

DEFORMATIONS OF DIHEDRAL REPRESENTATIONS

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ABSTRACT. G. Burde proved (1990) that the $SU_2(\mathbb{C})$ representation space of two-bridge knot groups is one-dimensional. The same holds for all torus knot groups.

The aim of this note is to prove the following:

Given a knot $k \subset S^3$ we denote by \hat{C}_2 its twofold branched covering space. Assume that there is a prime number p such that $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$. Then there exist representations of the knot group onto the binary dihedral group $D_p \subset SU_2(\mathbb{C})$ and these representations are smooth points on a one-dimensional curve of representations into $SU_2(\mathbb{C})$.

Let $k \subset S^3$ be a knot, let $G = \pi_1(S^3 - k)$, and assume $\rho : G \rightarrow SL_2(\mathbb{C})$ is an irreducible representation, i.e. the only subspaces of \mathbb{C}^2 which are invariant under $\rho(G)$ are $\{0\}$ and $\{\mathbb{C}^2\}$. According to a result of Thurston (see [5, Proposition 3.2.1]) it is possible to deform ρ non-trivially, i.e. ρ is contained in a component R of the $SL_2(\mathbb{C})$ representation space which is at least four-dimensional.

There is no general theorem which allows the deformation of representations $\rho : G \rightarrow SU_2(\mathbb{C})$. In [6] the authors proved that every non-abelian representation corresponding to a simple root of the Alexander polynomial on the complex unit circle is a limit point of an arc of non-abelian representations.

Given a knot $k \subset S^3$ we denote by \hat{C}_2 the twofold branched covering of the pair (S^3, k) and by $\Delta(t)$ the Alexander polynomial of the knot. The aim of this note is to prove the following theorem:

Theorem 1. *Let $k \subset S^3$ be given. Assume there exists a prime number p such that $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$. Then there exists a representation $\rho : G \rightarrow SU_2(\mathbb{C})$ such that $\text{Im } \rho$ is a binary dihedral group of order $4p$. Moreover, $[\rho] \in \mathfrak{R}(G)$ is a regular point and there is a neighborhood $U = U([\rho]) \subset \mathfrak{R}(G)$ which is diffeomorphic to an interval.*

Remark 1. Theorem 1 applies if $H_1(\hat{C}_2)$ is cyclic and non-trivial. For example it is valid for all 2-bridge knots (see [2]).

Remark 2. The converse of Theorem 1 is false. For example, in the case of the knot $k = 9_{35}$, the group $H_1(\hat{C}_2, \mathbb{Z}_3) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$, but all dihedral representations are smooth points on one-dimensional components of $\mathfrak{R}(G)$.

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1. INTRODUCTION

Given a finitely generated group G we denote

$$\mathcal{R}(G, \mathfrak{G}) := \{\varrho : G \rightarrow \mathfrak{G} \mid \varrho \text{ is a homomorphism}\}$$

where $\mathfrak{G} \in \{\mathrm{SL}_2(\mathbb{C}), \mathrm{SU}_2(\mathbb{C})\}$. The space $\mathcal{R}(G, \mathfrak{G})$ has the structure of a complex (resp. real) affine algebraic set if $\mathfrak{G} = \mathrm{SL}_2(\mathbb{C})$ (resp. $\mathfrak{G} = \mathrm{SU}_2(\mathbb{C})$). We call two representations $\varrho, \varrho' : G \rightarrow \mathfrak{G}$ equivalent ($\varrho \sim \varrho'$) iff they differ by an inner automorphism of \mathfrak{G} . Let $\mathcal{A}(G, \mathfrak{G}) := \{\varrho \in \mathcal{R}(G, \mathfrak{G}) \mid \mathrm{Im}(\varrho) \text{ is abelian}\}$.

If $\mathfrak{G} = \mathrm{SU}_2(\mathbb{C})$ we denote by $\mathfrak{R}(G)$ the space of conjugacy classes of non-abelian $\mathrm{SU}_2(\mathbb{C})$ representations, i.e.

$$\mathfrak{R}(G) := (\mathcal{R}(G, \mathrm{SU}_2(\mathbb{C})) \setminus \mathcal{A}(G, \mathrm{SU}_2(\mathbb{C}))) / \sim.$$

For each $\varrho \in \mathcal{R}(G, \mathrm{SL}_2(\mathbb{C}))$ there is a character $\chi_\varrho : G \rightarrow \mathbb{C}$ given by $\chi_\varrho : g \mapsto \mathrm{tr} \varrho(g)$. We have $\chi_\varrho = \chi_{\varrho'}$ if $\varrho \sim \varrho'$. The set $X(G)$ of these characters is a complex affine algebraic set; its ambient coordinates are given by $\{\chi(g_i)\}$ where $\chi \in X(G)$ is a character and $\{g_i\} \subset G$ is finite. Moreover, the map $t : \mathcal{R}(G, \mathrm{SL}_2(\mathbb{C})) \rightarrow X(G)$ given by $t : \varrho \mapsto \chi_\varrho$ is polynomial in the ambient coordinates (for details see [5]).

