A RENEWAL THEOREM IN THE FINITE-MEAN CASE

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Abstract. Let $F(\cdot)$ be a c.d.f. on $(0, \infty)$ such that $F(\cdot) \equiv 1 - F(\cdot)$ is regularly varying with exponent $-\alpha$, $1 < \alpha < 2$. Then $U(t) - \frac{t}{\mu} - \frac{1}{\pi^2} \int_0^\infty F(v)dv = O(t^2F(t)^2F(t^2(t)))$ as $t \to \infty$, where $U(t) = EN(t)$ is the renewal function associated with $F(t)$. Moreover similar estimates are given for distributions in the domain of attraction of the normal distribution and for the variance of $N(t)$. The estimates improve earlier results of Teugels and Mohan.

1. Introduction and results

In this paper we assume that $X_1, X_2, \ldots$ is a sequence of i.i.d. real-valued positive random variables with d.f. $F$. Define the associated random walk by $S_0 = 0, S_n = X_1 + \ldots + X_n$ for $n \geq 1$, $N(t) = \max \{n \geq 0; S_n \leq t\}$ and the renewal function $U$ by

$$U(t) \equiv EN(t) = \sum_{n=0}^\infty P(S_n \leq t).$$

If $F$ is not arithmetic, Blackwell’s theorem says that

$$U(t) - U(t - h) \to \frac{h}{\mu} \text{ as } t \to \infty$$

for every $h > 0$ where $\mu = EX_1$. See Feller [5] for a proof and an extension to the arithmetic case. Extensions such as

$$U(t) - \frac{t}{\mu} \to \frac{\sigma^2 + \mu^2}{2\mu^2} \text{ as } t \to \infty$$

in case $\sigma^2$ is finite are usually proved using so-called key renewal theorems which give the asymptotic behavior of the convolution

$$(U * Q)(t) = \int_{0-}^t Q(t-u)U(du)$$

as $t \to \infty$ under suitable hypothesis on $Q(\cdot)$ and $F(\cdot)$.
In this paper we consider the special case where $F$ is not arithmetic and the distribution function tail $\bar{F} \equiv 1 - F$ is regularly varying, i.e.
 solves
\begin{equation}
\frac{\bar{F}(tx)}{\bar{F}(t)} \to x^{-\alpha} \text{ as } t \to \infty.
\end{equation}

In this case we use the notation $\bar{F}(\cdot) \in RV_{-\alpha}$. Following earlier work by Feller [4] and Smith [10, 11], Teugels [12] addressed the question of the asymptotic behavior of the convolution $U \ast Q(\cdot)$ under the assumption $\bar{F} \in RV_{-\alpha}$ where $0 < \alpha < 2$. In case $\frac{1}{2} < \alpha < 1$ (which implies $\mu = \infty$) an improvement of Teugels’ result is given by Anderson and Athreya [1] using a result of Erickson [3]. In the case of a regularly varying tail function with $1 < \alpha < 2$ (in which case $\mu < \infty$ and $\sigma^2 = \infty$) Teugels [12] proved that
\begin{equation}
\tau(t) \equiv U(t) - \frac{t}{\mu} \sim \frac{t^2 \bar{F}(t)}{\mu^2 (\alpha - 1)(2 - \alpha)} \text{ as } t \to \infty
\end{equation}

(whence $\tau(\cdot) \in RV_{2-\alpha}$) under a supplementary condition. This condition was shown to be unnecessary in a paper by Mohan [8]. Besides this, Mohan proved that for $F \in D(\alpha)$, the domain of attraction of a stable law with exponent $\alpha$ where $1 < \alpha \leq 2$ (in case $\alpha = 2$ $F$ is assumed to have infinite variance), the above function $\tau(\cdot)$ satisfies the asymptotic relation
\begin{equation}
\tau(t) \sim \frac{1}{\mu^2} \int_0^t \int_x^{\infty} \bar{F}(v)dvdx \text{ as } t \to \infty.
\end{equation}

It should be observed that the above asymptotic relation holds for non-arithmetic $F$ with finite mean $\mu$ even without the assumption (1.1). See Frenk [6], Lemma 4.1.2. Note that in case $1 < \alpha < 2$ relation (1.3) is equivalent to (1.2). In this paper we show that the assumption $F \in D(\alpha)$ with $1 < \alpha \leq 2$ permits a stronger conclusion than (1.3). In particular we have the following results.

