ON 'CLIFFORD'S THEOREM'
FOR PRIMITIVE FINITARY GROUPS

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Abstract. Let \( V \) be an infinite-dimensional vector space over any division ring \( D \), and let \( G \) be an irreducible primitive subgroup of the finitary group \( \text{FGL}(V) \). We prove that every non-identity ascendant subgroup of \( G \) is also irreducible and primitive. For \( D \) a field, this was proved earlier by U. Meierfrankenfeld.

Let \( V \) be a left vector space over some division ring \( D \), and let \( G \) be an irreducible subgroup of the full finitary general linear group \( \text{FGL}(V) \) on \( V \). A subgroup \( H \) of a group \( G \) is ascendant if there is an ascending (possibly infinite) series

\[ H = H_0 \leq H_1 \leq \cdots \leq H_\alpha \leq \cdots \leq H_\gamma = G \]

of subgroups of \( G \), each normal in its successor, joining \( H \) to \( G \). In [1], see 7.6, U. Meierfrankenfeld proved that if \( D \) is a field, then every ascendant subgroup \( H \) of \( G \) is completely reducible, the classical Clifford’s Theorem being the case where \( V \) is finite dimensional and \( H \) is normal in \( G \). Independently, and using a substantially different approach, the author derived the same conclusion for any division ring \( D \); see Prop. 9 of [5].

Meierfrankenfeld’s approach studies primitive groups first and the major step in his proof ([1], 7.4) is to show that, if \( D \) is a field, if \( \dim_D V \) is infinite and if \( G \) is also primitive with \( H \neq \langle 1 \rangle \), then \( H \) is irreducible and primitive. The author’s approach did not consider primitive groups separately. Thus two obvious questions arise. Does this same conclusion hold for any division ring \( D \), and if so, can it be obtained relatively quickly from the results of [5] without, for example, repeating and adapting as necessary the lengthy analysis in [1]? (Incidentally I assume that the latter is feasible, though I have not seriously tried to carry it out.) The object of this note is to show that the answer to both questions is yes.

**Theorem.** Let \( V \) be an infinite-dimensional left vector space over the division ring \( D \), and let \( H \) be a non-trivial ascendant subgroup of the irreducible primitive subgroup \( G \) of \( \text{FGL}(V) \). Then \( H \) too is irreducible and primitive.

**Proof of the Theorem.** We prove first that \( H \) is irreducible. By ‘Clifford’s Theorem’ (see [5], Prop. 9) the group \( H \) is completely reducible. Suppose \( H \) is reducible. Let \( \{ H_\alpha : 0 \leq \alpha \leq \gamma \} \) be an ascending series with \( H = H_0 \) and \( H_\gamma = G \). By hypothesis \( H \neq \langle 1 \rangle \); choose \( h \in H \setminus \langle 1 \rangle \) and set \( d = \dim_D [V, h] \). Then \( 0 < d < \infty \). For each

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\( \alpha \leq \gamma \) set

\[
K_\alpha = \langle g \in H_\alpha : \dim_D[V, g] \leq d \rangle.
\]

Then \( K_\alpha \) is normal in \( H_\alpha \), indeed if \( \alpha < \gamma \) then \( K_\alpha \) is normal in \( H_{\alpha+1} \), and

\[
K = K_0 \leq K_1 \cdots K_\alpha \leq K_{\alpha+1} \cdots K_\gamma \leq G
\]
is an ascending series. Moreover \( h \in K \leq H \), so \( K \) is non-trivial and reducible.

Let \( U \) be an irreducible \(-K\) submodule of \( V \) and suppose \( \dim_D U > d \). If \( UK_\alpha = U \) and \( g \in K_{\alpha+1} \) with \( \dim_D[V, g] \leq d \), then \( g \) normalizes \( K_\alpha \), the modules \( U \) and \( Ug \) are both \(-K_{\alpha+1} \) irreducible, \( U \cap Ug \neq \{0\} \) and \( U = Ug \). Therefore \( UK_\alpha = U \). A simple induction yields that \( UK_\gamma = U \). But \( K_\gamma \) is normal in the primitive group \( G \), so \( K_\gamma \) is irreducible ([4], 3.1, and [5], Prop. 8). Consequently \( U = V \), a contradiction of the reducibility of \( K \). Therefore \( \dim_D U \leq d \). The same argument yields that \( V \) is a direct sum of irreducible \(-K_{\alpha} \) modules of dimension at most \( d \) for any \( \alpha \leq \gamma \) with \( K_\alpha \) reducible.