2. REAL POINTS IN $X(G)$

Because $\mathrm{SU}_2(\mathbb{C})$ is a subgroup of $\mathrm{SL}_2(\mathbb{C})$, there is an obvious map

$$t : \mathcal{R}(G, \mathrm{SU}_2(\mathbb{C})) \rightarrow X(G)$$

which associates to each representation the corresponding character. This map induces an injection $\hat{t} : \mathfrak{R}(G) \rightarrow X(G)$ (see [8]).

Definition 1. We call a representation $\varrho : G \rightarrow \mathrm{SL}_2(\mathbb{C})$ real iff $\mathrm{Im}(\varrho) \subset \mathrm{SL}_2(\mathbb{R})$ or $\mathrm{Im}(\varrho) \subset \mathrm{SU}_2(\mathbb{C})$.

If $\varrho : G \rightarrow \mathrm{SL}_2(\mathbb{C})$ is conjugate to a real representation, then $\chi_\varrho : G \rightarrow \mathbb{C}$ is a real-valued function. The converse is true for irreducible representations:

Lemma 1. *Let $\varrho : G \rightarrow \mathrm{SL}_2(\mathbb{C})$ be an irreducible representation. If $\chi_\varrho(G) \subset \mathbb{R}$, then ϱ is equivalent to a real representation.*

Proof. Choose $g \in G$ such that $\varrho(g) \neq \pm 1$. By [5, Lemma 1.5.1] there exists $h \in G$ such that the restriction of ϱ to the subgroup generated by g and h is irreducible and such that $\chi_\varrho(h) \neq \pm 2$.

By conjugation we assume that $\varrho(h) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ where $\lambda + \lambda^{-1} \in \mathbb{R}$ and $\lambda \neq \pm 1$. Moreover, we have $\varrho(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a + d \in \mathbb{R}$ and $cb \neq 0$.

If $|\lambda + \lambda^{-1}| > 2$ we have $\lambda \in \mathbb{R}$. Under the assumptions of the lemma we obtain $\chi_\varrho(hg) = \lambda a + \lambda^{-1} d \in \mathbb{R}$. This together with $a + d \in \mathbb{R}$ implies that $a, d \in \mathbb{R}$.

Let $X_\mu \in \mathrm{SL}_2(\mathbb{C})$ be the matrix $X_\mu = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$ where $\mu \in \mathbb{C}^*$. It is easy to see that

$$X_\mu \varrho(g) X_\mu^{-1} = \begin{pmatrix} a & \mu^2 b \\ \mu^{-2} c & d \end{pmatrix}.$$

By choosing $\mu^2 = b^{-1}$ we obtain $\varrho \sim \varrho'$ where

$$\varrho'(h) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad \varrho'(g) = \begin{pmatrix} a & 1 \\ c' & d \end{pmatrix}$$

with $c' \neq 0$. But $\varrho'(g) \in \mathrm{SL}_2(\mathbb{C})$ and therefore $c' \in \mathbb{R}$.

Now, let $f \in G$ be given and assume $\varrho'(f) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Again, $\alpha + \delta \in \mathbb{R}$ and $\chi_{\varrho'}(fh) \in \mathbb{R}$ which gives $\alpha, \delta \in \mathbb{R}$. Consider

$$\varrho'(fg) = \begin{pmatrix} \alpha a + \beta c' & * \\ * & \gamma + d\delta \end{pmatrix}.$$

By use of $\chi_{\varrho'}(fg) \in \mathbb{R}$ and $\chi_{\varrho'}(hfg) \in \mathbb{R}$ we obtain $\alpha a + \beta c' \in \mathbb{R}$ and $\gamma + d\delta \in \mathbb{R}$. As a result we get $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$; remember $a, d, c' \in \mathbb{R}$ and $c' \neq 0$. Therefore, we have $\varrho'(G) \subset \text{SL}_2(\mathbb{R})$.

If $|\lambda + \lambda^{-1}| < 2$ we have $\pm 1 \neq \lambda \in S^1 \subset \mathbb{C}$, i.e. $\varrho'(h) = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$. Again we have $\varrho(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $cb \neq 0$ and $a + d \in \mathbb{R}$. Therefore, $d = \bar{a} + h$ where $h \in \mathbb{R}$. Moreover, $\chi_{\varrho}(gh) = a\lambda + d\bar{\lambda} = a\lambda + \bar{a}\bar{\lambda} + \bar{\lambda}h \in \mathbb{R}$ and as a result we obtain $h = 0$ and $d = \bar{a}$ (see [9]).