**Theorem 1.1.** Suppose $F \in D(\alpha)$ with $1 < \alpha \leq 2$ (where in case $\alpha = 2$ we assume $\sigma^2$ infinite) is not arithmetic. Suppose $Q(t) = \int_t^{\infty} q(s)ds < \infty$, $t \geq 0$ where $q(\cdot)$ is nonnegative and nonincreasing. Suppose $\int_0^\infty Q(s)ds \in RV_{-\gamma+1}$ ($0 < \gamma \leq 1$, where in case $\gamma = 1$ we assume $tQ(t) \to \infty$ as $t \to \infty$. Then as $t \to \infty$

\begin{equation}
(U \ast Q)(t) = \begin{cases} 
\frac{1}{\mu} \int_0^t Q(s)ds + O(\tau(t)Q(\tau(t))) & \text{if } 0 < \gamma < 1, \\
\frac{1}{\mu} \int_0^t Q(s)ds + o(\int_0^{\tau(t)} Q(s)ds) & \text{if } \gamma = 1.
\end{cases}
\end{equation}

**Theorem 1.2.** Suppose $F \in D(\alpha)$ with $1 < \alpha \leq 2$ (where in case $\alpha = 2$ we assume $\sigma^2$ infinite) is not arithmetic. Then as $t \to \infty$

\begin{equation}
U(t) - \frac{t}{\mu} - \frac{1}{\mu^2} \int_0^t \int_x^{\infty} \bar{F}(v)dvdx = \begin{cases} 
O(\tau(\tau(t))) = O(t^3 \bar{F}(t)^2 \bar{F}(t^2)) & \text{if } 1 < \alpha < 2, \\
o(\tau(\tau(t))) & \text{if } \alpha = 2.
\end{cases}
\end{equation}

We may use the same technique as Smith [11] to find the variance of $N(t)$. Using (1.5) and the method used in Smith’s paper gives the following estimate which improves the estimates given in the papers of Teugels [12]) and Mohan [8]. We omit the details of the proof.
Theorem 1.3. Under the assumptions of Theorem 1.2 we have

\[
\var N(t) - \frac{4}{\mu^3} \int_0^t \int_0^u \int_v^\infty F(s)dsdvdu + \frac{2t}{\mu^3} \int_0^t \int_0^\infty F(s)dsdv = \\
\begin{cases} 
O(t^\beta F(t)^2 F(t^2)) & \text{if } 1 < \alpha < 2, \\
o(t\tau(t)), \text{where } \tau(\cdot) \text{ satisfies (1.3)} & \text{if } \alpha = 2.
\end{cases}
\]

(1.6)

2. Proofs

In the proofs below we write \( \beta = 2 - \alpha \). Before giving the proofs of the results we list some well-known properties of RV functions which are used in the sequel. For a proof the reader is referred to Bingham et al. [2], Geluk and de Haan [7] or Resnick [9].

Lemma 1. Suppose \( \phi \in RV_\delta \). It follows that

(i) Uniform convergence theorem for regularly varying functions. Convergence in \( \phi(tx)/\phi(t) \to x^\delta(t \to \infty) \) is uniform on compact intervals of \((0, \infty)\).

(ii) Karamata’s theorem. There exists \( t_0 > 0 \) such that \( \phi(t) \) is positive and locally bounded for \( t > t_0 \). If \( \delta \geq -1 \), then

\[
\lim_{t \to \infty} \frac{t \phi(t)}{\int_{t_0}^t \phi(s)ds} = \delta + 1.
\]

A similar result holds for \( \delta < -1 \).

