

COPRIMENESS AMONG IRREDUCIBLE CHARACTER DEGREES OF FINITE SOLVABLE GROUPS

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ABSTRACT. Given a finite solvable group G , we say that G has property P_k if every set of k distinct irreducible character degrees of G is (setwise) relatively prime. Let $k(G)$ be the smallest positive integer such that G satisfies property P_k . We derive a bound, which is quadratic in $k(G)$, for the total number of irreducible character degrees of G . Three exceptional cases occur; examples are constructed which verify the sharpness of the bound in each of these special cases.

1. INTRODUCTION

Suppose G is a finite solvable group and let $\text{cd}(G)$ denote the set $\{\chi(1) \mid \chi \in \text{Irr}(G)\}$. We say that G has property P_k if every set of k distinct elements of $\text{cd}(G)$ is (setwise) relatively prime. Every finite group G satisfies P_k at least for $k \geq |\text{cd}(G)|$, since $1 \in \text{cd}(G)$. The main result of this paper is the following:

Theorem A. *Let G be a nonabelian finite solvable group and let k be the smallest positive integer such that G satisfies property P_k . Then*

$$|\text{cd}(G)| \leq \begin{cases} 3 & \text{if } k = 2; \\ 6 & \text{if } k = 3; \\ 9 & \text{if } k = 4; \\ k^2 - 3k + 4 & \text{if } k \geq 5. \end{cases}$$

Following the proof of Theorem A, a collection of examples is presented. In each of the exceptional cases $k = 2, 3, 4$ the bound is attained. For $k = 2$, an example is provided by the group $SL(2, 3)$. This group satisfies P_2 and has 3 irreducible character degrees: $\text{cd}(SL(2, 3)) = \{1, 2, 3\}$. For $k = 3$, we construct a group Γ with $\text{cd}(\Gamma) = \{1, r, s, rs, q^4, q^5\}$, where q, r, s are any three primes satisfying $q \equiv 3 \pmod{4}$ and $q \equiv 1 \pmod{rs}$. The group Γ attains the bound in this case. Next, for infinitely many values of k , we construct a group which satisfies property P_k and has $3(k - 1)$ irreducible character degrees. Observe that, for $k = 4$, such a group satisfies P_4 and has 9 irreducible character degrees, verifying the sharpness of the bound in this case. It also follows from this infinite set of examples that the best possible bound for $|\text{cd}(G)|$ in terms of k cannot be better than the linear bound $3(k - 1)$.

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While this result belongs to a genre of problems and results concerning the irreducible character degrees of finite solvable groups (see §2 of [4]), it has a unique flavor. The investigation of property P_k was inspired by problem 12.3 of [1] which is, in fact, the $k = 2$ case of the result. At this point the author would like to express her appreciation to Professor Martin Isaacs for his direction and encouragement in this work, which is a portion of her thesis.

2. PRELIMINARIES

The purpose of this section is to restate facts about the structure and character degrees of a factor group G/K of a finite nonabelian solvable group G with K chosen to be maximal such that G/K remains nonabelian. Notice that, in this situation every proper factor group of G/K is abelian and thus $(G/K)'$ is the unique minimal normal subgroup of G/K .

(2.1) Lemma. *Let G be a finite solvable group and assume that G' is the unique minimal normal subgroup of G . Then all the nonlinear irreducible characters of G have equal degree f and one of the following situations obtains:*

- (a) G is a p -group, $\mathbf{Z}(G)$ is cyclic and $G/\mathbf{Z}(G)$ is elementary abelian of order f^2 .
- (b) G is a Frobenius group with a cyclic Frobenius complement of order f . Also, G' is the Frobenius kernel and is an elementary abelian p -group.

Proof. This is Lemma 12.3 of [1] with the observation that an abelian Frobenius complement is necessarily cyclic. \square

(2.2) Theorem. *Let $K \triangleleft G$ be such that G/K is a Frobenius group with kernel N/K , an elementary abelian p -group. Let $\psi \in \text{Irr}(N)$. Then one of the following holds:*

- (a) $|G : N| \psi(1) \in \text{cd}(G)$.
- (b) p divides $\psi(1)$.