Choose $\mu \in \mathbb{C}^*$ such that $|\mu^2 b| = |\mu^{-2} c|$. Let $\varrho'(f) := X_{\mu} \varrho(f) X_{\mu}^{-1}$ for all $f \in G$. We obtain:

$$\varrho \sim \varrho' \quad \text{such that} \quad \varrho'(h) = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \quad \text{and} \quad \varrho'(g) = \begin{pmatrix} a & \beta \\ \gamma & \bar{a} \end{pmatrix}$$

where $\beta\gamma \neq 0$ and $|\beta| = |\gamma|$. The fact that $\varrho'(g) \in \text{SL}_2(\mathbb{C})$ ($\beta\gamma = 1 - a\bar{a}$) implies $\gamma = -\bar{\beta}$ or $\gamma = \bar{\beta}$. Analogous to the case $|\chi_{\varrho}(h)| > 2$ it is easy to see that $\varrho'(G) \subset \text{SU}_2(\mathbb{C})$ if $\gamma = -\bar{\beta}$ and

$$\varrho'(G) \subset \mathcal{S} := \left\{ \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} \mid a\bar{a} - b\bar{b} = 1 \right\} \quad \text{if } \gamma = \bar{\beta}.$$

Let $X \in \text{SL}_2(\mathbb{C})$ be defined as $X := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. We get

$$\text{SL}_2(\mathbb{R}) = X\mathcal{S}X^{-1}$$

and therefore ϱ' is conjugate to a representation $\varrho'' : G \rightarrow \text{SL}_2(\mathbb{R})$. □

Again, let $\mathfrak{G} \in \{\text{SL}_2(\mathbb{C}), \text{SU}_2(\mathbb{C})\}$, and let \mathfrak{g} be its Lie algebra. The Lie group \mathfrak{G} acts on itself by conjugation and the differential of that action at the unit element $\mathbf{1} \in \mathfrak{G}$ defines the adjoint representation $\text{Ad} : \mathfrak{G} \rightarrow \text{Aut } \mathfrak{g}$. In other words, given $A \in \mathfrak{G}$ we have a map $c_A : \mathfrak{G} \rightarrow \mathfrak{G}$ defined by $c_A : B \mapsto ABA^{-1}$ and $\text{Ad}(A)(X) := d_{\mathbf{1}}(c_A)(X)$ for all $X \in \mathfrak{g}$.

Therefore, given a representation $\varrho : G \rightarrow \mathfrak{G}$ the Lie algebra turns into a G -module via $\text{Ad} \circ \varrho$, i.e.

$$gX := \text{Ad}(\varrho(g))(X) \quad \text{for all } g \in G \text{ and } X \in \mathfrak{g}.$$

We denote by \mathfrak{g}_{ϱ} the G -module \mathfrak{g} via $\text{Ad} \circ \varrho$. Let $B^1(G, \mathfrak{g}_{\varrho})$ (resp. $Z^1(G, \mathfrak{g}_{\varrho})$) (resp. $H^1(G, \mathfrak{g}_{\varrho})$) be the coboundaries (resp. cocycles) (resp. first cohomology group) of G with coefficients in \mathfrak{g}_{ϱ} .

Let $T_{\varrho}^{\text{Zar}}(\mathcal{R}(G, \mathfrak{G}))$ be the scheme-theoretic Zariski tangent space to $\mathcal{R}(G, \mathfrak{G})$ at a representation ϱ (see [8] and [10]). Following A. Weil (see [11]) $T_{\varrho}^{\text{Zar}}(\mathcal{R}(G, \mathfrak{G}))$ is isomorphic to the space of group 1-cocycles, i.e.

$$T_{\varrho}^{\text{Zar}}(\mathcal{R}(G, \mathfrak{G})) \cong Z^1(G, \mathfrak{g}_{\varrho}).$$

Let

$$\mathfrak{G}_{\varrho} := \{ \varrho' \in \mathcal{R}(G, \mathfrak{G}) \mid \exists A \in \mathfrak{G} \text{ such that } \forall g \in G, \varrho'(g) = A\varrho(g)A^{-1} \}$$

denote the orbit of ϱ under the action of \mathfrak{G} . We obtain (see [11]) $T_{\varrho}^{\text{Zar}}(\mathfrak{G}_{\varrho}) \cong B^1(G, \mathfrak{g}_{\varrho})$ and therefore we can make the identification:

$$(1) \quad T_{[\varrho]}^{\text{Zar}}(\mathcal{R}(G, \mathfrak{G}) / \sim) \cong H^1(G, \mathfrak{g}_{\varrho}).$$

