CALIBRATED THIN $\Pi^1_1$ $\sigma$-IDEALS ARE $G_\delta$

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Abstract. Let $E$ be a compact metric space, and let $I \subset \mathcal{K}(E)$ be a calibrated thin $\Pi^1_1$ $\sigma$-ideal. Then $I$ is $G_\delta$. This solves an open problem, which was posed by Kechris, Louveau and Woodin. Using our result we obtain a new proof of Kaufman’s theorem concerning $U$-sets and $U_0$-sets.

The study of $\sigma$-ideals of compact sets has been motivated by problems in harmonic analysis, namely by problems concerning $U$-sets and $U_0$-sets. The theory of $\sigma$-ideals of compact sets was developed by Kechris, Louveau and Woodin in [3]. Their results on the structure of $\sigma$-ideals of compact sets were used by Debs and Saint-Raymond ([1]) to give a positive answer to the old question of whether every Borel $U$-set is meager. The main aim of this paper is to give an answer to an open question which was posed in [3], namely to prove the assertion from the title. Using our result we obtain a new proof of Kaufman’s theorem concerning $U$-sets and $U_0$-sets.

Let $A \subset E$, then $\mathcal{K}(A)$ stands for the set of all compact subsets of $A$. A set $I \subset \mathcal{K}(E)$ is called $\sigma$-ideal if

(i) $K, L \in \mathcal{K}(E), K \subset I, L \subset K$, then $L \in I$,

(ii) $K, K_1, K_2, \ldots \in \mathcal{K}(E), K_n \in I$ for all $n \in \mathbb{N}$ and $K = \bigcup_{n=1}^{+\infty} K_n$, then $K \in I$.

We say that a $\sigma$-ideal $I$ is calibrated if, whenever $F \in \mathcal{K}(E), F_n \in I$ for every $n \in \mathbb{N}$, $\mathcal{K}(F \setminus \bigcup_{n=1}^{+\infty} F_n) \subset I$, then $F \in I$. A $\sigma$-ideal $I \subset \mathcal{K}(E)$ is said to be thin if $E$ contains no uncountable family of pairwise disjoint closed sets which are not in $I$. A $\sigma$-ideal $I$ is called locally non-Borel if for every closed set $F \not\in I$, $I \cap \mathcal{K}(F)$ is not Borel.

Let $X$ be a Polish space and $P \subset X$ be $\Pi^1_1$. A mapping $\varphi : X \to [0, \omega_1]$ is said to be $\Pi^1_1$-rank on $P$ if

(i) $\forall x \in X \setminus P : \varphi(x) = \omega_1$,

(ii) $\forall x \in P : \varphi(x) < \omega_1$.
(iii) \( \{(x, y) \in X \times X; \varphi(x) < \varphi(y)\} \) is \( \Pi^1_1 \) in \( X \times X \).

(iv) \( \{(x, y) \in X \times X; x \in P, \varphi(x) \leq \varphi(y)\} \) is \( \Pi^1_1 \) in \( X \times X \).

Remark. In the following, only properties described below in Theorems C, D and E of \( \Pi^1_1 \)-rank will be important for us.

A subset \( P \) of \( X \) is called \( \Pi^1_1 \)-complete if it is \( \Pi^1_1 \) and, for any Polish space \( Y \) and any \( \Pi^1_1 \) subset \( Q \) of \( Y \), there is a Borel mapping \( f: Y \to X \) such that \( Q = f^{-1}(P) \).

It is easy to see that no \( \Pi^1_1 \)-complete set is analytic.

The following problem was posed in [3]: Is every calibrated thin \( \Pi^1_1 \) \( \sigma \)-ideal \( I \subset \mathcal{K}(E) \), where \( E \) is a compact metric space, necessarily \( G_δ \)?

A partial answer was given in [5]. We will prove:

**Theorem 1.** Let \( E \) be a compact metric space and \( I \subset \mathcal{K}(E) \) be a calibrated thin \( \Pi^1_1 \) \( \sigma \)-ideal. Then \( I \) is a \( G_δ \) subset of \( \mathcal{K}(E) \).

We will use the following theorems.