Proof. This is immediate from Theorem 12.4 of [1]. \square

3. PROOF OF THEOREM A

We begin by proving a key lemma.

(3.1) Lemma. *Let G be a finite nonabelian solvable group with $G' \leq \mathbf{O}^p(G)$ for all primes p . Suppose that $K \triangleleft G$ and K is maximal such that G/K is nonabelian. Then G/K is a Frobenius group with Frobenius kernel N/K , an elementary abelian q -group for some prime q , and a cyclic Frobenius complement. Let f denote the order of the Frobenius complement and assume further that K is chosen so that f is minimal. Then for each linear character λ of N , either λ^G is irreducible or λ extends to G . In particular, if $\chi \in \text{Irr}(G)$ lies over a linear character of N , then χ must have degree 1 or f .*

It will be handy in the proof to use the standard notation $b(G)$ to denote the largest irreducible character degree of G ; that is, the maximum of the set $\text{cd}(G)$.

Proof. By hypothesis, if $M \triangleleft G$ with $K < M$, then the quotient G/M is abelian. Since G is solvable, it follows that $(G/K)'$ is the unique minimal normal subgroup of G/K . Now since G has no nonabelian p -factor groups for any prime p , the Frobenius structure of G/K follows from Lemma 2.1 (b).

Fix a linear character $\lambda \in \text{Irr}(N)$ and let $\chi \in \text{Irr}(G)$ lie over λ . Set $T = I_G(\lambda)$ and $t = |G : T|$. Since T/N is cyclic, λ extends to a character $\hat{\lambda} \in \text{Irr}(T)$ and further, by Corollary 6.17 of [1], every element of $\text{Irr}(T|\lambda)$ is an extension of λ . We may then assume without loss of generality that the extension $\hat{\lambda}$ is the Clifford correspondent between χ and λ . Thus $\chi = (\hat{\lambda})^G$ and $\chi_N = \sum_{i=1}^t \lambda_i$, labeled so that $\lambda_1 = \lambda$. Also note that $\chi(1) = t$.

We are done if $t = 1$ or $t = f$, so assume, for a contradiction, that neither happens. In this case $1 < t = \chi(1) < f$ and $N < T < G$. Let $M = \ker \lambda$. Since T fixes the linear character λ , it follows that T centralizes N/M and so $[T, N] \leq M$. Also $[T, N] \triangleleft G$, since both T and N are normal. Let $\bar{}$ denote quotients mod $[T, N]$. Then \bar{N} is central in \bar{T} and \bar{T}/\bar{N} is cyclic since it is isomorphic to T/N ; thus \bar{T} is abelian. We have \bar{T} is normal and abelian in \bar{G} . By Ito's Theorem (6.15 of [1]) $t = |\bar{G} : \bar{T}| \geq b(\bar{G})$. Also, since $\ker \chi \geq \text{core}_G(\ker \lambda) \geq [T, N]$, we may view χ as an element of $\text{Irr}(\bar{G})$. We have $t = \chi(1) \in \text{cd}(\bar{G})$ and thus \bar{G} is nonabelian.

Now, let \bar{G}/\bar{L} be a minimal nonabelian factor of \bar{G} . Clearly the hypothesis on p -factors of G holds for p -factors of \bar{G} . It follows from Lemma 2.1 that \bar{G}/\bar{L} is a Frobenius group with a Frobenius complement of order $\chi(1) = t < f$. Since factors of \bar{G} are factors of G , this contradicts the minimality of f . \square

