

## THE SLICE GENUS AND THE THURSTON-BENNEQUIN INVARIANT OF A KNOT

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ABSTRACT. For any knot  $K \subset S^3$ ,  $g_s(K) \geq (\text{TB}(K) + 1)/2$ .

Let  $K \subset S^3 = \partial D^4 \subset \mathbf{C}^2 \subset \mathbf{C}\mathbf{P}^2$  be a (smooth) knot. The *slice genus*  $g_s(K)$  is the smallest genus of a smooth, oriented surface in  $D^4$  with boundary  $K$ . If  $K$  is *Legendrian*, that is, everywhere tangent to the field of complex lines tangent to  $S^3$ , then the *Thurston-Bennequin invariant*  $\text{tb}(K)$  is the integer associated in the usual way (using linking numbers in  $S^3$ , with the orientation induced by  $\mathbf{C}^2$ ) to the normal linefield to  $K$  which assigns to each point of  $K$  the tangent line to  $K$  multiplied by  $\sqrt{-1}$ ; cf. [1]. For arbitrary  $K$ , the *maximal Thurston-Bennequin invariant*  $\text{TB}(K)$  is  $\sup\{\text{tb}(K') : K' \subset S^3 \text{ is a Legendrian knot ambient isotopic to } K\}$ .

Appealing to a theorem of Donaldson, I showed [4] that if  $g_s(K) = 0$ , then  $\text{TB}(K) \leq -1$ . The following inequality is more general; the proof requires more than Donaldson's theorem, namely, the truth of the local Thom Conjecture [2].

**Theorem.**  $g_s(K) \geq (\text{TB}(K) + 1)/2$ .

*Proof.* Using the construction of *entire totally-tangential C-links* given in [4], and elementary rational approximation theory, it is easily seen that, for any integer  $t \leq \text{TB}(K)$ , there is a rational immersion  $F : \mathbf{C}\mathbf{P}^1 \rightarrow \mathbf{C}\mathbf{P}^2$ , in general position, such that  $A = F^{-1}(D^4) \subset \mathbf{C} \subset \mathbf{C}\mathbf{P}^1$  is a neighborhood of  $S^1$  diffeomorphic to an annulus,  $F|_A$  is an embedding, and  $\partial F(A) = \partial A(K, t) \subset S^3$ , where  $\partial A(K, t) \subset S^3$  denotes an (embedded) annulus of type  $K$  with  $t$  twists (i.e., each component of  $\partial A(K, t)$  is ambient isotopic to  $K$ , and the linking number of the components is  $-t$ ).

Let  $D_i \subset \mathbf{C}\mathbf{P}^1$ ,  $i = 1, 2$ , be the two 2-disks complementary to  $\text{Int } A$ . Let the smoothly immersed 2-disk  $F(D_i) \subset \mathbf{C}\mathbf{P}^2 \setminus \text{Int } D^4$  have  $d_i$  doublepoints; let  $S_i \subset \mathbf{C}\mathbf{P}^2$  be a smoothly immersed surface which is the union along a component of  $\partial A(K, t)$  of  $F(D_i)$  and a suitable smoothly embedded surface in  $D^4$  of genus  $g_s(K)$ . Let  $n_i$  be the degree of  $S_i$ . Then the degree of  $F(\mathbf{C}\mathbf{P}^1)$  is  $n_1 + n_2$ , and, by the local Thom Conjecture,  $g_s(K) + d_i \geq (n_i - 1)(n_i - 2)/2$ .

Let  $p$  be the intersection number of the relative cycles  $F(D_1)$  and  $F(D_2)$  in  $\mathbf{C}\mathbf{P}^2 \setminus \text{Int } D^4$ . Then  $p - t = n_1 n_2$  is the intersection number of  $S_1$  and  $S_2$  in  $\mathbf{C}\mathbf{P}^2$ , and  $p + d_1 + d_2 = (n_1 + n_2 - 1)(n_1 + n_2 - 2)/2$ .

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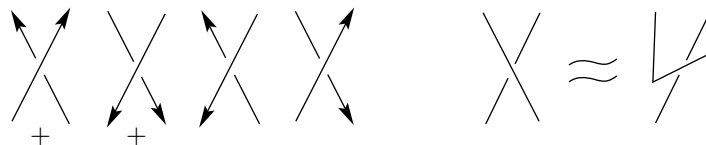


FIGURE 1. Left, global signs of the four types of crossings with local sign  $+$ . Right, changing a local sign from  $-$  to  $+$ .

By combining these equations and inequalities, we find that  $2g_s(K) - 1 \geq t$ .  $\square$

*Remarks.* (1) An advantage of the Theorem over the slice-Bennequin inequality is that a lower bound (however crude) for  $\text{TB}(K)$ , and thus for  $g_s(K)$ , can be read off from any (smooth or piecewise-linear) knot diagram  $\mathbf{D}(K) \subset \mathbf{R}^2$  for  $K$ , without a need for preliminary braiding. The *global sign* of a doublepoint  $\mathbf{P}$  of  $\mathbf{D}(K)$  is its sign ( $+$  or  $-$ ) in the usual sense (cf. Figure 1); the *writhe*  $w(\mathbf{D}(K))$  is the number of doublepoints of global sign  $+$ , less the number of doublepoints of global sign  $-$ . A linear form  $\ell : \mathbf{R}^2 \rightarrow \mathbf{R}$  with  $2m_\ell(\mathbf{D}(K)) < \infty$  local extrema on  $\mathbf{D}(K)$ , of which none is a doublepoint, assigns a *local sign*  $s_\ell(\mathbf{P}) \in \{+, -\}$  to each doublepoint  $\mathbf{P}$ . If  $s_\ell(\mathbf{P}) = +$  for all  $\mathbf{P}$ , then  $\mathbf{D}(K)$  is piecewise-smoothly isotopic to a *front* [4] and, as described in [4],  $\text{TB}(K) \geq w(\mathbf{D}(K)) - m_\ell$ ; in general (cf. Figure 1),  $\text{TB}(K) \geq w(\mathbf{D}(K)) - (m_\ell + \text{card } s_\ell^{-1}(-))$ .

(2) Of course the genus  $g(K)$  is greater than or equal to  $g_s(K)$ . Is there a direct, elementary, 3-dimensional proof that  $g(K) \geq (w(\mathbf{D}(K)) - m_\ell - \text{card } s_\ell^{-1}(-) + 1)/2$ ?

(3) Since this note was first circulated, Kronheimer and Mrowka [3] have improved the estimate for any Legendrian knot  $K$  with  $\text{TB}(K) > 0$ ; by using the adjunction inequality rather than just the local Thom Conjecture, they show that for any such  $K$ ,  $g_s(K) \geq (\text{tb}(K) + |r(K)| + 1)/2$ , where  $r(K)$  denotes the *rotation number* of  $K$  (cf. [1]). (The restriction that  $\text{TB}(K) > 0$  may be unnecessary.)

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