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ON CONVEX CLASS OF PAIRS OF CONVEX BODIES

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Abstract. In this paper we introduce a quotient class of pairs of convex bodies in which every member have convex union.

The space of pairs of convex bodies has been investigated in a number of papers [3], [8], [9], and [12]. This space has found an application in quasidifferential calculus (*cf.* [1], [5], [7], [10]). A quasidifferential is represented as a pair of convex bodies and it is essential to find the minimal representation of this pair. The notion of minimal pairs was introduced in [5] and investigated in [2], [6], [7] and [11]. Some criteria of minimality are given in [6]. In this paper we investigate pairs of convex bodies with convex union. We introduced a quotient class of pairs of convex compact sets in which every member has convex union. Moreover some criteria for the convex class are given.

In this paper $X = (X, \tau)$ stands for a real locally convex vector space, and X^* denotes the dual space of X. Denote by $\mathcal{K}(X)$ the family of all convex bodies in X, i.e., of all nonempty compact convex subsets of X. If A, B are nonempty subsets of X, then $A+B$ is the usual algebraic Minkowski sum of A and B. It may be showed that $\mathcal{K}(X)$ satisfies the order cancellation law; i.e. for every $A, B, C \in \mathcal{K}(X)$ the inclusion $A + B \subset B + C$ implies $A \subset C$ (*cf.* [12]).Hence it follows that $\mathcal{K}(X)$ endowed with the Minkowski sum is a commutative semigroup satisfying the law of cancellation.

Now let $\mathcal{K}^2(X) = \mathcal{K}(X) \times \mathcal{K}(X)$; the equivalence relation between pairs of convex bodies is given by: $(A, B) \sim (C, D)$ if and only if $A + D = B + C$. For $A, B \in \mathcal{K}(X)$ we will use the notation $A \vee B := conv\,(A \cup B)$, where the operation *"conv"* denotes the convex hull. If $A, B, C \in \mathcal{K}(X)$, and $b \in X$, then $A \vee B + C = (A \vee B) + C$ and $A + b = A + \{b\}$. We have $[a, b] = \{a\} \vee \{b\}$.

Let $f \in X^*$, $A \in \mathcal{K}(X)$ and $c \in \mathbb{R}$. We denote by $p_A(f) := max_{x \in A} f(x)$ the support function of the set A. Moreover, $H_f^c := \{x \in X \mid f(x) = c\}$ and $H_f A := \{x \in A \mid f(x) = p_A(f)\}$, where H_f^c is the hyperplane generated by the functional f and the number c, and $H_f A$ is the face of A with respect to f. For the sum of the faces of two convex bodies $A, B \subset X$ with respect to $f \in X^*$ the identity $H_f(A + B) = H_f + H_f B$ holds true. For $A \subset X$ we denote by ∂A the boundary $\bar{A} \setminus A^o$ of the set A, where $\bar{A} := cl\, A$ and $A^o := int\, A$.

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Let $A, B \in \mathcal{K}(X)$. The class $[A, B] := \{(C, D) \in \mathcal{K}^2(X) \mid (A, B) \sim (C, D)\}$ is called *convex* if for every member $(C, D) \in [A, B]$ the set $C \cup D$ is convex.

Proposition 1. *If $dimX > 1$, and $A, B \in \mathcal{K}(X)$, then $A \cup B$ is convex if and only if $\partial(A \vee B) \subset A \cup B$.*

Proof. Necessity. For arbitrary sets $A, B \subset X$ we have $\partial(A \cup B) \subset \partial A \cup \partial B \subset A \cup B$. But $A \vee B = A \cup B$. Hence $\partial(A \vee B) \subset A \cup B$.

Sufficiency. Let $dimX < \infty$. Given any $x \in A \vee B$ with $x \notin A$, there exist $f \in X^*$ and $c \in \mathbb{R}$ such that the hyperplane H_f^c separates the set $\{x\}$ and A. We can assume that $x \in H_f^c$ and $H_f^c \cap A = \emptyset$. Take any line $l \subset H_f^c$ passing through the point x. Then $l \cap (A \vee B) = [p, q]$ for some $p, q \in \partial(A \vee B)$. But $\partial(A \vee B) \subset A \cup B$, $H_f^c \cap A = \emptyset$. Hence $p, q \in B$ and we get $x \in [p, q] \subset B$.

Now, let $dimX = \infty$. Then $\partial(A \vee B) = A \vee B$, and $\partial(A \vee B) \subset A \cup B$ implies $A \vee B \subset A \cup B$. Hence $A \vee B = A \cup B$. □

If $dimX = 1$, then for $A := \{0\}, B := \{1\}$ we have $\partial(A \vee B) = \{0, 1\} = A \cup B$ but $A \cup B$ is not convex.