We denote by \mathfrak{sl} (resp. \mathfrak{su}) the Lie algebra of $SL_2(\mathbb{C})$ (resp. $SU_2(\mathbb{C})$). In fact we obtain the complex 3-dimensional Lie algebra \mathfrak{sl} by tensoring the real 3-dimensional Lie algebra \mathfrak{su} with \mathbb{C} , i.e. there is an isomorphism

$$(2) \quad \mathfrak{su} \otimes \mathbb{C} \cong \mathfrak{sl} \quad \text{given by} \quad X \otimes a \mapsto Xa.$$

Now, given $\varrho : G \rightarrow SU_2(\mathbb{C}) \subset SL_2(\mathbb{C})$ the algebras \mathfrak{su} and \mathfrak{sl} turn into G -modules via ϱ and from (2) we obtain

$$(3) \quad H^1(G, \mathfrak{su}_\varrho) \otimes \mathbb{C} \cong H^1(G, \mathfrak{sl}_\varrho).$$

Let $k \subset S^3$ be given and let $G := \pi_1(C)$ where C is the complement of a regular neighborhood of the knot k . Let $\varrho : G \rightarrow SU_2(\mathbb{C}) \hookrightarrow SL_2(\mathbb{C})$ be a non-abelian representation. The representation $\varrho : G \rightarrow SL_2(\mathbb{C})$ is irreducible and $\varrho(\text{Im}(\pi_1(\partial C) \rightarrow \pi_1(C))) \not\subset \{\pm \mathbf{1}\}$. By a theorem of Thurston (see [5, Proposition 3.2.1]) $t(\varrho) \in X(G)$ is contained in a component $X_0 \subset X(G)$ of complex dimension at least one.

Lemma 2. *Let $\varrho : G \rightarrow SU_2(\mathbb{C})$ be non-abelian and assume that $\dim_{\mathbb{R}} H^1(G, \mathfrak{su}_\varrho) = 1$. Then $t(\varrho) \in X(G)$ is a non-singular point and it is contained in a one-dimensional component of $X(G)$.*

Proof. Let X_0 , where $\dim_{\mathbb{C}} X_0 \geq 1$, be an algebraic component of $X(G)$ which contains $t(\varrho)$. Using (1) and (3) we get

$$(4) \quad 1 \leq \dim_{\mathbb{C}} X_0 \leq \dim_{\mathbb{C}} T_{t(\varrho)}^{\text{Zar}}(X_0) \leq \dim_{\mathbb{C}} T_{t(\varrho)}^{\text{Zar}}(X(G)) \\ = \dim_{\mathbb{C}} H^1(G, \mathfrak{sl}_\varrho) = \dim_{\mathbb{R}} H^1(G, \mathfrak{su}_\varrho) = 1.$$

This results in $1 = \dim_{\mathbb{C}} X_0 = \dim T_{t(\varrho)}^{\text{Zar}}(X(G))$ and therefore $t(\varrho) \in X(G)$ is a regular point. □

Let τ be the mapping that associates with every point $x \in \mathbb{C}^N$ the point $\tau(x)$ with complex conjugate coordinates. Let $\mathcal{C} \in \mathbb{C}^N$ be a complex algebraic curve; we denote by $\mathcal{C}_{\mathbb{R}}$ the set of real points of \mathcal{C} , i.e. $\mathcal{C}_{\mathbb{R}} := \{x \in \mathcal{C} | \tau(x) = x\}$.

Lemma 3. *Let $\mathcal{C} \subset \mathbb{C}^N$ be an affine algebraic curve which is invariant under complex conjugation. Let $x \in \mathcal{C}_{\mathbb{R}} \subset \mathcal{C}$ be a regular point of \mathcal{C} . Then x has a neighborhood in $\mathcal{C}_{\mathbb{R}}$ which is diffeomorphic to \mathbb{R} .*

Proof. Let $\tau : \mathcal{C}_r \rightarrow \mathcal{C}_r$ denote complex conjugation, where $\mathcal{C}_r \subset \mathcal{C}$ is the manifold of regular points of \mathcal{C} . Then τ generates a smooth action of \mathbb{Z}_2 on the manifold \mathcal{C}_r , of which x is a fixed point.

It is a standard result that x has an invariant neighborhood U on which this action is smoothly equivalent to the linear action of \mathbb{Z}_2 on $T_x(\mathcal{C}_r)$ generated by the derivative $d_x(\tau)$. (To prove this, one first chooses a \mathbb{Z}_2 -invariant Riemannian metric and then uses the exponential map at x to relate the two \mathbb{Z}_2 -actions.)