(iii) Potter’s inequality. If \( \varepsilon_1, \varepsilon_2 > 0 \) are arbitrary, there exists \( t_0 = t_0(\varepsilon_1, \varepsilon_2) \) such that for \( t \geq t_0, tx \geq t_0 \)

\[
(1 - \varepsilon_1)x^\delta - \varepsilon_2 < \frac{\phi(tx)}{\phi(t)} < (1 + \varepsilon_1)x^{\delta + \varepsilon_2}.
\]

(iv) Monotone density theorem. If \( \delta \geq 0 \) and \( \phi(t) = \int_0^t f(s)ds \) for \( t \geq 0 \) with \( f \) monotone, then \( \lim_{t \to \infty} tf(t)/\phi(t) = \delta \). Hence in case \( \delta > 0 \) we have \( f(\cdot) \in RV_{\delta-1} \).

Proof of Theorem 1.1. We write

\[
\int_{0-}^t Q(t - y)U(dy) = I_1 + I_2 + I_3,
\]

where \( I_1, I_2 \) and \( I_3 \) are the integrals over \((0, [L(t)])\), \([L(t)], [t]\) and \([t], t)\) respectively, where \([t]\) denotes the greatest integer not exceeding \( t \). Take \( L(t) \to \infty (t \to \infty) \) a slowly varying (i.e. in \( RV_\delta \)) function. In case \( \beta = 0 \) take in addition \( L(t) = o(\tau(t)) \) (\( t \to \infty \)), which is possible since \( \tau(t) \to \infty \) as \( t \to \infty \). (Note that \( F \) has infinite variance.)

First we estimate \( I_1 = \int_0^{[L(t)]} Q(t - y)U(dy) \). By monotonicity of \( Q \) we have as \( t \to \infty \)

\[
0 \leq I_1 \leq Q(t - [L(t)])U(L(t)).
\]

Since \( L(t) = o(t) \) and \( \tau(.) \in RV_\beta, 0 \leq \beta < 1 \), we have \( \tau(t) = o(t - L(t)) \) as \( t \to \infty \); hence \( Q(t - [L(t)]) \leq Q(\tau(t)) \) for \( t \) sufficiently large. It follows that \( I_1 = O(L(t)Q(\tau(t))) = o(\tau(t)Q(\tau(t))) \).
The second integral is estimated as follows.

\[ I_2 = \sum_{j=[L(t)]+1}^{[t]} \int_{j-1}^{j} Q(t - y) U(dy) \]

\[ \leq \sum_{j=[L(t)]+1}^{[t]} Q(t - j)(U(j) - U(j - 1)) \]

(2.1)

and similarly

\[ I_2 \geq \sum_{j=[L(t)]+1}^{[t]} Q(t - j + 1)(U(j) - U(j - 1)) \]

(2.2)

Application of the Lemmas 2, 4 and 5 below shows that \( I_2 = \frac{1}{\mu} \int_{[t]}^{t} Q(s) ds + O(\tau(t)Q(\tau(t))) + o(\int_{0}^{\tau(t)} Q(s) ds) \) as \( t \to \infty \). From Lemma 1 (ii) and (iv) in case \( 0 < \gamma < 1 \) it follows that \( \int_{0}^{t} Q(s) ds \sim (1 - \gamma)^{-1} t Q(t) \) and \( t Q(t) \to \infty (t \to \infty) \).

In case \( \gamma = 1 \) we have \( t Q(t) = o(\int_{0}^{t} Q(s) ds) \). The proof is complete since \( I_3 = \int_{[t]}^{t} Q(t - y) U(dy) = O(U(t) - U([t])) = O(1) \) by Blackwell’s theorem. \( \square \)

In the Lemmas 2 to 5 below the assumptions of Theorem 1.1 are supposed to be satisfied.