**Theorem A** ([2], p. 133). Let \( E \) be a compact metric space. Let \( P, B \) be two disjoint subsets of \( E \) with \( P \) in \( \Sigma^1_1 \). If there is no \( F_\sigma \) set \( C \) separating \( P \) from \( B \) (i.e. \( P \subset C, B \cap C = \emptyset \)), then there is a homeomorphic copy \( F \) of \( 2^\mathbb{N} \) with \( F \subset P \cup B \), and \( F \cap B \) is countable dense in \( F \).

**Theorem B** ([2], p. 132). Let \( E \) be a compact metric space. Then every \( \Pi^1_1 \) \( \sigma \)-ideal \( I \subset \mathcal{K}(E) \) is either \( G_δ \) or else \( \Pi^1_1 \)-complete.

**Theorem C** ([2], p. 144). Any \( \Pi^1_1 \) subset of a Polish space admits a \( \Pi^1_1 \)-rank.

**Theorem D** ([2], p. 144). Let \( X \) be a Polish space, \( P \subset X \) be a \( \Pi^1_1 \) set and \( \varphi \) be a \( \Pi^1_1 \)-rank on \( P \). Then for every ordinal \( \alpha < \omega_1 \) the set \( \{x \in X; \varphi(x) < \alpha\} \) is Borel.

**Theorem E** ([2], p. 148). Let \( X \) be a Polish space, \( P \subset X \) be a \( \Pi^1_1 \) set and \( \varphi \) be a \( \Pi^1_1 \)-rank on \( P \). If \( Q \subset P \) is \( \Sigma^1_1 \), then \( \varphi \) is bounded on \( Q \), i.e. \( \sup\{\varphi(x); x \in Q\} < \omega_1 \).

Let \( E \) be a compact metric space, \( K \in \mathcal{K}(E) \), \( r > 0 \). Then a ball in the space \( \mathcal{K}(E) \) with the center \( K \) and with the radius \( r > 0 \) is denoted by \( B(K, r) \).

We start with an easy observation.

**Lemma 2.** Let \( E \) be a compact metric space, \( K, L \in \mathcal{K}(E) \), \( K \neq L \), \( K \cap L = \emptyset \). Then there exists \( \varepsilon > 0 \) such that \( B(K, \varepsilon) \cap B(L, \varepsilon) = \emptyset \) and if \( K' \in B(K, \varepsilon) \), \( L' \in B(L, \varepsilon) \), then \( K' \cap L' = \emptyset \).

**Proof.** If \( K = \emptyset \) or \( L = \emptyset \), then we put \( \varepsilon = \frac{1}{4}(\text{diam}(E) + 1) \) and we are done. Otherwise \( \varepsilon = \frac{1}{2}\text{dist}(K, L) \). If \( K' \in B(K, \varepsilon) \), \( L' \in B(L, \varepsilon) \), then \( \text{dist}(K', L') \geq \frac{1}{2}\varepsilon > 0 \) and the proof is complete. \( \Box \)

**Theorem 3.** Let \( E \) be a compact metric space, \( I \subset \mathcal{K}(E) \), \( I \neq \mathcal{K}(E) \) be a calibrated locally non-Borel \( \Pi^1_1 \) \( \sigma \)-ideal. Then there exists a family of pairwise disjoint elements from \( \mathcal{K}(E) \setminus I \) with the cardinality of the continuum.

**Proof.** There exists an unbounded \( \Pi^1_1 \)-rank \( \varphi \) on \( I \) (see Theorems B, C and D). We will construct a transfinite sequence \( (K_\alpha)_{\alpha < \omega_1} \) such that

(i) \( K_\alpha \in I \),

(ii) \( K_\alpha \cap K_\beta = \emptyset \) for every \( \alpha, \beta < \omega_1 \), \( \alpha \neq \beta \),
Choose $K_0 \in I$ such that $\varphi(K_0) > 0$. Suppose that for $\lambda < \omega_1$ we have constructed $K_{\alpha}$, $\alpha < \lambda$ fulfilling (i), (ii) and (iii). There exists $T \in \mathcal{K}(E) \setminus I$ such that $T \cap \bigcup_{\alpha < \lambda} K_{\alpha} = \emptyset$; otherwise the calibration of $I$ gives $E \in I$ and it implies $I = \mathcal{K}(E)$, a contradiction. If

$$\sup\{\varphi(K); K \in I, K \subset T\} < \xi < \omega_1,$$

then

$$I \cap \mathcal{K}(T) \subset \{K \in \mathcal{K}(E); \varphi(K) < \xi\} \subset I.$$  