It remains only to verify that if $L \subset \mathbb{C}^N$ (in our case, $L = T_x(\mathcal{C}_r)$) is a one-dimensional complex linear subspace invariant under τ , then the points on L which are fixed by τ consist precisely of a real line through the origin. We leave this easy exercise to the reader. □

Let $\varrho_0 : G \rightarrow SU_2(\mathbb{C})$ be a non-abelian representation. Assume that $t(\varrho_0) \in X(G)$ is a regular point which is contained in a one-dimensional irreducible component X_0 of $X(G)$. Since $X(G)$ is defined over \mathbb{Q} (see [4, 2.3]) it is invariant under complex conjugation. Therefore, Lemma 3 implies that there is a smooth arc

$\chi_s : \mathbb{R} \rightarrow X_0$, $s \in \mathbb{R}$, of characters such that $\chi_0 = t(\varrho_0)$ and $\chi_s(G) \subset \mathbb{R}$ for all $s \in \mathbb{R}$. Moreover, there exists $\varepsilon > 0$ such that if $|s| < \varepsilon$, the character χ_s corresponds to an irreducible representation.

Now, let $E \subset \mathcal{R}(G, \text{SL}_2(\mathbb{C}))$ be the set of all irreducible representations. The restriction $t|_E : E \rightarrow t(E) \subset X(G)$ is a fiber bundle with fiber $\text{PSL}_2(\mathbb{C})$. Hence any map $\mathbb{R} \rightarrow t(E) \subset X(G)$ can be lifted (since \mathbb{R} is contractible).

This implies that there is a smooth arc $\varrho_s : G \rightarrow \text{SL}_2(\mathbb{C})$, $s \in \mathbb{R}$, such that $t(\varrho_s) = \chi_s$ for all $s \in \mathbb{R}$. In order to get an arc $\varrho_s : G \rightarrow \text{SU}_2(\mathbb{C})$ we need the following lemma.

Lemma 4. *Let $\varrho_s : G \rightarrow \text{SL}_2(\mathbb{C})$, $s \in \mathbb{R}$, be a smooth arc of representations such that $\varrho_0 : G \rightarrow \text{SL}_2(\mathbb{R})$ (resp. $\text{SU}_2(\mathbb{C})$) is irreducible and $\chi_{\varrho_s}(G) \subset \mathbb{R}$ for all $s \in \mathbb{R}$. Then there exists an $\varepsilon > 0$ and a smooth arc $A_s \in \text{SL}_2(\mathbb{C})$ such that for each s , $|s| < \varepsilon$, $\varrho' : G \rightarrow \text{SL}_2(\mathbb{R})$ (resp. $\text{SU}_2(\mathbb{C})$) where $\varrho'(g) = A_s \varrho(g) A_s^{-1}$.*

Proof. As in the proof of Lemma 1 we choose $g, h \in G$ such that $\varrho_0(g) \neq \pm 1$, $\chi_{\varrho_0}(h) \neq \pm 2$ and ϱ_0 restricted to the subgroup generated by g and h is irreducible. Each of the conditions above is an open condition, i.e. we can assume that there is an $\varepsilon > 0$ such that the $\varrho_s(g) \neq \pm 1$, $\chi_{\varrho_s}(h) \neq \pm 2$ and ϱ_s restricted to the subgroup generated by g and h is irreducible.

For that reason, there is a smooth arc of matrices $A_s \in \text{SL}_2(\mathbb{C})$ such that

$$A_s \varrho_s(h) A_s^{-1} = \begin{pmatrix} \lambda_s & 0 \\ 0 & \lambda_s^{-1} \end{pmatrix} \quad \text{and} \quad A_s \varrho_s(g) A_s^{-1} = \begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix}$$

where $\lambda_s \neq \pm 1$ and $b_s c_s \neq 0$ for every $|s| < \varepsilon$.

By following the proof of Lemma 1 we can find a smooth arc $\varrho'_s : G \rightarrow \text{SL}_2(\mathbb{R})$ (resp. $\text{SU}_2(\mathbb{C})$) such that $\varrho_s \sim \varrho'_s$ for all $|s| < \varepsilon$. □

Proposition 1. *Let G be a knot group and $\varrho_0 : G \rightarrow \text{SU}_2(\mathbb{C})$ a non-abelian representation. If $\dim_{\mathbb{R}} H^1(G, \mathfrak{su}_{\varrho_0}) = 1$, then a neighborhood of $[\varrho_0]$ in $\mathfrak{R}(G)$ is a one-dimensional manifold, i.e. there is a neighborhood of $[\varrho_0] \in \mathfrak{R}(G)$ which is diffeomorphic to an open interval.*

Proof. By Lemma 2, $t(\varrho_0)$ is a regular point of a one-dimensional component X_0 of $X(G)$. Lemma 3 implies that there is a smooth arc of non-conjugate representations $\varrho_s : G \rightarrow \text{SL}_2(\mathbb{C})$ through ϱ_0 such that $\chi_{\varrho_s}(G) \subset \mathbb{R}$, and by Lemma 4 we may assume that $\varrho_s : G \rightarrow \text{SU}_2(\mathbb{C})$. Therefore, we get

$$1 = \dim_{\mathbb{R}}(X_0)_{\mathbb{R}} \leq \dim_{\mathbb{R}} T_{[\varrho_0]}^{\text{Zar}}(\mathfrak{R}(G)) = \dim_{\mathbb{R}} H^1(G, \mathfrak{su}_{\varrho_0}) = 1.$$