In [4] the following is proved:

Lemma. *If X is finite-dimensional and $A \subset X$ is a convex set, then at any point x of the boundary ∂A of A there is a supporting hyperplane for A.*

In the infinite-dimensional case, the above lemma does not hold true. For example let $X = l^2$ and we consider the Hilbert cube

$$A := \{x = (\xi_n) \mid \xi_n \in \mathbb{R}, \text{ and } \mid \xi_n \mid \leq \frac{1}{n}\}.$$

The set A is compact and convex. It is easy to observe that $p_A(f) > 0$ for every nontrivial $f \in X^*$. Since A is compact $\partial A = A$. Moreover, $f(0) = 0$ for any $f \in X^*$. So, there is no supporting hyperplane at 0.

Proposition 2. *If $1 < dimX < \infty$, $A, B \in \mathcal{K}(X)$, then $A \cup B$ is convex if and only if $H_f(A \vee B) \subset H_f A \cup H_f B$ for every $f \in X^\star \setminus \{0\}$.*

Proof. Necessity. Given any $f \in X^* \setminus \{0\}$, we have

$$p_{A \vee B}(f) = max\{p_A(f), p_B(f)\}.$$

Now let $p_A(f) < p_B(f)$. Then $H_f(A \vee B) = H_f B$. Analogously, if $p_A(f) > p_B(f)$, we obtain $H_f(A \vee B) = H_f A$. Suppose $p_A(f) = p_B(f)$. If $x \in H_f(A \vee B) \subset A \cup B$, then $x \in A$ or $x \in B$. If $x \in A$, then $x \in H_f A$. It follows from the above that $H_f(A \vee B) \subset H_f A \cup H_f B$ for every $f \in X^* \setminus \{0\}$.

Sufficiency. Let $x \in \partial(A \vee B)$. Since $dimX < \infty$, so from the lemma it follows that $x \in H_f(A \vee B)$ for some nontrivial $f \in X^*$. And we obtain from assumption $x \in A \cup B$. Hence $\partial(A \vee B) \subset A \cup B$. Now, it follows from Proposition 1 that $A \cup B$ is convex. □

Theorem 1. *Let $1 < dimX < \infty$, $A, B \in \mathcal{K}(X)$. If $H_f(A \vee B) = H_f A$ or $H_f B$ for every $f \in X^* \setminus \{0\}$ then the class $[A, B]$ is convex.*

Proof. We observe that

$$H_f(A \vee B) \subset H_f A \cup H_f B \text{ for every nontrivial } f \in X^*.$$

Hence from Proposition 2 we have that $A \cup B$ is convex.

Now, given any pair $(C,D) \in \mathcal{K}^2(X)$ equivalent to (A,B), we have

$$A + C \vee D = (A+C) \vee (A+D) = (A+C) \vee (B+C) = C + A \vee B.$$

Analogously

$$B + C \vee D = D + A \vee B.$$

We also have

$$H_f A + H_f(C \vee D) = H_f C + H_f(A \vee B),$$

$$H_f B + H_f(C \vee D) = H_f D + H_f(A \vee B) \text{ for every } f \in X^* \setminus \{0\}.$$

But

$$H_f(A \vee B) = H_f A \text{ or } H_f B.$$

Hence from the law of cancellation we have

$$H_f(C \vee D) = H_f C \text{ or } H_f D \text{ for every } f \in X^* \setminus \{0\}.$$

This implies that

$$H_f(C \vee D) \subset H_f C \cup H_f D \text{ for every } f \in X^* \setminus \{0\}.$$

Hence we obtain from Proposition 2 that $C \cup D$ is convex. □

The condition $H_f(A \vee B) \subset H_f A \cup H_f B$ in Theorem 1 is not sufficient. For example, let $A, B \in \mathcal{K}(\mathbb{R}^2)$,

$$A := \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\},\ B := \{(1,0)\} + A.$$

Then $A \vee B = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 1\}$, $H_f(A \vee B) \subset H_f A \cup H_f B$ for $f \in X^* \setminus \{0\}$. Define $C := \{(0,0)\}$, $D := \{(1,0)\}$. Then $A + D = B + C$, but $C \cup D = \{(0,0),(1,0)\}$ is not convex.

Example 1. Convex classes

i) Take any $(A,B) \in \mathcal{K}^2(X)$ such that $A \subset B$ and any $(C,D) \in \mathcal{K}^2(X)$ being an equivalent pair to (A,B). Then $B + C = A + D \subset B + D$ and from the order law of cancellation, we have $C \subset D$. Hence $C \cup D = D$. It is obvious that $\partial(A \vee B) \subset A \cup B$.

ii) Let $X = \mathbb{R}^2$ and $R > 0$. Consider the closed ball $\mathbb{B}((0,0),R)$ and let

$$a := (-\frac{1}{2} \cdot \sqrt{2} \cdot R,\ \frac{1}{2} \cdot \sqrt{2} \cdot R),\ \ b := (\frac{1}{2} \cdot \sqrt{2} \cdot R,\ \frac{1}{2} \cdot \sqrt{2} \cdot R).$$

Define

$$A := \{(x,y) \in \mathbb{B}((0,0),R) \mid -\frac{1}{2} \cdot \sqrt{2} \cdot R \le x \le \frac{1}{2} \cdot \sqrt{2} \cdot R\},$$

$$B := \{(x,y) \in \mathbb{B}((0,0),R) \mid -\frac{1}{2} \cdot \sqrt{2} \cdot R \le y \le \frac{1}{2} \cdot \sqrt{2} \cdot R\}$$

(see Figure 1).