It will now be convenient to establish some notation for the proof of Theorem A. For a group G , we define $k(G)$ to be the smallest integer such that G satisfies property P_k . Note that $k(G) \leq |\text{cd}(G)|$ and if G is a q -group, for q prime, then equality holds. Given a positive integer q , we define $\text{cd}_q(G) = \{n \in \text{cd}(G) \mid (q, n) = 1\}$ and $\text{cd}^q(G) = \{n \in \text{cd}(G) \mid q|n\}$. If q is prime, then $\text{cd}(G)$ is the disjoint union of these two sets. Also, for $N \triangleleft G$ with $m \in \text{cd}(N)$ if there exists $\psi \in \text{Irr}(N)$ and $\chi \in \text{Irr}(G|\psi)$ with $\chi(1) = n$ and $\psi(1) = m$, then we will say that n lies over m . Further, for each such m , define a subset of $\text{cd}(G)$ by $s(m) = \{n \in \text{cd}(G) \mid n \text{ lies over } m\}$. Note that a given n may lie over many different m and each element of $s(m)$ is divisible by m .

To prove Theorem A, we bound each of $|\text{cd}^q(G)|$ and $|\text{cd}_q(G)|$ separately in terms of $k(G)$ and add the results. Note that if $k = k(G)$ for a group G , then $|\text{cd}^q(G)| \leq k - 1$ for any positive integer q . On the other hand, given k , examples are available among q -groups, where q is prime, which satisfy $|\text{cd}(G)| = k$. For instance, let Q be the direct product of $k - 1$ copies of a q -group A having $\text{cd}(A) = \{1, q\}$. Then $\text{cd}(Q) = \{1, q, q^2, \dots, q^{k-1}\}$; thus $k = k(Q)$ and $|\text{cd}^q(Q)| = k - 1$. It follows that $k - 1$ is the best possible bound for $|\text{cd}^q(G)|$. Our challenge in proving Theorem A will be to bound $|\text{cd}_q(G)|$.

Proof of Theorem A. Let G be a nonabelian finite solvable group and let $k = k(G)$.

Suppose first that G has a nonabelian p -factor group G/K for some prime p . As we have observed, $|\text{cd}^p(G)| \leq k - 1$. Now we consider $|\text{cd}_p(G)|$. Fix $\psi \in \text{Irr}(G/K)$ with $\psi(1) = p^a > 1$. For each character $\chi \in \text{Irr}(G)$ with $(p, \chi(1)) = 1$ we have $\chi_K \in \text{Irr}(K)$. By Corollary 6.17 of [1], we have $\chi\psi \in \text{Irr}(G)$ with degree $\chi\psi(1)$ divisible by p , since $\chi\psi(1) = \chi(1)p^a$. This gives an injection from $\text{cd}_p(G)$ into $\text{cd}^p(G)$. Thus $|\text{cd}_p(G)| \leq k - 1$ and $|\text{cd}(G)| \leq 2(k - 1)$. In this case the conclusion of the theorem holds. Henceforth we assume that $G' \leq \mathbf{O}^p(G)$ for all primes p .

Now fix $K \triangleleft G$ so that K is maximal with G/K nonabelian. By Lemma 2.1, G/K is a Frobenius group with kernel N/K , an elementary abelian q -group, and with a

cyclic complement H/K of order f . Also $\text{cd}(G/K) = \{1, f\}$. Assume further that K is chosen so that f is minimal. As before, we have $|\text{cd}^q(G)| \leq k - 1$.

To assess $|\text{cd}_q(G)|$ we will examine how many distinct elements of $\text{cd}_q(G)$ lie over each element of $\text{cd}(N)$. If we write $\text{cd}(N) = \text{cd}^q(N) \dot{\cup} \text{cd}_q(N)$, then notice that elements of $\text{cd}^q(G)$ can lie over only elements of $\text{cd}^q(N)$, since $(q, f) = 1$, and elements of $\text{cd}_q(G)$ lie over only elements of $\text{cd}_q(N)$. Also, by Theorem 2.2 (a), for each element $z \in \text{cd}_q(N)$ we must have $fz \in \text{cd}(G)$. This gives an injection from $\text{cd}_q(N)$ into $\text{cd}^f(G)$. Again, by hypothesis, $|\text{cd}^f(G)| \leq k - 1$; thus $|\text{cd}_q(N)| \leq k - 1$. It follows that all the elements of $\text{cd}_q(G)$ lie over the, at most $k - 1$, elements of $\text{cd}_q(N)$.