We obtain

$$I \cap \mathcal{K}(T) \subset \{K \in \mathcal{K}(E); \varphi(K) < \xi\} \subset I \cap \mathcal{K}(T).$$

This and Theorem D imply that $I \cap \mathcal{K}(T)$ is a Borel set. It is a contradiction with locally non-Borelness of $I$. Thus we have

$$\sup\{\varphi(K); K \in I, K \subset T\} = \omega_1.$$  

We choose $K_{\lambda} \in I \cap \mathcal{K}(T)$ such that $\varphi(K_{\lambda}) > \lambda$. It finishes the construction of $(K_{\alpha})_{\alpha < \omega_1}$. Denote $B = \{K_{\alpha}; \alpha < \omega_1\}$. Let $C$ be a Borel set such that $B \subset C \subset I$. Then we have

$$\sup\{\varphi(K); K \in C\} \geq \sup\{\varphi(K); K \in B\} = \omega_1.$$  

It is a contradiction, since $\varphi$ must be bounded on each Borel subset of $I$ (Theorem E). Now Theorem A gives that there exists a homeomorphic copy $F$ of $2^\mathbb{N}$ such that $F \subset B \cup \mathcal{K}(E) \setminus I$ and $Q = F \cap B$ is countable and dense in $F$.

Following [2], p. 111, we denote the set of all finite sequences from $\{0, 1\}$ by $\text{Seq}(0, 1)$ and if $i \in \text{Seq}(0, 1)$, then $i^\tau (i^\tau$, respectively) stands for the concatenation of the sequences $i$ and $0$ ($i$ and $1$, respectively).

Now for every $i \in \text{Seq}(0, 1)$ we construct $P_i \in \mathcal{K}(E)$, $\varepsilon_i > 0$ such that

(i) $P_i \in Q$,
(ii) for every $K_0 \in B(P_{i^0}, \varepsilon_{i^0})$, $K_1 \in B(P_{i^1}, \varepsilon_{i^1})$ we have $K_0 \cap K_1 = \emptyset$,
(iii) $\varepsilon_i < 2^{-|i|}$ ($|i|$ stands for the length of $i$),
(iv) $B(P_{i^0}, \varepsilon_{i^0}) \cap B(P_{i^1}, \varepsilon_{i^1}) = \emptyset$,
(v) $B(P_{i^0}, \varepsilon_{i^0}) \cup B(P_{i^1}, \varepsilon_{i^1}) \subset B(P_i, \varepsilon_i)$.

Choose $P_\emptyset \in Q$, $\varepsilon_\emptyset \in (0, 1)$. Now suppose that $P_i$, $\varepsilon_i$, were defined. Since $F$ has no isolated point and $Q$ is dense in $F$, we can choose $P_{\iota^0}$, $P_{\iota^1} \in Q \cap B(P_i, \varepsilon_i)$ such that $P_{\iota^0} \neq P_{\iota^1}$. As $Q \subset B$ we have $P_{\iota^0} \cap P_{\iota^1} = \emptyset$. According to Lemma 2 there exists $\varepsilon \in (0, 2^{-|\iota| - 1})$ such that the conditions (ii) and (iv) are fulfilled for $P_{\iota^0}$, $P_{\iota^1}$ and $\varepsilon_{\iota^0} = \varepsilon_{\iota^1} = \varepsilon$. If $\varepsilon$ is sufficiently small, then the condition (v) is also fulfilled. Put

$$W = \bigcap_{n=1}^{+\infty} \bigcup_{|\iota| = n} B(P_{\iota}, \varepsilon_{\iota}).$$

