j \in T(i)} y_j^{n+k-1} = y_1 - \sum_{i=2}^{m(n)} \alpha_i y_i - \sum_{i=1}^{m(n)} v_i \\ &= y_1 - v_1 - \sum_{i=2}^{m(n)} (\alpha_i y_i + v_i) = - \sum_{i=1}^{m(n)} (\alpha_i y_i + v_i), \text{ where } \alpha_1 = -1. \end{aligned}$$

Since  $z_2 \in E_{n+k-1}$ , we can write  $-z_2 = \sum_{j=1}^{m(n+k-1)} w_j$ , where

$$w_j = \lambda_j m(n+k-1) \chi_{S_j^{n+k-1}}, \lambda_j \in \mathbb{R}, j = 1, \dots, m(n+k-1).$$

Denote  $\varphi_i = \sum_{S_j^{n+k-1} \subset S_i^n} w_j$ . Since

$$\varphi_i \in E_{n+k-1} \text{ and } \text{supp } \varphi_i \subset S_i^n \text{ for every } i = 1, \dots, m(n),$$

we get

$$-z_2 = \sum_{i=1}^{m(n)} \varphi_i = \sum_{i=1}^{m(n)} (\alpha_i y_i + v_i).$$

Therefore

$$(49) \quad \sum_{i=1}^{m(n)} (\alpha_i y_i + v_i - \varphi_i) = \theta.$$

From (42) and (48) we have

$$\text{supp}(\alpha_i y_i + v_i) \subset S_i^n \text{ for every } i = 1, \dots, m(n).$$

Hence

$$\text{supp}(\alpha_i y_i + v_i - \varphi_i) \subset S_i^n \text{ for every } i = 1, \dots, m(n).$$

Therefore from (49) we obtain

$$\alpha_i y_i + v_i - \varphi_i = \theta \text{ for } i = 1, \dots, m(n).$$

It follows that

$$\alpha_i y_i + v_i = \varphi_i \in E_{n+k-1} \text{ for } i = 1, \dots, m(n).$$

Since  $\alpha_1 = 1$  we get

$$\alpha_1 y_1 + v_1 = -y_1 + u_1 = -(y_1 - u_1) \in E_{n+k-1}.$$

Consequently  $y_1 - u_1 \in E_{n+k-1}$ , and the claim is proved.

*Remark 7.* Denote  $u_i^1 = v_i$  and  $z_i^2 = z_i^1 + v_i = z_i^1 + u_i^1$  for  $i = 1, \dots, m(n)$ . Then from (44) and (48) we get

$$y_1 - u_1 = z_2 + \sum_{i=2}^{m(n)} z_i^2 \in E_{n+k-1};$$

moreover

$$(50) \quad \text{supp}(y_1 - u_1) \subset S_1^n \text{ and } \text{supp } z_i^2 \subset S_i^n \text{ for } i = 2, \dots, m(n).$$

Therefore (43) and (44) hold true if  $y_1, z_i^1$  are replaced by  $y_1 - u_1$  and  $z_i^2$ , respectively.

**The induction step.** We continue to cut a small element  $u_2$  off from  $y_1 - u_1$  so that  $y_1 - u_1 - u_2 \in E_{n+k-2}$  and so on. In the  $k$ -th step we get

$$y_1 - u_1 - u_2 - \dots - u_k \in E_n.$$

This process is done by induction as follows: Since

$$z_2 \in E_{n+k-1} \text{ and } N_{n+k-1}(z_2) < \varepsilon - \varepsilon_1 < 4,$$

we can obtain  $z_3 \in E_{n+k-2}$  and  $y_i^{n+k-2} \in E_{n+k-1}(S_i^{n+k-2}), i = 1, \dots, m(n+k-1)$ , such that

$$z_2 = z_3 + \sum_{i=1}^{m(n+k-2)} y_i^{n+k-2},$$

and

$$(51) \quad N_{n+k-2}(z_3) + \sum_{i=1}^{m(n+k-2)} N_{n+k-2}(S_i^{n+k-2})(y_i^{n+k-2}) < \varepsilon - \varepsilon_1.$$

Denote

$$T(2, i) = \{t : S_t^{n+k-2} \subset S_i^n\}, \quad i = 1, \dots, m(n);$$

$$u_i^2 = \sum_{t \in T(2, i)} y_t^{n+k-2}; \quad i = 1, \dots, m(n).$$

Notice  $u_2 = u_1^2$ . Let  $z_i^3 = z_i^2 + u_i^2$  for  $i = 2, \dots, m(n)$ . Then it is easy to see that  $y_1 - u_1, z_2, z_i^2$  and  $E_{n+k-1}$  in Remark 7 can be replaced by  $y_1 - u_1 - u_2, z_3, z_i^3$  and  $E_{n+k-2}$  respectively.