If $z \in \text{cd}_q(N)$, how many elements of $\text{cd}_q(G)$ can lie over z ? By Lemma 3.1, if $z = 1$, then $s(z) = \{1, f\}$. If $z > 1$, then $|s(z)| \leq k - 1$, since $s(z) \subseteq \text{cd}^z(G) \leq k - 1$, by hypothesis. It follows that $|\text{cd}_q(G)| \leq 2 + (k - 2)(k - 1)$ and thus we have:

$$(*) \quad |\text{cd}(G)| \leq |\text{cd}^q(G)| + |\text{cd}_q(G)| \leq (k - 1) + 2 + (k - 2)(k - 1) = k^2 - 2k + 3.$$

Observe that, when $k = 2$ the bound $(*)$ yields $|\text{cd}(G)| \leq 3$ and when $k = 3$ the bound $(*)$ yields $|\text{cd}(G)| \leq 6$. Thus the first two special cases of Theorem A have been proved. Henceforth we assume that $k \geq 4$ and will improve $(*)$. We continue as before with the Frobenius factor group G/K .

If $|\text{cd}_q(N)| < k - 1$, then each of the, at most $k - 3$, nonlinear character degrees of $\text{cd}_q(N)$ has at most $k - 1$ elements of $\text{cd}_q(G)$ lying over it; thus $|\text{cd}_q(G)| \leq 2 + (k - 3)(k - 1)$. This observation along with our bound on $\text{cd}^q(G)$ yields $|\text{cd}(G)| \leq (k - 1) + 2 + (k - 3)(k - 1) = k^2 - 3k + 4$ and there is nothing further to prove in this case.

We may now assume that $|\text{cd}_q(N)| = k - 1$. In this case $\{fx|x \in \text{cd}_q(N)\}$ is a subset of $\text{cd}(G)$ of size $k - 1$. We will show that $s(z) \subseteq \{z\} \cup \{fx|x \in \text{cd}_q(N)\}$ for each $z \in \text{cd}_q(N)$. Recall that an arbitrary member of $s(z)$ has the form rz , where $r|f$. If $rz \in s(z)$ with $r > 1$, then r divides every member of $\{rz\} \cup \{fx|x \in \text{cd}_q(N)\}$. Since the latter set in this union has size $k - 1$, it follows that $rz \in \{fx|x \in \text{cd}_q(N)\}$ and thus we conclude that $s(z) \subseteq \{z\} \cup \{fx|x \in \text{cd}_q(N)\}$ as claimed. It follows that all the members of $\text{cd}_q(G)$ lie in $\text{cd}_q(N) \cup \{fx|x \in \text{cd}_q(N)\}$; hence $|\text{cd}_q(G)| \leq 2(k - 1)$. Since $|\text{cd}^q(G)| \leq k - 1$, we have $|\text{cd}(G)| \leq 3k - 3$, in this case.

For $k \geq 4$ (and $\mathbf{O}^p(G) = 1$), it follows that $|\text{cd}(G)|$ is bounded by the maximum of the bounds derived in the two preceding paragraphs. That is,

$$|\text{cd}(G)| \leq \max \begin{cases} (k - 1) + 2 + (k - 3)(k - 1) = k^2 - 3k + 4, \\ (k - 1) + 2(k - 1) = 3k - 3. \end{cases}$$

Observe that, for $k = 4$, the second formula yields a maximum of 9, giving $|\text{cd}(G)| \leq 9$. In the cases $k \geq 5$, the maximum is $k^2 - 3k + 4$. Thus Theorem A is proved. \square

In the next section we give constructions which verify the sharpness of the bound in the exceptional cases $k = 3$ and $k = 4$.