Consequently there is neighborhood $U = U(t(\varrho_0)) \subset X_0$ such that $(X_0)_{\mathbb{R}} \cap U = \hat{t}(\mathfrak{R}(G)) \cap U$. So \hat{t} is a diffeomorphism from a neighborhood of $[\varrho_0]$ in $\mathfrak{R}(G)$ to a smooth real arc in X_0 . □

3. DEFORMING BINARY DIHEDRAL REPRESENTATIONS

During this section it is more convenient to work with unit quaternions. Therefore, we identify $\text{SU}_2(\mathbb{C})$ with S^3 (see [8]). The Lie algebra of S^3 is the set \mathbb{E} of pure quaternions and S^3 acts via Ad, i.e. $\text{Ad}(q)X = qXq^{-1}$ for $q \in S^3$ and $X \in \mathbb{E}$.

The binary dihedral group $N \subset S^3$ is given by $N = S_A^1 \cup S_B^1$ where $S_A^1 := \{a + b\mathbf{i} \mid a, b \in \mathbb{R}, a^2 + b^2 = 1\}$ and $S_B^1 := \{c\mathbf{j} + d\mathbf{k} \mid c, d \in \mathbb{R}, c^2 + d^2 = 1\}$. The set $S_A^1 \subset N$ is a subgroup of index 2 and hence normal (see [8]). Each element of S_B^1 can be expressed as

$$e^{i\theta} \mathbf{j} = \cos \theta \mathbf{j} + \sin \theta \mathbf{k}.$$

Let

$$(5) \quad G = \langle S_1, \dots, S_n | S_{j_l}^{\epsilon_l} S_l S_{j_l}^{-\epsilon_l} = S_{l+1}, \quad l = 1, \dots, n-1 \rangle, \quad \epsilon_l \in \{\pm 1\},$$

be a Wirtinger presentation of G .

Every non-abelian representation $\varrho : G \rightarrow N \subset S^3$ is given by

$$(6) \quad \varrho : S_l \mapsto e^{i\theta_l} \mathbf{j}, \quad 1 \leq l \leq n,$$

where the $\zeta_l := e^{i\theta_l}$ satisfy $\zeta_{j_l}^2 = \zeta_l \zeta_{l+1}$.

Remark 3. It is implicit from [8] and [3] that all binary dihedral representations of a knot group have finite image. Furthermore if there exists a binary dihedral representation, it is easy to see there exists one of order $4p$ for some $p \in \mathbb{N}$ prime. By [3, Chapter 13] there exists a binary dihedral representation of order $4p$ if and only if $H_1(\hat{C}_2, \mathbb{Z}_p) \neq 0$ which is equivalent to the condition $p|\Delta(-1)$.

We would like to use Proposition 1 in order to deform binary dihedral representations of order $4p$ where p is a prime number. Let $\varrho : G \rightarrow N$ be such a binary dihedral representation. By Proposition 1 it is enough to prove that $\dim H^1(G, \mathfrak{su}_\varrho) = 1$.

Remark 4. Following an argument given by Hodgson (for details see [7]) we have $\dim H^1(G, \mathfrak{su}_\varrho) \geq 1$. Namely, let C be the complement of k in S^3 ; we have $H^i(G, \mathfrak{su}_\varrho) \cong H^i(C, \mathfrak{su}_\varrho) =: H_\varrho^i(C)$ for $i = 0, 1$. The long exact sequence in cohomology gives:

$$H_\varrho^1(C) \xrightarrow{\alpha} H_\varrho^1(\partial C) \xrightarrow{\beta} H_\varrho^2(C, \partial C).$$

We have $\dim H_\varrho^1(\partial C) = \dim H^1(\mathbb{Z} \oplus \mathbb{Z}, \mathfrak{su}_\varrho) = 2$ and $\dim \text{Im}(\beta) + \dim \ker(\beta) = \dim H_\varrho^1(\partial C)$. Moreover, α and β are dual maps via Poincaré duality. Hence, $\dim \text{Im}(\alpha) = \dim \text{Im}(\beta)$ and by $\text{Im}(\alpha) = \ker(\beta)$ we obtain

$$2 \dim \text{Im}(\alpha) = 2 \Rightarrow \dim H_\varrho^1(C) \geq 1.$$

Remark 5. The fact that $\dim H^1(G, \mathfrak{su}_\varrho) = 1$ if $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$ is implicitly contained in [3, Proposition 14.10]. Nevertheless, to make this paper more complete we include the proof here.