Choose \( \alpha \) and an irreducible \(-K_{\alpha} \) submodule \( U \) of \( V \) with \( [U, K_\alpha] \neq \{0\} \), with \( \dim_D U \leq d \) and with \( \dim_D U \) maximal. By the above such \( \alpha \) and \( U \) exist. From all such choices pick \( \alpha \) and \( U \) such that the \(-K_\alpha \) homogeneous component \( W \) of \( V \) containing \( U \) has \( \dim_D W \) minimal; note that by finitariness \( \infty > \dim_D W \geq \dim_D U > 0 \).

Now suppose \( K_{\alpha+1} \) is reducible. If \( U_1 \) is an irreducible \(-K_{\alpha+1} \) submodule of \( V \) containing a copy of \( U \), then the choice of \( \alpha \) and \( U \) ensures that \( \dim_D U = \dim_D U_1 \) and that \( U \) and \( U_1 \) are isomorphic as \(-K_{\alpha} \) modules. It follows that \( W \) is a direct sum of \(-K_{\alpha+1} \) homogeneous components of \( V \) and the minimal choice of \( W \) yields that \( W \) is a \(-K_{\alpha+1} \) homogeneous component of \( V \). If \( \lambda \leq \gamma \) is a limit ordinal with \( W \) a \(-K_{\alpha+1} \) homogeneous component of \( V \) for all \( \beta \) with \( \alpha \leq \beta < \lambda \), then \( WK_\lambda = \bigcup_{\beta<\lambda} WK_\beta = W \), so \( K_\lambda \) is reducible, the \(-K_\beta \) irreducible submodules of \( W \) are \(-K_\beta \) irreducible and isomorphic for \( \alpha \leq \beta < \lambda \) (by the choice of \( \alpha \) and \( U \)), \( W \) is a sum of \(-K_\lambda \) homogeneous components of \( V \) and, by the minimal choice of \( W \), \( W \) is a \(-K_\lambda \) homogeneous component of \( V \). Since \( K_\gamma \) is irreducible ([4], 3.1 again), so \( WK_\gamma \neq W \) and the above yields the existence of \( \beta < \gamma \) with \( K_\beta \) reducible and \( K_{\beta+1} \) irreducible.

To simplify notation assume \( \beta = 0 \); that is, assume \( K_1 \) is irreducible. Let \( \alpha \geq 1 \), let \( L \neq \{1\} \) be a reducible normal subgroup of \( K_\alpha \) and let \( U \) be an irreducible \(-L \) submodule of \( V \) with \( [U, L] \neq \{0\} \). (Note that such \( \alpha \) and \( L \) exist, for example \( 1 \) and \( K_1 \).) By an argument we have seen before \( \dim_D U \leq d \). Let \( W \) be the \(-L \) homogeneous component of \( V \) containing \( U \). Choose \( \alpha \), \( L \) and \( U \) so firstly that \( \dim_D U \) is maximal and secondly that \( \dim_D W \) is minimal. For simplicity of notation assume \( \alpha = 1 \).

The \(-L \) homogeneous components of \( V \) form a system \( V = \bigoplus_{\omega \in \Omega} V_\omega \) of imprimitivity for \( K_1 \). Here \( \Omega \) is infinite and permuted transitively and finitarily by \( K_1 \) and the \( \dim_D V_\omega \) are finite. By definition \( K_1 \) is generated by elements with support of bounded dimension \(( \leq d \) \). Consequently \( K_1|_{\Omega} \) is generated by elements with support of bounded cardinality (at most \( 2d \)). It follows, in the terminology of P. M. Neumann ([3], p. 563), that \( K_1|_{\Omega} \) cannot be totally imprimitive and therefore must be almost primitive. Thus there is a \( K_1 \)-invariant congruence \( q \) on \( \Omega \) such that \( K_1|_{\Omega/q} \) is primitive and hence is either \( \text{Alt}(\Omega/q) \) or \( \text{FSym}(\Omega/q) \); see [3], 2.3. Moreover each \( \omega q = \{ \sigma \in \Omega : \omega \sigma q \} \) is finite. Let \( L_1 \) be the kernel of the action of \( K_1 \) on \( \Omega/q \), so \( L \leq L_1 \leq K_1 \). For any subset \( \Sigma \) of \( \Omega \), write \( V_\Sigma \) for \( \bigoplus_{\sigma \in \Sigma} V_\sigma \).
Let \( g \in K_2 \) with \( L^0_1 \neq L_1 \). Then \( (K_1: L^0_1L_1) \leq 2 \), since \( \text{Alt}(\Omega/q) \) is simple of index 2 in \( \text{FSym}(\Omega/q) \). Also for any \( \omega \) in \( \Omega \) there is a finite subset \( \Sigma \) of \( \Omega \) with
\[
V_{\omega q}g(L^0_1)\mid L_1 = V_{\omega q}g \mid L_1 \leq V_{\Sigma q} = V_{\Sigma}.
\]
But \( V_{\omega}gK_1 = V \), so
\[
\dim_D V \leq 2 \cdot \dim_D(V_{\omega q}g(L^0_1)\mid L_1) \leq 2 \cdot |\Sigma| \cdot \dim_D V_{\omega} < \infty.
\]
This contradiction yields that \( L_1 \) is normal in \( K_2 \). Hence the \( D-L_1 \) homogeneous components of \( V \) form a system of imprimitivity for \( K_2 \) in \( V \). But the choice of \( L, U \) and \( W \) above ensures that these are just the \( V_{\omega} \). Therefore \( V = \bigoplus_{\omega} V_{\omega} \) is also a system of imprimitivity for \( K_2 \). We may repeat the above arguments with \( K_2 \) in place of \( K_1 \). A simple transfinite induction yields that \( V = \bigoplus_{\omega} V_{\omega} \) is a system of imprimitivity for \( G \). This contradiction of the primitivity of \( G \) completes the proof that \( H \) is irreducible. The primitivity of \( H \) follows at once from the following lemma.