Now, for every  $j = 1, \dots, k$  and  $i = 1, \dots, m(n+k-j)$  we choose  $y_i^{n+k-j} \in E_{n+k-j+1}(S_i^{n+k-j})$  and  $z_{j+1} \in E_{n+k-j}$  such that

$$z_j = z_{j+1} + \sum_{i=1}^{m(n+k-j)} y_i^{n+k-j};$$

and

$$(52) \quad N_{n+k-j}(z_{j+1}) + \sum_{i=1}^{m(n+k-j)} N_{n+k-j+1}(S_i^{n+k-j})(y_i^{n+k-j}) < \varepsilon - \sum_{t=1}^{j-1} \varepsilon_t,$$

where

$$\varepsilon_t = \sum_{i=1}^{m(n+k-t)} N_{n+k-t+1}(S_i^{n+k-t})(y_i^{n+k-t}).$$

(Observe that  $\varepsilon - \varepsilon_1 > 0$ , by (47), and  $\varepsilon - \sum_{t=1}^{j-1} \varepsilon_t > 0$  by induction.) Then from (52) we have  $N_{n+k-j}(z_{j+1}) < \varepsilon - \sum_{t=1}^j \varepsilon_t$ . Observe that

$$(53) \quad \sum_{j=1}^k \sum_{i=1}^{m(n+k-j)} N_{n+k-j+1}(S_i^{n+k-j})(y_i^{n+k-j}) = \sum_{j=1}^k \varepsilon_j < \varepsilon.$$

Denote

$$T(j, i) = \{t : S_t^{n+k-j} \subset S_i^n\}, \quad i = 1, \dots, m(n);$$

$$u_i^j = \sum_{t \in T(j, i)} y_t^{n+k-j}.$$

Notice  $u_j = u_1^j$ . Let  $z_i^j = z_i^{j-1} + u_i^{j-1}$  for  $i = 2, \dots, m(n)$ . Then from Theorem 4(ii)–(iv) we get

$$(54) \quad \begin{aligned} \|u_j\| &\leq \sum_{i \in T(j, 1)} N_{n+k-j+1}(S_i^{n+k-j})(y_i^{n+k-j}) \\ &\leq \sum_{i=1}^{m(n+k-j)} N_{n+k-j+1}(S_i^{n+k-j})(y_i^{n+k-j}) = \varepsilon_j. \end{aligned}$$

Let  $z = \sum_{j=1}^k u_j$ . Since  $\{N_n(x)\}$  is a decreasing sequence for every  $x \in E$ , from (53) and (54) we have

$$\|z\| \leq \sum_{j=1}^k \|u_j\| \leq \sum_{j=1}^k \varepsilon_j < \varepsilon.$$

Then it is easy to see that  $y_1 - u_1, z_2, z_i^2$  and  $E_{n+k-1}$  in Remark 7 can be replaced by  $y_1 - \sum_{j=1}^k u_j = y_1 - z, z_{k+1}, z_i^{k+1}$  and  $E_{n+k-k} = E_n$  respectively. Therefore we get  $y_1 - z = y_1 - \sum_{j=1}^k u_j \in E_n$ . Observe that

$$\|x_1 - (y_1 - z)\| \leq \|x_1 - y_1\| + \|z\| < 2\varepsilon.$$

Therefore  $\|x_1 - E_n\| < 2\varepsilon$ . Since  $\varepsilon$  is arbitrarily small we infer that  $x_1 \in E_n$ . So,  $x_1 \in E_n \cap \text{span } C_n(a_1^n)$ . Hence from Lemma 6 we get

$$x_1 = t_1 a_1^n = t_1 m(n) \chi_{S_1^n} \text{ for some } t_1 \in \mathbb{R}.$$

Thus we have shown that if  $\lambda_i \neq 0$ , then  $x_i = t_i m(n) \chi_{S_i^n}$  for some  $t_i \in \mathbb{R}$ . Since  $\{m(n) \chi_{S_i^n} : i = 1, \dots, m(n)\}$  is linearly independent, it follows from (40) that  $t_i = 0$ , if  $\lambda_i \neq 0$ . This contradicts the fact that  $x_i \in \text{span } C_n(a_i^n) \setminus \{\theta\}$  for every  $i = 1, \dots, m(n)$ . Theorem 5 is proved.

From Theorem 5 and from [CDM] we obtain the following result which answers affirmatively a problem of Dobrowolski and Mogilski; see [DM], Question 5-5 (Question 575).

**Corollary.** *For any dense  $\sigma$ -fd-compact convex subset  $W$  of  $C$ , we have  $(C, W) \simeq (Q, Q^f)$ , where  $Q$  denotes the Hilbert cube, and*

$$Q^f = \{x = (x_n) \in Q : x_n = 0 \text{ for almost all } n \in \mathbb{N}\}$$

and “ $\sigma$ -fd-compact” means a countable union of finite dimensional compact sets.

Related to Theorem 5 we ask:

**Question 1** [NT1]. Let  $F$  denote the linear metric space constructed by Roberts [R1]:

- (i) Is every convex subset of  $F$  an AR?
- (ii) Does every compact convex subset of  $F$  have the fixed point property?

We do not know even if the *whole space*  $F$  is an AR. It is of interest to know whether Theorem 2 still holds in the general case:

**Question 2.** Is the compact convex set  $C$  defined by Theorem 1 an AR?

It should be noted that Condition (i) is essential in our proof of Theorem 2. However, as we observed in Remark 6, Condition (ii) can be dropped. By [NT2]  $C$  has the fixed point property. So we may ask Question 2 in the following more general situation:

**Question 3** [NT1]. Assume that  $X$  is a compact convex set with the fixed point property. Is  $X$  an AR?

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