4. CONSTRUCTIONS

For any three primes q, r, s satisfying $q \equiv 3 \pmod{4}$ and $q \equiv 1 \pmod{rs}$, we construct a group Γ as the semidirect product of a normal Sylow q -subgroup Q and a cyclic group H of order rs such that $\text{cd}(\Gamma) = \{1, r, s, rs, q^4, q^5\}$. The group Γ satisfies P_3 and has 6 irreducible character degrees; thus providing an example for

the sharpness of the bound in the case $k = 3$. Note that $r = 2, s = 3, q = 7$ satisfy the conditions.

First we construct Q . Let q be prime with $q \equiv 3 \pmod{4}$. Define the group Q of exponent q as follows, where all unspecified commutators are trivial:

$$\begin{aligned} Q &= \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \mid \\ &[x_1, x_2] = [x_3, x_4] = [x_5, x_6] = [x_7, x_8] = [x_9, x_{10}], \\ &[x_1, x_4] = [x_2, x_3] = [x_5, x_8] = [x_6, x_7] \rangle. \end{aligned}$$

A few observations about the group Q are helpful. First, for notational convenience, label $z_1 = [x_1, x_2] = [x_3, x_4] = [x_5, x_6] = [x_7, x_8] = [x_9, x_{10}]$ and $z_2 = [x_1, x_4] = [x_2, x_3] = [x_5, x_8] = [x_6, x_7]$. Notice that $\mathbf{Z}(Q) = Q' = \langle z_1, z_2 \rangle$. From this we see that $|\mathbf{Z}(Q)| = q^2$ and $|Q/\mathbf{Z}(Q)| = q^{10}$. Thus the group Q is q -special of order q^{12} with exponent q . Further, using additive notation, we may view $Q/\mathbf{Z}(Q)$ and $\mathbf{Z}(Q)$ as vector spaces over $GF(q)$ with bases $\{\overline{x_1}, \overline{x_2}, \dots, \overline{x_{10}}\}$ and $\{z_1, z_2\}$, respectively. (Here $\overline{}$ denotes quotient mod $\mathbf{Z}(Q)$.)

What are the degrees of the irreducible characters of Q ? Since Q is q -special with $|Q : \mathbf{Z}(Q)| = q^{10}$ it follows that Q has q^{10} linear characters. Notice that $Q/\langle z_1 \rangle$ is isomorphic to the direct product of an extra-special group of order q^9 having exponent q with $Z_q \times Z_q$. Also $Q/\langle z_2 \rangle$ is an extra-special group of order q^{11} having exponent q . These quotients give some information about $\text{Irr}(Q)$ and about $\text{cd}(Q)$. In particular, we have $\{1, q^4, q^5\} \subseteq \text{cd}(Q)$. In fact, with the assumption $q \equiv 3 \pmod{4}$, we can show that these are the only irreducible character degrees of Q . The following fact is required:

Claim. For each nonlinear character $\theta \in \text{Irr}(Q)$ we have

- (i) $Q/\ker(\theta)$ is an extra-special q -group with center $\mathbf{Z}(\theta)/\ker(\theta)$.
- (ii) $\mathbf{Z}(\theta) \leq \langle z_1, z_2, x_9, x_{10} \rangle$.

In particular, we have $\theta(1)$ is q^4 or q^5 and every automorphism that centralizes $\langle z_1, z_2, x_9, x_{10} \rangle$ fixes θ .

Fix a nonlinear character $\theta \in \text{Irr}(Q)$. Since $Q' = \mathbf{Z}(Q)$ we have $\mathbf{Z}(Q) \not\leq \ker(\theta)$; also $\mathbf{Z}(Q) \cdot \ker(\theta) \leq \mathbf{Z}(\theta)$. The nontrivial group $\mathbf{Z}(\theta)/\ker(\theta)$ is cyclic of exponent q ; thus it has order q . Further $\mathbf{Z}(\theta) = \ker(\theta) \cdot \mathbf{Z}(Q)$. It follows that $Q/\ker(\theta)$ has a center of order q and that the factor group modulo the center is elementary q -abelian. Thus part (i) holds and from this we have $\theta(1)^2 = |Q : \mathbf{Z}(\theta)|$. Now since $|\langle z_1, z_2, x_9, x_{10} \rangle| = q^4$, the final statement of the claim will hold once part (ii) is established.

