ON CONVEX CLASS OF PAIRS OF CONVEX BODIES

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Abstract. In this paper we introduce a quotient class of pairs of convex bodies in which every member have convex union.

The space of pairs of convex bodies has been investigated in a number of papers [3], [8], [9], and [12]. This space has found an application in quasidifferential calculus (cf. [1], [5], [7], [10]). A quasidifferential is represented as a pair of convex bodies and it is essential to find the minimal representation of this pair. The notion of minimal pairs was introduced in [5] and investigated in [2], [6], [7] and [11]. Some criteria of minimality are given in [6]. In this paper we investigate pairs of convex bodies with convex union. We introduced a quotient class of pairs of convex compact sets in which every member has convex union. Moreover some criteria for the convex class are given.

In this paper $X = (X, \tau)$ stands for a real locally convex vector space, and $X^*$ denotes the dual space of $X$. Denote by $\mathcal{K}(X)$ the family of all convex bodies in $X$, i.e., of all nonempty compact convex subsets of $X$. If $A, B$ are nonempty subsets of $X$, then $A + B$ is the usual algebraic Minkowski sum of $A$ and $B$. It may be showed that $\mathcal{K}(X)$ satisfies the order cancellation law; i.e. for every $A, B, C \in \mathcal{K}(X)$ the inclusion $A + B \subset B + C$ implies $A \subset C$ (cf. [12]). Hence it follows that $\mathcal{K}(X)$ endowed with the Minkowski sum is a commutative semigroup satisfying the law of cancellation.

Now let $\mathcal{K}^2(X) = \mathcal{K}(X) \times \mathcal{K}(X)$; the equivalence relation between pairs of convex bodies is given by: $(A, B) \sim (C, D)$ if and only if $A + D = B + C$. For $A, B \in \mathcal{K}(X)$ we will use the notation $A \vee B := \text{conv}(A \cup B)$, where the operation "conv" denotes the convex hull. If $A, B, C \in \mathcal{K}(X)$, and $b \in X$, then $A \vee B + C = (A \vee B) + C$ and $A + b = A + \{b\}$. We have $[a, b] = \{a\} \vee \{b\}$.

Let $f \in X^*$, $A \in \mathcal{K}(X)$ and $c \in \mathbb{R}$. We denote by $p_A(f) := \max_{x \in A} f(x)$ the support function of the set $A$. Moreover, $H_f^c := \{x \in X \mid f(x) = c\}$ and $H_f A := \{x \in A \mid f(x) = p_A(f)\}$, where $H_f^c$ is the hyperplane generated by the functional $f$ and the number $c$, and $H_f A$ is the face of $A$ with respect to $f$. For the sum of the faces of two convex bodies $A, B \subset X$ with respect to $f \in X^*$ the identity $H_f (A + B) = H_f A + H_f B$ holds true. For $A \subset X$ we denote by $\partial A$ the boundary $A \setminus A^o$ of the set $A$, where $A := d A$ and $A^o := \text{int} A$. 

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Proof. Necessity. For arbitrary sets \( A, B \subseteq X \) we have \( \partial(A \cup B) \subseteq \partial A \cup \partial B \subseteq A \cup B \). But \( A \cup B = A \cup B \). Hence \( \partial(A \cup B) \subseteq A \cup B \).

Sufficiency. Let \( \dim X < \infty \). Given any \( x \in A \cup B \) with \( x \not\in A \) there exist \( f \in X^* \) and \( c \in \mathbb{R} \) such that the hyperplane \( H^*_f \) separates the set \( \{x\} \) and \( A \). We can assume that \( x \in H^*_f \) and \( H^*_f \cap A = \emptyset \). Take any line \( l \subseteq H^*_f \) passing through the point \( x \). Then \( f \cap (A \cup B) = [p, q] \) for some \( p, q \in \partial(A \cup B) \). But \( \partial(A \cup B) \subseteq A \cup B \), \( \partial(A \cup B) = \emptyset \). Hence \( p, q \in B \) and we get \( x \in [p, q] \subseteq B \).

Now, let \( \dim X = \infty \). Then \( \partial(A \cup B) = A \cup B \), and \( \partial(A \cup B) \subseteq A \cup B \) implies \( A \cup B = A \cup B \). Hence \( A \cup B = A \cup B \).

If \( \dim X = 1 \), then for \( A := \{0\}, B := \{1\} \) we have \( \partial(A \cup B) = \{0, 1\} = A \cup B \) but \( A \cup B \) is not convex.

In [4] the following is proved:

Lemma. If \( X \) is finite-dimensional and \( A \subseteq X \) is a convex set, then at any point \( x \) of the boundary \( \partial A \) of \( A \) there is a supporting hyperplane for \( A \).