Since ϱ is non-abelian it is irreducible and the orbit $\text{SU}_2(\mathbb{C})_\varrho$ is 3-dimensional. By definition of the coboundary operator δ^0 (see [1]) we get $\dim B^1(G, \mathfrak{su}_\varrho) = 3$ because $\text{Ad}(\varrho(G)) \subset \text{Aut } \mathfrak{su}$ has no non-trivial fixed vector. Ergo,

$$\dim H^1(G, \mathfrak{su}_\varrho) = 1 \Leftrightarrow \dim Z^1(G, \mathfrak{su}_\varrho) = 4.$$

Remark 6. According to [8, Proposition 18] we are able to calculate $Z^1(G, \mathfrak{su}_\varrho)$ as follows. Specifically, each relator

$$S_{j_l}^{\epsilon_l} S_l S_{j_l}^{-\epsilon_l} = S_{l+1}$$

gives the equations:

$$(7) \quad (1 - \text{Ad} \circ \varrho(S_{i+1}))X_{j_i} + \text{Ad} \circ \varrho(S_{j_i})X_i - X_{i+1} = 0 \quad \text{if } \epsilon_{j_i} = 1,$$

$$(8) \quad (1 - \text{Ad} \circ \varrho(S_i))X_{j_i} + \text{Ad} \circ \varrho(S_{j_i})X_{i+1} - X_i = 0 \quad \text{if } \epsilon_{j_i} = -1.$$

Altogether, we obtain a system of linear equations over \mathbb{E} and

$$Z^1(G, \mathfrak{su}_\varrho) \cong \{(X_1, \dots, X_n) \in (\mathbb{E})^n | (X_1, \dots, X_n) \text{ is a solution of the system}\}.$$

By choosing the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ of \mathbb{E} we obtain that

$$\text{Ad}(\mathbf{j}) \text{ acts as } \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$\text{Ad}(e^{\theta \mathbf{i}}) \text{ acts as } \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix}.$$

By setting $X_i = x_i \mathbf{i} + y_i \mathbf{j} + z_i \mathbf{k}$ and defining

$$u_i := \begin{pmatrix} \cos 2\theta_i & -\sin 2\theta_i \\ \sin 2\theta_i & \cos 2\theta_i \end{pmatrix} \text{ and } t := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we obtain from (7) and (8) the following system of equations:

$$(9) \quad 2x_{j_i} - x_i - x_{i+1} = 0, \quad 1 \leq i \leq n-1,$$

and

$$(10) \quad (\mathbf{1} - u_{i+1}t) \begin{pmatrix} y_{j_i} \\ z_{j_i} \end{pmatrix} + u_{j_i}t \begin{pmatrix} y_i \\ z_i \end{pmatrix} - \begin{pmatrix} y_{i+1} \\ z_{i+1} \end{pmatrix} = 0 \text{ if } \epsilon_i = 1,$$

$$(11) \quad (\mathbf{1} - u_it) \begin{pmatrix} y_{j_i} \\ z_{j_i} \end{pmatrix} + u_{j_i}t \begin{pmatrix} y_{i+1} \\ z_{i+1} \end{pmatrix} - \begin{pmatrix} y_i \\ z_i \end{pmatrix} = 0 \text{ if } \epsilon_i = -1.$$

Let $J(t)$ be the $n \times (n-1)$ Jacobian obtained from the Wirtinger presentation (5) of G (see [3, Chapter 9]). A presentation matrix $A(t)$ for the Alexander module is obtained from any $(n-1) \times (n-1)$ minor of $J(t)$. Equation (9) is equivalent to

$$J(-1) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0.$$

This system of equations has only the solution $x_i = x \in \mathbb{R}$ for all $1 \leq i \leq n$ because $|\det A(-1)| = |\Delta(-1)|$ is an odd integer.

In order to solve the equations (10) and (11) we introduce the following complex notation:

$$b_j := y_j + z_j \mathbf{i}; \quad t : \mathbb{C} \rightarrow \mathbb{C}, t : z \mapsto \bar{z} \quad \text{and} \quad u_i : \mathbb{C} \rightarrow \mathbb{C}, u_i : z \mapsto \zeta_i^2 z,$$

where $\zeta_i = e^{i\theta_i}$. Then equation (10) is equivalent to

$$(12) \quad b_{j_i} - \zeta_{i+1}^2 \bar{b}_{j_i} + \zeta_{j_i}^2 \bar{b}_i - b_{i+1} = 0 \text{ if } \epsilon_i = 1$$

and equation (11) transforms into

$$(13) \quad b_{j_i} - \zeta_i^2 \bar{b}_{j_i} + \zeta_{j_i}^2 \bar{b}_{i+1} - b_i = 0 \text{ if } \epsilon_i = -1.$$