As before, let $\overline{}$ denote quotient mod $\mathbf{Z}(Q)$. Using additive notation in each of the abelian groups \overline{Q} and $\mathbf{Z}(Q)$, for an element $y \in \mathbf{Z}(\theta)$ we may write:

$$\overline{y} = c_1\overline{x_1} + c_2\overline{x_2} + \dots + c_{10}\overline{x_{10}} \quad \text{with } c_i \in GF(q).$$

For each element $z \in \mathbf{Z}(Q)$ and for each generator x_i we have $[x_i, yz] = [x_i, y]$. Using the defining relations for the group Q , it follows that:

$$[x_1, y] = c_2z_1 + c_4z_2 \quad \text{and} \quad [x_3, y] = c_4z_1 - c_2z_2.$$

Observe that for each element $y \in \mathbf{Z}(\theta)$ and $x \in Q$ we have $[y, x] \in Q' \cap \ker(\theta) = \mathbf{Z}(Q) \cap \ker(\theta)$; thus each of $[x_1, y]$ and $[x_3, y]$ lie in $\mathbf{Z}(Q) \cap K$. Viewing $\mathbf{Z}(Q) \cap K$ as a 1-dimensional subspace of $\mathbf{Z}(Q)$, the vectors $c_2z_1 + c_4z_2$ and $c_4z_1 - c_2z_2$ are dependent. If either c_2 or c_4 is nonzero, then both are nonzero and $c_4 = \alpha c_2$ and $c_2 = -\alpha c_4$ for some $\alpha \in GF(q)$; in which case $-\alpha^2 = 1$. Invoking the hypothesis

$q \equiv 3 \pmod 4$, there is no solution for $-\alpha^2 = 1$ in $GF(q)$. This forces $c_2 = c_4 = 0$. Under the same assumption, similar comparisons (e.g. $[x_2, y]$ and $[x_4, y]$) force $c_1 = c_3 = c_5 = c_6 = c_7 = c_8 = 0$. Thus we have $\mathbf{Z}(\theta) \leq \langle x_9, x_{10}, z_1, z_2 \rangle$ and the claim is proved.

Now let r and s be distinct primes dividing $q - 1$. We will define the action of a cyclic group H of order rs on the group Q . First, we define actions on Q by automorphisms a and b of orders r and s respectively. Let δ and ϵ be primitive r and s roots of unity in $GF(q)$ respectively. The actions of a and b are defined on the generators of Q , where all unspecified generators are fixed:

$$\text{for } a : x_1^a = x_1^\delta, x_2^a = x_2^{\delta^{-1}}, x_3^a = x_3^\delta, x_4^a = x_4^{\delta^{-1}};$$

$$\text{for } b : x_5^b = x_5^\epsilon, x_6^b = x_6^{\epsilon^{-1}}, x_7^b = x_7^\epsilon, x_8^b = x_8^{\epsilon^{-1}}.$$