Clearly $W \subset F$ and $W$ is a homeomorphic copy of $2^\mathbb{N}$. If $L_0$, $L_1 \in W$, $L_0 \neq L_1$, then there exist $\iota$, $\tau \in \text{Seq}(0, 1)$ such that $|\iota| = |\tau|$, $\iota \neq \tau$, $L_0 \in B(P_{\iota}, \varepsilon_{\iota})$, $L_1 \in B(P_{\tau}, \varepsilon_{\tau})$. The property (ii) implies that $L_0 \cap L_1 = \emptyset$. Thus the elements of $W$ form a family of pairwise disjoint sets. This implies that $W \setminus Q$ is a family of pairwise disjoint sets from $\mathcal{K}(E) \setminus I$ with the cardinality of the continuum and we are done. \[\square\]
We introduce the notion of a $I$-thin set to exhibit an interesting corollary of Theorem 3.

Let $E$ be a compact metric space, and let $I \subset \mathcal{K}(E)$ be a $\sigma$-ideal. A set $A \subset E$ is $I$-thin if there is no uncountable family $\Phi \subset \mathcal{K}(A)$ of pairwise disjoint sets which are not in $I$.

**Corollary 4.** Let $E$ be a compact metric space, and let $I \subset \mathcal{K}(E)$ be a calibrated locally non-Borel $\Pi_1^1$-$\sigma$-ideal. Then $K \in \mathcal{K}(E)$ is $I$-thin if and only if $K \in I$.

Since $\sigma$-ideals $U$ (of closed sets of uniqueness in the unit circle $\mathbb{T}$) and $U_0$ (of closed sets of extended uniqueness in the unit circle $\mathbb{T}$) (cf. [2], p. 117, [4]) fulfill the conditions in Theorem 3 we have the following corollary.

**Corollary 5** (Kaufman, see [2], p. 235). Let $K \in \mathcal{K}(\mathbb{T}) \setminus U$. Then we can find a family $\{K_x\}_{x \in 2^\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{K}(\mathbb{T}) \setminus U$ contained in $K$. Similarly replace $U$ by $U_0$.

Corollary 4 and Corollary 5 follow from Theorem 3 and the following easy observation.

**Observation 6.** Let $E$ be a metric compact space, $I \subset \mathcal{K}(E)$ be a calibrated locally non-Borel $\Pi_1^1$-$\sigma$-ideal and $F \in \mathcal{K}(E) \setminus I$. Then the $\sigma$-ideal $I \cap \mathcal{K}(F)$ is a calibrated locally non-Borel $\Pi_1^1$-$\sigma$-ideal in the space $\mathcal{K}(F)$.

**Lemma 7.** Let $E$ be a compact metric space and $K \in \mathcal{K}(E)$. Then

$$f_K : L \mapsto L \cap K$$

is a Borel mapping from $\mathcal{K}(E)$ to $\mathcal{K}(E)$.

**Proof.** It is well-known that $\mathcal{K}(E)$ is a compact metric space and that the topology induced by the Hausdorff metric and the topology generated by the sets of the form $\{L \in \mathcal{K}(E) ; L \subset V\}$, $\{L \in \mathcal{K}(E) ; L \cap V \neq \emptyset\}$ for $V$ open in $E$ coincide on $\mathcal{K}(E)$ (cf. [2], p. 117, [4]). Therefore it is sufficient to prove that the sets

$$X = \{L \in \mathcal{K}(E) ; L \cap K \cap G \neq \emptyset\}, \quad Y = \{L \in \mathcal{K}(E) ; L \cap K \cap G \neq \emptyset\}$$

are Borel for every $G$ open in $E$. There exist closed sets $F_n$, $n \in \mathbb{N}$, such that $K \cap G = \bigcup_{n=1}^{+\infty} F_n$. We have

$$\{L \in \mathcal{K}(E) ; L \cap K \cap G \neq \emptyset\} = \bigcup_{n=1}^{+\infty} \{L \in \mathcal{K}(E) ; L \cap F_n \neq \emptyset\}$$

and therefore $X$ is Borel. The set $Y$ is also Borel, since

$$\{L \in \mathcal{K}(E) ; L \cap K \subset G\} = \mathcal{K}(E) \setminus \{L \in \mathcal{K}(E) ; L \cap K \cap (E \setminus G) \neq \emptyset\}. \quad \Box$$