**Lemma 2.**

\[ \sum_{j=[L(t)]+1}^{[t]} Q(t - j) = \int_{0}^{t} Q(s) ds + O(\tau(t)Q(\tau(t))) \quad (t \to \infty). \]

The same estimate holds for \( \sum_{j=[L(t)]}^{[t]-1} Q(t - j) \).

**Proof of Lemma 2.** Since the second statement is equivalent to the first (note that \( Q(0+) < \infty \)) we only prove the first statement. Note that by the monotonicity of \( Q \)

\[ \sum_{j=[L(t)]+1}^{[t]} Q(t - j) \geq \int_{t-[L(t)]}^{t-[L(t)+1]} Q(s) ds = \int_{0}^{t} Q(s) ds - \int_{t-[L(t)+1]}^{t} Q(s) ds + O(1). \]

The last integral can be estimated by \( 0 \leq \int_{t-[L(t)+1]}^{t} Q(s) ds \leq Q(t - [L(t)]L([t])); \) hence the integral is \( O(\tau(t)Q(\tau(t))) \), \( t \to \infty \) using the same argument as in the proof above.

Similarly we have

\[ \sum_{j=[L(t)]+1}^{[t]} Q(t - j) \leq Q(0+) + \int_{0}^{t-[L(t)+1]} Q(s) ds = \int_{0}^{t} Q(s) ds + O(\tau(t)Q(\tau(t))). \]

\( \square \)
Lemma 3. As \( t \to \infty \)

\[
\phi(t) \equiv \int_{[L(t)]}^{[t]-[\tau(t)]} q(t-s)\tau(s)ds = O(\tau(t)Q(\tau(t)))
\]

Proof of Lemma 3. We estimate

\[
\phi(t) = t \int_{[L(t)]}^{[t]-[\tau(t)]} q(t(1-u))\tau(tu)du
\]
as follows. Note that \( L(t) \to \infty \) and \( \tau(t) \to \infty \) \((t \to \infty)\); hence \( \tau(.) \) is positive on the specified interval of integration for \( t \) sufficiently large. From (1.3) it follows that \( \tau(.) \) is asymptotic to a non-decreasing function. It follows that for \( \varepsilon > 0 \) arbitrary there exists \( t_0 = t_0(\varepsilon) \) such that \( \tau(tu) \leq (1+\varepsilon)\tau([t] - [\tau(t)]) \) uniformly for \( u \in \left[ \frac{[L(t)]}{t}, \frac{[t]-[\tau(t)]}{t} \right] \) as \( t > t_0(\varepsilon) \). Hence we obtain

\[
\phi(t) \leq (1+\varepsilon)t\tau([t] - [\tau(t)]) \int_{[L(t)]}^{[t]-[\tau(t)]} q(t(1-u))du.
\]

Regular variation of the function \( \tau(.) \) with exponent \( \beta \in [0,1) \) implies

\[
\tau([t] - [\tau(t)]) \sim \tau(t) \quad \text{as} \quad t \to \infty;
\]
hence

\[
\phi(t) \leq (1+\varepsilon)^2\tau(t) \int_{t-[t]+[\tau(t)]}^{t-[L(t)]} q(s)ds \leq (1+\varepsilon)^2\tau(t)Q([\tau(t)])
\]
for \( t \) sufficiently large. \( \Box \)

Lemma 4.

\[
S_1(t) \equiv \sum_{j=[L(t)]+1}^{[t]-[\tau(t)]} Q(t-j)\tau(j) - \tau(j-1)) = O(\tau(t)Q(\tau(t)))
\]
as \( t \to \infty \). The same estimate holds if \( Q(t-j) \) is replaced with \( Q(t-j+1) \).