**Lemma.** Let \( V \) be an infinite-dimensional left vector space over the division ring \( D \) and \( G \) a subgroup of \( \text{FGL}(V) \). The following are equivalent.

a) \( G \) is irreducible and primitive.

b) \( G \neq (1) \) and every non-trivial normal subgroup of \( G \) is irreducible.

**Proof.** As we have seen above a) implies b) by [4], 3.1, and [5], Prop. 8. Suppose b) holds. Clearly \( G \) is irreducible. Consider a non-trivial system \( V = \bigoplus_{\omega} V_{\omega} \) of imprimitivity for \( G \). Then \( G \) acts transitively and finitarily on \( \Omega \) and each \( \dim_D V_{\omega} \) is finite. Also \( N = \bigcap_{\omega} N_{G}(V_{\omega}) \) is a reducible normal subgroup of \( G \) and hence by b) is \((1)\). This holds for any such system of imprimitivity. If \( G|_{\Omega} \) is totally imprimitive, then every element of \( G \) lies in some such \( N \). Hence \( G|_{\Omega} \) is almost primitive and hence for some such system of imprimitivity \( G|_{\Omega} \) is \( \text{Alt}(\Omega) \) or \( \text{FSym}(\Omega) \).

Let \( \omega \in \Omega \) and \( g \in N_{G}(\omega) \). Then the support \( \text{supp}_{\Omega}(g) \) of \( g \) in \( \Omega \) is finite and \( C_{G}(g) \geq \text{Alt}(\Omega \setminus \text{supp}_{\Omega}(g)) \). The latter is simple and does not lie in \( N_{G}(\omega) \). Hence
\[
|\omega C_{G}(g)| = (C_{G}(g) : C_{G}(g) \cap N_{G}(\omega)),
\]
which is infinite. If \( |V_{\omega},g| \neq \{0\} \), then \( |V_{\omega}x,g| \neq \{0\} \) for all \( x \) in \( C_{G}(g) \) and \( \dim_D[V,g] \geq |\omega C_{G}(g)| \) is infinite. Consequently \( g \) and \( N_{G}(\omega) \) centralize \( V_{\omega} \). Let \( v_{\omega} \in V_{\omega} \{0\} \) and let \( X \) be a right transversal of \( N_{G}(\omega) \) to \( G \). Then \( U = \bigoplus_{x \in X} Dv_{\omega}x \leq V \) is a permutation module for \( G \) and \( \sum_{x,g \in X} D(v_{\omega}x - v_{\omega}y) \) is a proper \( D-G \) submodule of \( U \) and hence of \( V \). This contradicts the irreducibility of \( G \) and so \( G \) is primitive as claimed.

Now suppose \( \dim_D V \) is finite but otherwise assume the notation of the theorem. Clearly \( H \) now need not be irreducible; just let \( H \) be the center of \( G = \text{GL}(V) \). If \( H \) is normal, then \( H \) is homogeneous and hence has no non-zero fixed-points in \( V \). However if \( H \) is only subnormal, then \( H \) can have non-zero fixed-points. For example let \( G \) be the group \((E)\) of [2], p. 239. Then \( G \) is an irreducible primitive subgroup of \( \text{GL}(3,\mathbb{C}) \) of order 108 and is the split extension of a non-abelian normal subgroup \( N \) of order 27 and exponent 3 (it is a copy of \( \text{Tr}_{1}(3,3) \)) containing \( a = \text{diag}(1,\exp(2\pi/3),\exp(4\pi/3)) \) and a cyclic group of order 4. Also \( N \) is nilpotent of class 2, the subgroup \((a)\) is subnormal in \( G \) of subnormal depth 3 and \((1,0,0) \) is a non-zero fixed-point of \( a \).
REFERENCES


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