In the infinite-dimensional case, the above lemma does not hold true. For example let \( X = \ell^2 \) and we consider the Hilbert cube \( A := \{x = (\xi_n) \mid \xi_n \in \mathbb{R}, \text{ and } |\xi_n| \leq \frac{1}{n}\} \). The set \( A \) is compact and convex. It is easy to observe that \( p_A(f) > 0 \) for every nontrivial \( f \in X^* \). Since \( A \) is compact \( \partial A = A \). Moreover, \( f(0) = 0 \) for any \( f \in X^* \). So, there is no supporting hyperplane at 0.

Proposition 2. If \( 1 < \dim X < \infty, A, B \in K(X) \), then \( A \cup B \) is convex if and only if \( H_f(A \cup B) \subseteq H_f A \cup H_f B \) for every \( f \in X^* \setminus \{0\} \).

Proof. Necessity. Given any \( f \in X^* \setminus \{0\} \), we have

\[
p_{A \cup B}(f) = \max\{p_A(f), p_B(f)\}.
\]

Now let \( p_A(f) < p_B(f) \). Then \( H_f(A \cup B) = H_f B \). Analogously, if \( p_A(f) > p_B(f) \), we obtain \( H_f(A \cup B) = H_f A \). Suppose \( p_A(f) = p_B(f) \). If \( x \in H_f(A \cup B) \subseteq A \cup B \), then \( x \in A \) or \( x \in B \). If \( x \in A \), then \( x \in H_f A \). It follows from the above that \( H_f(A \cup B) \subseteq H_f A \cup H_f B \) for every \( f \in X^* \setminus \{0\} \).

Sufficiency. Let \( x \in \partial(A \cup B) \). Since \( \dim X < \infty \), so from the lemma it follows that \( x \in H_f(A \cup B) \) for some nontrivial \( f \in X^* \). And we obtain from assumption \( x \in A \cup B \). Hence \( \partial(A \cup B) \subseteq A \cup B \). Now, it follows from Proposition 1 that \( A \cup B \) is convex.

Theorem 1. Let \( 1 < \dim X < \infty, A, B \in K(X) \). If \( H_f(A \cup B) = H_f A \) or \( H_f B \) for every \( f \in X^* \setminus \{0\} \) then the class \( [A, B] \) is convex.

Proof. We observe that

\[
H_f(A \cup B) \subseteq H_f A \cup H_f B \quad \text{for every nontrivial } f \in X^*.
\]

Hence from Proposition 2 we have that \( A \cup B \) is convex.
Now, given any pair \((C, D) \in K^2(X)\) equivalent to \((A, B)\), we have
\[
A + C \vee D = (A + C) \vee (A + D) = (A + C) \vee (B + C) = C + A \vee B.
\]
Analogously
\[
B + C \vee D = D + A \vee B.
\]
We also have
\[
H_f A + H_f (C \vee D) = H_f C + H_f (A \vee B),
\]
\[
H_f B + H_f (C \vee D) = H_f D + H_f (A \vee B) \text{ for every } f \in X^* \setminus \{0\}.
\]
But
\[
H_f (A \vee B) = H_f A \text{ or } H_f B.
\]
Hence from the law of cancellation we have
\[
H_f (C \vee D) = H_f C \text{ or } H_f D \text{ for every } f \in X^* \setminus \{0\}.
\]
This implies that
\[
H_f (C \vee D) \subset H_f C \cup H_f D \text{ for every } f \in X^* \setminus \{0\}.
\]
Hence we obtain from Proposition 2 that \(C \cup D\) is convex.

The condition \(H_f (A \vee B) \subset H_f A \cup H_f B\) in Theorem 1 is not sufficient. For example, let \(A, B \in K(\mathbb{R}^2)\),
\[
A := \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}, \ B := \{(1, 0)\} + A.
\]
Then \(A \vee B = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 1\}, \ H_f (A \vee B) \subset H_f A \cup H_f B\) for \(f \in X^* \setminus \{0\}\). Define \(C := \{(0, 0)\}, \ D := \{(1, 0)\}\). Then \(A + D = B + C\), but \(C \cup D = \{(0, 0), (1, 0)\}\) is not convex.

**Example 1.** Convex classes

i) Take any \((A, B) \in K^2(X)\) such that \(A \subset B\) and any \((C, D) \in K^2(X)\) being an equivalent pair to \((A, B)\). Then \(B + C = A + D \subset B + D\) and from the order law of cancellation, we have \(C \subset D\). Hence \(C \cup D\). It is obvious that \(\partial (A \vee B) \subset A \cup B\).

ii) Let \(X = \mathbb{R}^2\) and \(R > 0\). Consider the closed ball \(\mathbb{B}((0, 0), R)\) and let
\[
a := (-\frac{1}{2}, \sqrt{2} \cdot R, \frac{1}{2}, \sqrt{2} \cdot R), \ b := (\frac{1}{2}, \sqrt{2} \cdot R, \frac{1}{2}, \sqrt{2} \cdot R).
\]
Define
\[
A := \{(x, y) \in \mathbb{B}((0, 0), R) \mid -\frac{1}{2} \cdot \sqrt{2} \cdot R \leq x \leq \frac{1}{2} \cdot \sqrt{2} \cdot R\},
\]
\[
B := \{(x, y) \in \mathbb{B}((0, 0), R) \mid -\frac{1}{2} \cdot \sqrt{2} \cdot R \leq y \leq \frac{1}{2} \cdot \sqrt{2} \cdot R\}
\]
(see Figure 1).