Define new variables t_j and s_j in \mathbb{R} by $b_j = \zeta_j(t_j + s_j \mathbf{i})$. Substituting in (12) (and (13)) yields

$$(\zeta_{j_i} + \zeta_{i+1}^2 \bar{\zeta}_{j_i})s_{j_i} \mathbf{i} - \zeta_{j_i}^2 \bar{\zeta}_i s_i \mathbf{i} - \zeta_{i+1} s_{i+1} \mathbf{i} + (\zeta_{j_i} - \zeta_{i+1}^2 \bar{\zeta}_{j_i})t_{j_i} + \zeta_{j_i}^2 \bar{\zeta}_i t_i - \zeta_{i+1} t_{i+1} = 0$$

if $\epsilon_{j_i} = 1$ and an analogous equation if $\epsilon_{j_i} = -1$. Now, multiplication of this equation with $-\bar{\zeta}_{i+1} \mathbf{i}$ if $\epsilon_{j_i} = 1$ (and multiplication of the analogous equation with $-\bar{\zeta}_i \mathbf{i}$ if $\epsilon_{j_i} = -1$) gives

$$(\bar{\zeta}_{j_i} \zeta_i + \bar{\zeta}_i \zeta_{j_i})s_{j_i} - s_i - s_{i+1} - \epsilon_{j_i} t_{j_i} (\bar{\zeta}_{j_i} \zeta_i - \zeta_{j_i} \bar{\zeta}_i) \mathbf{i} - \epsilon_{j_i} (t_i - t_{i+1}) \mathbf{i} = 0$$

(use the equation $\zeta_{j_i}^2 = \zeta_i \zeta_{i+1}$). Note that $\bar{\zeta}_{j_i} \zeta_i + \bar{\zeta}_i \zeta_{j_i} \in \mathbb{R}$ and $(\bar{\zeta}_{j_i} \zeta_i - \zeta_{j_i} \bar{\zeta}_i) \mathbf{i} \in \mathbb{R}$. The imaginary part yields $t_i =: t \in \mathbb{R}$ for $1 \leq i \leq n$ and from the real part we obtain the following system of linear equations over \mathbb{R} :

$$(14) \quad \alpha(j_i, i) s_{j_i} - s_i - s_{i+1} - \epsilon_{j_i} \beta(j_i, i) t = 0$$

where $\alpha(j_i, i) := (\bar{\zeta}_{j_i} \zeta_i + \bar{\zeta}_i \zeta_{j_i})$ and $\beta(j_i, i) := (\bar{\zeta}_{j_i} \zeta_i - \bar{\zeta}_i \zeta_{j_i}) \mathbf{i}$ are real numbers.

We summarize the discussion above in the following proposition:

Proposition 2. *Let $k \subset S^3$ be a knot and G its group. Assume there is a non-abelian representation $\rho : G \rightarrow N$ given by (6). (By Remark 3, this assumption is equivalent to $|\Delta(-1)| \neq 1$.)*

The point $[\rho] \in \mathfrak{R}(G)$ has a neighborhood in $\mathfrak{R}(G)$ which is diffeomorphic to an open interval if the system (14) has a three-dimensional solution space (over \mathbb{R}).

We are ready to give a proof of Theorem 1.

Proof of Theorem 1. (This part of the proof uses exactly the argument given in [3, Proposition 14.10].) We would like to give a lower bound for the rank of the linear system

$$(15) \quad \alpha(j_i, i) s_{j_i} - s_i - s_{i+1} = 0.$$

Since a lower bound for the rank of (15) is also a lower bound for the rank of (14) we will then be done.

The system (15) is a linear system of equations with coefficients in $\mathbb{Z}(\zeta)$ where ζ is a primitive p -th root of unity and $\mathbb{Z}(\zeta) \subset \mathbb{Q}(\zeta)$ denotes the ring of algebraic integers. There is a homomorphism

$$\psi_p : \mathbb{Z}(\zeta) \rightarrow \mathbb{Z}_p,$$

given by $\psi_p : \zeta \mapsto [1]_p$ and $\psi_p : n \mapsto [n]_p$ where $[n]_p$ denotes the coset of n modulo p . Now, (15) maps under ψ_p onto the system

$$2s_{j_i} - s_i - s_{i+1} = 0$$

over \mathbb{Z}_p and the coefficient matrix of this system is the matrix $A(-1)$ where the coefficients are reduced mod p . Now, $A(-1)$ is an $(n - 1) \times (n - 1)$ presentation matrix for the first homology group of the double branched cover of (S^3, k) (see [3]). So its rank (mod p) is just $(n - 1) - \dim_{\mathbb{Z}_p} H_1(\hat{C}_2, \mathbb{Z}_p)$.

But $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$ by the assumption. Therefore, we have $\text{rk}_{\mathbb{Z}_p} A(-1) = n - 2$ and hence we obtain that the rank of (14) is at least $n - 2$. □

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