We must verify that the proposed definitions interact well with the defining relations among the generators. That is, it must be shown that: if $[x_i, x_j] = [x_k, x_l]$, then $[x_i, x_j]^a = [x_k, x_l]^a$ and $[x_i, x_j]^b = [x_k, x_l]^b$. In fact, more is true. If we write $x_i^a = x_i^{\alpha_i}$ and $x_j^a = x_j^{\alpha_j}$, then $[x_i, x_j]^a = [x_i^{\alpha_i}, x_j^{\alpha_j}] = [x_i, x_j]^{\alpha_i \alpha_j}$; and whenever $[x_i, x_j] \neq 1$ we have $\alpha_i \alpha_j = 1$. Thus every commutator $[x_i, x_j]$ is fixed by a . The same holds for b . It follows that a and b act as automorphisms on Q and that the subgroup $\langle z_1, z_2, x_9, x_{10} \rangle$ is centralized by both a and b . It is also clear that $ab = ba$. Since a and b are commuting automorphisms of Q of relatively prime orders, the cyclic group $H = \langle a \rangle \times \langle b \rangle$ acts on Q . Note that $|H| = rs$ and H centralizes $\langle z_1, z_2, x_9, x_{10} \rangle$.

Now we consider the action of H on $\mathbf{Z}(Q)$ and on $Q/\mathbf{Z}(Q)$. As observed, H centralizes $\mathbf{Z}(Q)$. To understand the action of H on $Q/\mathbf{Z}(Q)$, we return to a vector space point of view. As before, let $\bar{}$ denote quotients mod $\mathbf{Z}(Q)$ and use additive notation. From this perspective, the set $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{10}\}$ is a basis for $Q/\mathbf{Z}(Q)$ consisting of eigenvectors of a and b with corresponding eigenvalues—

$$\text{for } a : \delta, \delta^{-1}, \delta, \delta^{-1}, 1, 1, 1, 1, 1, 1;$$

$$\text{for } b : 1, 1, 1, 1, \epsilon, \epsilon^{-1}, \epsilon, \epsilon^{-1}, 1, 1.$$

It follows that a and b act diagonally on \bar{Q} and that the orbits of the action of H on \bar{Q} have sizes $1, r, s$, and rs .

To complete the construction, define the group: $\Gamma = Q \rtimes H$.

Now we consider $\text{cd}(\Gamma)$. The coprime action of H on the abelian group $Q/\mathbf{Z}(Q)$ is permutation isomorphic to the action of H on $\text{Irr}(Q/\mathbf{Z}(Q))$. The orbit sizes of the former action are $\{1, r, s, rs\}$. Since H is cyclic, it follows that $\text{cd}(\Gamma/\mathbf{Z}(Q)) = \{1, r, s, rs\}$. On the other hand, we see from the claim that each nonlinear irreducible character of Q is fixed by H , since H centralizes $\langle z_1, z_2, x_9, x_{10} \rangle$. Again, since H is cyclic and since the action of H on Q is coprime, the remaining elements of $\text{cd}(\Gamma)$ are exactly the nonlinear irreducible character degrees of $\text{cd}(Q)$. It follows that $\text{cd}(\Gamma) = \{1, r, s, rs, q^4, q^5\}$.

The next construction will show that for infinitely many values of k there exist groups which satisfy property P_k and have $3(k - 1)$ irreducible character degrees. It follows that the bound of Theorem A cannot be better than the linear bound $3(k - 1)$.

For any two distinct primes p and q and for an appropriate n , one can let a cyclic group of order p act on an extra-special q -group of order $q^{(2n+1)}$ such that the

irreducible character degrees of the resulting semidirect product are $\{1, p, q^n\}$. Let the group G be the direct product of m groups of this sort such that all of the primes involved are distinct. Then $|\text{cd}(G)| = 3^m$; further, since a single prime divides no more than $3^{(m-1)}$ irreducible character degrees of G , we have $k = k(G) = 3^{(m-1)} + 1$. It follows that $|\text{cd}(G)| = 3(k - 1)$, showing that the bound of Theorem A can be no better than $3(k - 1)$.

Finally, observe that if, as in the preceding construction, we let $\Delta = A \times B$ for groups A and B with $\text{cd}(A) = \{1, 2, 3\}$ and $\text{cd}(B) = \{1, 5, 11\}$, then Δ satisfies P_4 and has $|\text{cd}(\Delta)| = 9$. Thus Δ verifies the sharpness of the bound in the case $k = 4$ and completes our collection of examples for each of the exceptional cases of Theorem A.

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