**Proof of Theorem 1.** If $E \in I$, then we are done. Suppose that $E \notin I$. Let $\mathcal{F} \subset \mathcal{K}(E)$ be a maximal system (with respect to the inclusion) of pairwise disjoint elements from $\mathcal{K}(E) \setminus I$ such that $I \cap \mathcal{K}(F)$ is $G_6$ in $\mathcal{K}(E)$, whenever $F \in \mathcal{F}$. The system $\mathcal{F}$ is countable since $I$ is thin. Put

$$I^* = \{K \in \mathcal{K}(E) ; \text{ for every } F \in \mathcal{F} \text{ we have } K \cap F \in I\}.$$

We claim $I = I^*$. Clearly $I \subset I^*$. Suppose that $K \in I^*$. Let $L \in \mathcal{K}(E)$, $L \cap \bigcup\{F ; F \in \mathcal{F}\} = \emptyset$. If $L$ is an element from $\mathcal{K}(E) \setminus I$, then the $\sigma$-ideal $\mathcal{K}(L) \cap I$ must be locally non-Borel in the space $\mathcal{K}(L)$; otherwise we obtain a contradiction with the maximality of $\mathcal{F}$. Using Observation 6 and Theorem 3 we obtain that the
σ-ideal $K(L) \cap I$ contains an uncountable family of pairwise disjoint closed sets which are not in $I$. This contradicts the thinness of $I$. Thus we can conclude

$$K(K \setminus \bigcup\{F; F \in \mathcal{F}\}) \subset I.$$  

Since $K \setminus \bigcup\{F; F \in \mathcal{F}\} = K \setminus \bigcup\{F \cap K; F \in \mathcal{F}\}$, $\mathcal{F}$ is countable and $I$ is calibrated, we obtain $K \in I$. Using Lemma 7 and the following fact

$$I = I^* = \bigcap\{f^{-1}_F(I); F \in \mathcal{F}\} = \bigcap\{f^{-1}_F(I \cap K(F)); F \in \mathcal{F}\}$$

we can conclude that $I$ is Borel, therefore $G_δ$ (Theorem B) and the proof is complete. \qed

Remark. Let us note that the σ-ideal of countable compact subsets of the interval $[0, 1]$ is calibrated $\mathbf{Π}^1_1$ and is not $G_δ$. This means that thinness cannot be omitted in our theorem. We will show that the same holds for calibration. We will give only a sketch of the proof. Let $E = [0, 1]$ and let $(V_n)^{+∞}_{n=1}$ be a sequence of non-empty open sets forming an open base of $E$. Let $(F_n)^{+∞}_{n=1}$ be a sequence of pairwise disjoint closed subsets of $E$ satisfying the following conditions for every $n \in \mathbb{N}$:

1. $F_n \subset V_n$,
2. $F_n$ has positive Lebesgue measure.

(Of course $F_n$’s are nowhere dense.) Put

$$B = \{K \in K(E); K has null Lebesgue measure\} \cup \bigcup_{n=1}^{+∞} K(F_n).$$

The set $B$ is Borel and hereditary (i.e. if $K, L \in K(E), K \in B, L \subset K,$ then $L \in B$). Put

$$I = B_σ = \{K \in K(E); K can be covered by countably many elements from B}\}.$$  

Following [2] (pp. 197–198) we can consider the Cantor-Bendixson rank $rk_B$ associated with the $B$-derivative. We will use the following theorem.

**Theorem F** ([2], p. 202). Let $E$ be a compact metric space, $B$ be a Borel hereditary subset of $K(E)$ consisting of nowhere dense sets and $I = B_σ$. If every non-empty open subset of $E$ contains $K \in I$ with $rk_B(K) > 1$, then $I$ is $\mathbf{Π}^1_1$-complete.

It is not difficult to check that $B$ fulfills the conditions in the above theorem. It shows that $I$ is $\mathbf{Π}^1_1$ and non-$G_δ$. The σ-ideal $I$ is thin since $I$ contains thin σ-ideal of closed sets with Lebesgue null measure.

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REFERENCES


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