We have \(A \cup B = \mathbb{B}((0, 0), R)\) and \(A \cap B = -a \lor (-b) \lor a \lor b\). Since \(H_f (A \vee B) = H_f A \text{ or } H_f B\) for every nonzero \(f \in X^*\), it follows from the above theorem that the class \([A, B]\) is convex.

**Theorem 2.** Let \(X = \mathbb{R}^2\), \(A, B \in K(X)\). Then the class \([A, B]\) is convex if and only if \(H_f (A \vee B) = H_f A \text{ or } H_f B\) for all \(f \in X^* \setminus \{0\}\).
Proof. Necessity. Assume that $H_f A \neq H_f (A \cup B) \neq H_f B$ for some $f \in X^* \setminus \{0\}$. It follows from the assumption that $p_A (f) = p_B (f)$ and the faces $H_f A$ and $H_f B$ are parallel segments and not one-point sets. In fact, $H_f A$ and $H_f B$ are contained in one line. Denote $H_f A := a \vee b$, $H_f B := c \vee d$, where $a, b, c, d \in \mathbb{R}^2$ and assume that $d - c = k \cdot (b - c)$ for some $k \geq 1$. Let $e \in \mathbb{R}^2$ be a point $H_f T = e$ where $I = a \vee b$ and $T = I \vee e$. Then $H_{-f} T = I$. Denote $J := (c - a) \vee (d - b)$.

We have

\[
H_f (A + T) = I + e, \quad H_f (B + T) = I + J + e, \\
H_{-f} (A + T) = I + H_{-f} A, \quad H_{-f} (B + T) = I + H_{-f} B.
\]

Therefore, the segment $I$ is a summand of both $A + T$ and $B + T$. Let $A', B' \in \mathcal{K} (X)$, $A + T = A' + I$ and $B + T = B' + I$.

We have

\[
H_f A' + I = H_f A' + H_f I = H_f (A + T) = I + e, \\
H_f B' + I = I + J + e.
\]

It follows from these equations that $H_f A' = e$ and $H_f B' = J + e$. Since $H_f B$ does not contain $H_f A$ then $0 \not\in J$, and $e \not\in J + e$. Therefore, $H_f A' \cap H_f B' = \emptyset$. Since $p_A (f) = p_B (f)$ then $H_f (A' \vee B') = H_f A' \vee H_f B' \neq H_f A' \cup H_f B' = H_f (A' \cup B')$. According to Proposition 2, the pair $(A', B')$ is not convex while $(A', B') \in [A, B]$. This contradicts the assumption of our theorem.

Sufficiency. It follows immediately from Theorem 1.

\[\Box\]

Example 2. Let $X := \mathbb{R}^3$, $A := \{(x, y, z) \in B((0, 0), R) \mid x \leq 0, z \leq 0\}$, $B := \{(x, y, z) \in B((0, 0), R) \mid x \geq 0, z \leq 0\}$ (see Figure 2). Denote $f(x, y, z) := z$. 

\[\Box\]
The functional $f \in X^* \setminus \{0\}$ and $A' := H_f A, B' := H_f B \subset Y := \mathbb{R}^2 \times \{0\}$. Notice that $A'$ and $B'$ are half-discs and $A' \cup B'$ is a disc. Therefore for all $F' \in Y^* \setminus \{0\}, H_f F' (A' \vee B') = H_f F' A'$ or $H_f F' B'$. According to Theorem 1, the class $[A', B'] \in K_2(Y)/\sim$ is convex. Therefore, for any pair $(C, D) \in [A, B]$, the pair $(H_f C, H_f D)$ is convex. Then $H_f (C \vee D) \subset H_f C \cup H_f D$. Now, if $g \in X^*, g \neq kf$, where $k \geq 0$ then $H_g (A \vee B)$ is one-point set equal to $H_g A$ or $H_g B$. Therefore, $H_g (C \vee D)$ must be equal to $H_g C$ or $H_g D$. Now, $H_g (C \vee D) \subset H_g C \cup H_g D$ and, according to Proposition 2, $H_f A \neq H_f (A \vee B) \neq H_f B$. Therefore, we may not replace the space $X = \mathbb{R}^2$ in Theorem 2 any other more-than-two dimensional space.

REFERENCES