Proof of Lemma 4. We only prove the first estimate, the second can be proved similarly. Using partial summation we have

\[
\begin{align*}
|S_1(t)| & = | - \Sigma + Q(t-[t]+[\tau(t)])\tau([t] - [\tau(t)]) - Q(t-[L(t)])\tau([L(t)]) | \\
& \leq \Sigma + Q(t-[t]+[\tau(t)])\tau([t] - [\tau(t)]) + Q(t-[L(t)])\tau([L(t)])
\end{align*}
\]
(2.3)

where

\[
\Sigma = \sum_{j=[L(t)]+1}^{[t]-[\tau(t)]} (Q(t-j) - Q(t-j+1))\tau(j-1).
\]

The middle term on the right-hand side in equation (2.3) is dominated by \((1+\varepsilon)\tau(t)Q(\tau(t)-1)\) and the last term is asymptotic to

\[
\tau(L(t))Q(t-[L(t)]) = O(\tau(t)Q(\tau(t)))
\]
by the same argument as in the proof of Theorem 1.1. Note that since \( L(t) \to \infty \) and \( \tau(t) \to \infty \) \((t \to \infty)\), monotonicity of \( Q \) implies that \( \Sigma \geq 0 \) for \( t \) sufficiently
large. We proceed with the upper estimate. For $\varepsilon > 0$ arbitrary and $t$ sufficiently large we have

\[
\Sigma = \sum_{j=[t]-[\tau(t)\varepsilon]}^{[t]-[\tau(t)]} \tau(j-1) \int_{t-j}^{t-j+1} q(s)ds \\
\leq (1 + \varepsilon) \sum_{j=[t]-[\tau(t)]}^{[t]-[\tau(t)\varepsilon]} \int_{t-j}^{t-j+1} \tau(t-s)q(s)ds \\
= (1 + \varepsilon) \int_{[t]-[\tau(t)]}^{[t]-[\tau(t)\varepsilon]} \tau(s)q(t-s)ds.
\]

Note that the above inequality follows from (1.3) as in the proof of Lemma 3 above. Application of Lemma 3 completes the proof.

Lemma 5.

\[
S_2(t) = \sum_{j=[t]-[\tau(t)]}^{[t]} Q(t-j+1)(\tau(j) - \tau(j-1)) = o\left(\int_0^{\tau(t)} Q(s)ds\right)
\]

as $t \to \infty$. The same estimate holds if $Q(t-j+1)$ is replaced with $Q(t-j)$.

Proof of Lemma 5. Since $\tau(t) - \tau(t-1) = U(t) - U(t-1) - \frac{1}{\mu} \to 0$ as $t \to \infty$ by Blackwell’s theorem, it follows that for $\varepsilon > 0$ and $t > t(\varepsilon)$

\[
|S_2(t)| \leq \varepsilon \sum_{j=[t]-[\tau(t)\varepsilon]}^{[t]} Q(t-j+1) \\
\leq \varepsilon \{Q(0+) + \int_{[t]-[\tau(t)\varepsilon]}^{[t]-[\tau(t)]} Q(s)ds\}.
\]

Since $\int_0^{t} Q(s)ds$ is regularly varying, the integral on the right-hand side is asymptotic to $\int_0^{\tau(t)} Q(s)ds$. The statement of the lemma is proved since $\varepsilon > 0$ is arbitrary and $\int_0^{\tau(t)} Q(s)ds \geq \tau(t)Q(\tau(t)) \to \infty$ as $t \to \infty$. The proof of the second statement is similar.

Proof of Theorem 1.2. Integrating the renewal equation $\int_{0^-}^{s} F(s-y)U(dy) = 1$ over the interval $[0,t]$ ($t > 0$) gives $\int_0^{t} \int_0^{t-y} F(u)duU(dy) = t$. It follows that

\[
U(t) - \frac{t}{\mu} = \int_{0^-}^{t} Q(t-y)U(dy)
\]

where $Q(t) = \frac{1}{\mu} \int_t^{\infty} F(s)ds$.

Since the asymptotic behavior of $\tau$ is given by (1.3), application of Theorem 1.1 completes the proof. (Note that $t^2 F(t) \geq \int_0^t x^2 F(dx) \to \infty$; hence $tQ(t) \to \infty$ as $t \to \infty$ in case $\alpha = 2$.)

\[
\square
\]
REFERENCES


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