We have $A \cup B = \mathbb{B}((0,0),R)$ and $A \cap B = -a \vee (-b) \vee a \vee b$. Since $H_f(A \vee B) = H_f A$ or $H_f B$ for every nonzero $f \in X^*$, it follows from the above theorem that the class $[A,B]$ is convex.

Theorem 2. *Let $X = \mathbb{R}^2$, $A, B \in \mathcal{K}(X)$. Then the class $[A,B]$ is convex if and only if $H_f(A \vee B) = H_f A$ or $H_f B$ for all $f \in X^* \setminus \{0\}$.*

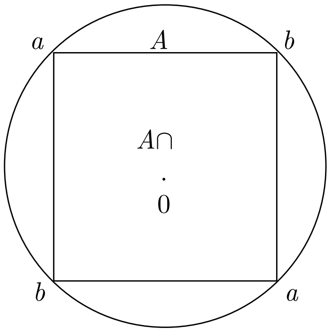

FIGURE 1

Proof. Necessity. Assume that $H_f A \neq H_f(A \cup B) \neq H_f B$ for some $f \in X^* \setminus \{0\}$. It follows from the assumption that $p_A(f) = p_B(f)$ and the faces $H_f A$ and $H_f B$ are parallel segments and not one-point sets. In fact, $H_f A$ and $H_f B$ are contained in one line. Denote $H_f A := a \vee b$, $H_f B := c \vee d$, where $a, b, c, d \in \mathbb{R}^2$ and assume that $d - c = k \cdot (b - c)$ for some $k \geq 1$. Let $e \in \mathbb{R}^2$ be a point $H_f T = e$ where $I = a \vee b$ and $T = I \vee e$. Then $H_{-f} T = I$. Denote $J := (c - a) \vee (d - b)$.

We have

$$H_f(A+T) = I + e, \qquad H_f(B+T) = I + J + e,$$
$$H_{-f}(A+T) = I + H_{-f}A, \quad H_{-f}(B+T) = I + H_{-f}B.$$

Therefore, the segment I is a summand of both $A + T$ and $B + T$. Let $A', B' \in \mathcal{K}(X)$, $A + T = A' + I$ and $B + T = B' + I$.

We have

$$H_f A' + I = H_f A' + H_f I = H_f(A+T) = I + e,$$
$$H_f B' + I = I + J + e.$$

It follows from these equations that $H_f A' = e$ and $H_f B' = J + e$. Since $H_f B$ does not contain $H_f A$ then $0 \notin J$, and $e \notin J + e$. Therefore, $H_f A' \cap H_f B' = \emptyset$. Since $p_{A'}(f) = p_{B'}(f)$ then $H_f(A' \vee B') = H_f A' \vee H_f B' \neq H_f A' \cup H_f B' = H_f(A' \cup B')$. According to Proposition 2, the pair (A', B') is not convex while $(A', B') \in [A, B]$. This contradicts the assumption of our theorem.

Sufficiency. It follows immediately from Theorem 1. $\square$

Example 2. Let $X := \mathbb{R}^3$, $A := \{(x, y, z) \in \mathbb{B}((0,0), R) \mid x \leq 0, z \leq 0\}$, $B := \{(x, y, z) \in \mathbb{B}((0,0), R) \mid x \geq 0, z \leq 0\}$ (see Figure 2). Denote $f(x, y, z) := z$.

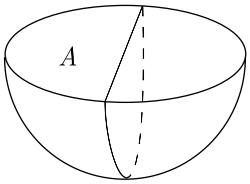

FIGURE 2

The functional $f \in X^* \setminus \{0\}$ and $A' := H_f A, B' := H_f B \subset Y := \mathbb{R}^2 \times \{0\}$. Notice that A' and B' are half-discs and $A' \cup B'$ is a disc. Therefore for all $F' \in Y^* \setminus \{0\}$, $H_{f'}(A' \vee B') = H_{f'}A'$ or $H_{f'}B'$. According to Theorem 1, the class $[A', B'] \in \mathcal{K}^2(Y)/_\sim$ is convex. Therefore, for any pair $(C, D) \in [A, B]$. the pair $(H_f C, H_f D)$ is convex. Then $H_f(C \vee D) \subset H_f C \cup H_f D$. Now, if $g \in X^*$, $g \neq kf$, where $k \geq 0$ then $H_g(A \vee B)$ is one-point set equal to $H_g A$ or $H_g B$. Therefore, $H_g(C \vee D)$ must be equal to $H_g C$ or $H_g D$. Now, $H_g(C \vee D) \subset H_g C \cup H_g D$ and, according to Proposition 2, $H_f A \neq H_f(A \vee B) \neq H_f B$. Therefore, we may not replace the space $X = \mathbb{R}^2$ in Theorem 2 any other more-than-two dimensional space.

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